1

Introduction to Abstractionism

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1.1 WHAT IS ABSTRACTIONISM?

Abstractionism in philosophy of mathematics has its origins in Gottlob Frege's logicism—a position Frege developed in the late nineteenth and early twentieth century. Frege's main aim was to reduce arithmetic and analysis to logic in order to provide a secure foundation for mathematical knowledge. As is well known, Frege's development of logicism failed. The infamous Basic Law V—one of the six basic laws of logic Frege proposed in his magnum opus *Grundgesetze der Arithmetik*—is subject to Russell's Paradox. The striking feature of Frege's Basic Law V is that it takes the form of an abstraction principle. The general form of an abstraction principle can by symbolised like this:[1]

$$\S(\alpha) = \S(\beta) \leftrightarrow \alpha \sim \beta$$

where '§' is a term-forming operator applicable to expression of the type of α and β, and $\sim$ is an equivalence relation on entities denoted by expressions of that type. Accordingly, abstraction principles are biconditionals that feature an equivalence relation on the right-hand side and an identity statement on the left-hand side. The *abstracta* denoted by the terms featuring in the identity statement on the left are taken to be introduced, in some sense, by the abstraction principle, giving the equivalence on the right-hand side conceptual priority over them. More on this below.

Frege's ill-fated Basic Law V, involves *co-extentionality* (of functions) as the relevant equivalence relation on the right-hand side, introducing, what Frege termed *value-ranges*, $\dot{\varepsilon}\varphi(\varepsilon)$, on the left:[2]

[1] Here and below, we will omit prefixed universal quantifiers in abstraction principles. We are thereby in effect neglecting the distinction between *schematic* and *axiomatic* (or *universal*) formulations of abstraction principles. In the context of full impredicative second-order logic, these formulations are equivalent, but in systems with weaker second-order comprehension (see p. 19 below), these come apart: the schematic formulations entail the axiomatic ones, but not *vice versa*; see e.g. Heck (1996, §1), Fine (2002, pp. 36–38), or Linnebo (2004, p. 158).

[2] In words: The value-range of function f is identical to the value-range of function g if and only if f and g have the same value for any argument. The value-range of a function is roughly

$$\dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}g(\varepsilon) \leftrightarrow \forall x(f(x) = g(x))$$

Ultimately, Frege was unable to find a suitable alternative for his Basic Law V (more on this in §2 below) and gave up on his logicist project.[3] In the latter half of the twentieth century, logicism enjoyed a revival. Its main catalyst was Crispin Wright's *Frege's Conception of Numbers as Object*, published in 1983. Wright soon joined forces with Bob Hale and together they developed and defended a view now often referred to as "neo-Fregeanism" or sometimes more specifically as the "Scottish school of neo-Fregeanism".[4] The main tenet of neo-Fregeanism is to revive a version of Frege's logicism by substituting the inconsistent Basic Law V with a principle called *Hume's Principle*. Like Basic Law V, Hume's Principle is an abstraction principle:

Hume's Principle (HP)

$$Nx{:}Fx = Nx{:}Gx \leftrightarrow F \approx G$$

where '$Nx{:}Fx$' stands for "the (cardinal) number of Fs" and '$\approx$' denotes the equivalence relation of equinumerosity between concepts.[5] Hume's Principle was, of course, considered by Frege in his *Grundlagen der Arithmetik* in 1884. He rejected it, however, considering it ill-suited as a foundation for arithmetic (more on this below). It was not until Geach (1955) that this way to revive a version of Frege's Logicism was reconsidered. Geach claimed that Frege's decision to identify numbers with extensions was questionable and suggested that one could prove the infinity of the number series without drawing on "any

what we would call its graph today. In the special case of concepts, the value-range is the *extension* of the concept. Concepts F and G have the same *extensions* if and only they are co-extensional (i.e., the same objects fall under them): $\dot{\varepsilon}F\varepsilon = \dot{\varepsilon}G\varepsilon \leftrightarrow \forall x(Fx \leftrightarrow Gx)$

[3] As we now know, if embedded in a weaker logic—e.g., predicative second-order logic—Basic Law V does not entail a contradiction. This has generated some very interesting research on identifying consistent fragments of Frege's *Grundgesetze* logic that retain Basic Law V—for further details see §1.3 below. For all we know, Frege never considered a weakening of the logic as a way out of the paradox. Indeed, it might seem to go against Frege's general conception of logic.

[4] In his excellent critical survey, MacBride (2003) distinguishes "neo-logicism" from "neo-Fregeanism". Neo-logicism stands for "the doctrine that Frege's judgement was premature [...] Frege should not have abandoned (HP)" (p. 106) while "neo-Fregeanism" stands for the general conception of the relation between language and reality that Hale and Wright are interpreted to have adopted. We here use the term "neo-Fregeanism" to stand for Hale and Wright's version of Abstractionism generally.

[5] HP may be glossed as: the cardinal number belonging to the concept F is identical to the cardinal number belonging to the concept G if and only if there is a one-to-one correspondence between the objects falling under F and those falling under G. The equivalence relation of equinumerosity (one-to-one correspondence, bijection) can be formulated in purely (second-order) logical vocabulary. In full detail, HP is the following statement:

$$\begin{aligned} &Nx{:}Fx = Nx{:}Gx \leftrightarrow \\ &\exists R(\forall x[Fx \supset \exists y(Gy \wedge Rxy \wedge \forall z(Gz \wedge Rxz \supset z = y))] \wedge \\ &\quad \forall y[Gy \supset \exists x(Fx \wedge Rxy \wedge \forall z(Fz \wedge Rzy \supset z = x))]) \end{aligned}$$

special set theory” (Geach, 1955, p. 569). In 1965, Charles Parson went a step further and noted explicitly that the derivation of the axioms of arithmetic “could be carried out by taking [Hume’s Principle] as an axiom” (Parsons, 1965, p. 194).

It was, however, not until Wright (1983) that such proof was presented. Here, Wright proves that the axioms of arithmetic can be derived from Hume’s Principle using second-order logic and Frege’s definition of zero, predecession, and natural number. Following Boolos (1990), the proof is now known as *Frege’s Theorem*.[6]

Wright not merely establishes Frege’s Theorem; he also offers the first robust philosophical defense of Hume’s Principle as a foundational principle. In contrast to Frege, who considered his Basic Law V to be a *logical* law, Wright does not take Hume’s Principle to be purely logical but regards it more akin to a definition or explanation of the concept CARDINAL NUMBER.[7] What is important here in particular is the status of Hume’s Principle as an abstraction principle: the fact that Hume’s Principle takes the form of an abstraction principle makes it especially suited as an explanation of the concept CARDINAL NUMBER and thus as a foundational principle.

So understood, we can regard neo-Fregeans among the main proponents of *Abstractionism*: the view that abstraction principles play a crucial role in the proper foundation of arithmetic, analysis, and possibly other areas of mathematics. Abstractionism therefore has two main aspects, a *mathematical* and a *philosophical* one. The main aim of the mathematical aspect of any abstractionist programme is the *mathematics of abstraction*—bluntly put: proving mathematical theorems about abstraction principles or taking abstraction principles as basic axioms and investigating the resulting theories. A primary aim is to capture various mathematical theories, such as arithmetic, analysis, complex analysis, or set theory as deriving from a few basic abstraction principles and (versions of) higher-order logic. Frege’s Theorem is one of the most important result for a mathematical Abstractionist, and numerous other interesting results have been discovered since.[8]

The philosophical aspect of Abstractionism is to offer a philosophical account of why it is beneficial to adopt abstraction principles as foundations for arithmetic or other mathematical theories. Broadly speaking, we can distinguish three kinds of philosophical themes concerning philosophical Abstractionism: semantic, epistemological, and ontological.

What we call *semantic Abstractionism*[9] is the thesis that our capacity to

[6]Compare also Heck (2011a, 2012), who provides an insightful and detailed account of the history of Frege’s Theorem and discusses the question whether Frege himself was aware of this theorem.

[7]In fact, this is in stark contrast to Frege. In his *Grundgesetze*, Frege notes that Basic Law V is not to be understood as a definition of the concept VALUE-RANGE; see Frege (1903, §146).

[8]See §1.3 below, were we will provide a short overview of some technical results.

[9]Semantic Abstractionism is sometimes identified with *Abstractionism* simpliciter, see e.g. Heck (2011a), p. 14.

have singular thoughts about objects of a certain type derives from and is constituted by an appreciation of the truth-conditions of identity judgements about objects of that type. The identity judgements involve the fundamental way of referring to objects of that type. The crucial claim of semantic Abstractionism is then that the truth-conditions of such identity judgements can be given by means of an abstraction principle, involving an equivalence relation of the relevant kind. Connected to this thesis is the claim often made by semantic abstractionists that abstraction principles are not only ideally suited to provide for our capacity of singular thought but also that they can introduce us to a *new* concept.

To explain the main claim of semantic Abstractionism, let us briefly consider a more "mundane" type of abstraction principle:

Abstraction Principle for Directions (AP_d)

The *direction* of line *a* equals the *direction* of line *b*
if and only if line *a* and line *b* are parallel.

$$d(x) = d(y) \leftrightarrow x \,//\, y$$

Based on this abstraction principle (featuring parallelism of lines as the relevant equivalence relation), we can grasp the concept DIRECTION. Given that identity judgements involve the fundamental way of referring to objects of this kind, it is by means of AP_d that we can have singular thoughts about the objects falling under the concept DIRECTION. Abstraction principles thus provide a way of grasping and apprehending objects, in particular abstract objects such as directions or numbers.[10] Contributions in Part II of this volume discuss various issues relating to *semantic* Abstractionism.

Epistemic Abstractionism is the view that abstraction principles are, in some sense, epistemically *innocent*. The basic claim is that abstraction principles, or at least those abstraction principles that fulfill certain criteria for being "good", are ideally suited to be warrantedly accepted as basic principles.[11] As early as 1983, Wright made the claim that Hume's Principle *qua* abstraction principle, is epistemically innocent given its status as a kind of definition:

"The fundamental truths of number theory would be revealed as consequences of an *explanation*: a statement whose role is to fix the character of a certain concepts."

[10] As we will discuss later, Frege himself rejected abstraction principles as genuinely concept-constitutive by raising what is now known as the *Julius Caesar* problem, see §1.2 below. The neo-Fregeans have offered numerous solutions to the problem, such as Wright (1983) and Hale and Wright (2001b). For more recent challenges against semantic abstractionism compare Part II of this volume. For further discussion of the neo-Fregean solution of the Caesar problem in particular, see Sullivan and Potter (1997), Stirton (2003), Potter and Sullivan (2005), Pedersen (2009), and Kim (2011), amongst others. For a response to Potter and Sullivan see Hale and Wright (2008).

[11] Establishing the correct criteria for distinguishing "good" from "bad" abstraction principle is a difficulty known as the *Bad Company objection*, which we will discuss in a little more detail in §1.5 below.

(Wright, 1983, p. 153)

As a result, Hume's Principle itself was regarded as a definition and so as an *analytic* truth that merely fixes the truth conditions of the concept of number. George Boolos and others opposed this conception of Hume's Principle as an analytic definition due to its substantial ontological commitments.[12]

There are other attempts to justify the foundational status of Hume's Principle, e.g. in Hale and Wright (2000), it is regarded as a specific type of stipulative implicit definition which explains our non-inferential *a priori* knowledge of Hume's Principle.[13] In the latest development of epistemic Abstractionism, it is argued that we have an *entitlement*, i.e. a certain type of non-evidential and non-inferential warrant, to accept Hume's Principle. What combines all these approaches is the underlying thought that Hume's Principle *qua* abstraction principle is concept-constituting or analytic of the concept CARDINAL NUMBER: we are (defeasibly) warranted to accept the principle because of its meaning-constituting character. It is here where epistemic Abstractionism draws on views defended by semantic Abstractionism. Contributions in Part III of this volume develop and discuss new forms of *epistemic* Abstractionism.

The thesis we call *ontological Abstractionism* is the view that good abstraction principles introduce new terms referring to *sui generis* objects. That is, in the case of Hume's Principle, the number-terms so introduced refer to *sui generis* abstract objects, namely cardinal numbers.[14] Thus, an ontological abstractionist defends a broadly platonist metaphysical picture of mathematical objects. However, this view has not gone unchallenged: Michael Dummett, one of the staunchest critics of the Neo-Fregean programme, repeatedly takes issue with the platonist aspects of ontological Abstractionism.[15] Contributions in Part II of this volume continue this line of criticism and question to what extend abstraction principles are indeed compatible with a broadly platonist conception of mathematical objects.

1.2 HISTORY OF ABSTRACTIONISM

Philosophers and mathematicians have entertained abstraction principles before Frege. Also his contemporaries, such as Dedekind, von Staudt, Plücker, Stolz, Klein, Schlömilch, and Grassmann have used ideas that underlie, or are similar to, Frege's method of abstraction.[16] Given that Frege's work on abstraction principles has been the most influential in recent debates and given

[12] Compare Field (1984a,b) and Boolos (1997); see Wright (1999), Hale (1994a), and Ebert (2008) for responses. See also Shapiro and Weir (2000) and Potter and Smiley (2001).

[13] See e.g. Rayo (2003), Ebert (2005b), or MacFarlane (2009) for criticism.

[14] There is a stronger view in the vicinity, first defended in Hale (1987) that all *sui generis* abstract objects can be captured by appropriate abstraction principles.

[15] See his Dummett (1981a), Dummett (1981b), and Dummett (1991) for a discussion of some his criticism, see for example MacBride (2003) and Ebert (2015).

[16] To what extend Frege was or might have been influenced by his contemporaries and teachers is discussed in Wilson (1992, 2010), Tait (1996), and Mancosu (2015a, forthcoming). For a general

that current forms of Abstractionism take Frege's work as their starting point, it is here where we begin our overview.

Frege develops his logicist account of arithmetic in *Die Grundlagen der Arithmetik* (1884). His project started with the publication of *Begriffsschrift* in 1879, which also marks the birth of modern logic. What led him to the development of his logic, *concept-script*, was the need for an appropriate tool to develop his logicist programme. Frege writes:

> The approach was here the following: first I attempted to reduce the concept of ordering in a series to that of *logical* sequence in order to proceed from here to the concept of number. To prevent anything intuitive from penetrating here unnoticed, everything had to depend on the gaplessness of the chain of inference. In striving to comply with this demand in the strictest possible way, I found an obstacle in the inadequacy of language; the more complex the relations became, the less I was able, given the resulting unwieldiness of expressions, to attain the precision that my aim required. This need then led me to the idea of the present concept-script.
>
> (Frege, 1879, p. IV)

The requirement of gaplessness of the chains of inference is a theme that remained at the heart of the logicist project. *Begriffsschrift*, however, was not all too well received by his contemporaries—a trend that continued with the poor reception of his main work, *Grundgesetze der Arithmetik* (1893/1903).[17] Before publishing his *Grundgesetze*, a precursor of which already existed as a nearly complete manuscript around 1882,[18] Frege published *Die Grundlagen der Arithmetik* intended as a more accessible introduction.[19] It contains in its first part criticisms of well-known approaches to the philosophy of arithmetic, proffered by Mill and Kant, and Frege's contemporaries such as Schröder, Cantor, Hankel, and others. In the second part of *Grundlagen*, Frege develops his logicism: the thesis that arithmetic is reducible to logic plus explicit definitions and that therefore our arithmetical knowledge is grounded in our logical knowledge.

We will here focus only on §§62–69 of *Grundlagen*, which Dummett regards as the "most brilliant and philosophically most fruitful [passages] in the book and the most important for Frege's philosophy of mathematics, and, indeed, his philosophy generally" (Dummett, 1991, p. 111). It is here that Frege performs the often celebrated linguistic turn, and it is here that he first considers an abstraction principle—Hume's Principle—as an answer to the most basic question: "How, then, is a number given to us, if we do not have any

account of the role of abstraction in the nineteenth century and its relation to Greek mathematics see Stein (1990).

[17] See in particular Frege's Foreword to *Grundgesetze*, pp. X–XI, where he complains about the lack of reception of his work and suggests that his requirement of gaplessness may seem off-putting to many of his contemporaries.

[18] Compare Frege's letter to Marty in Frege (1976), XXX/1, p. 163, and for discussion Heck (2012), ch. 1.

[19] He seemed to have followed the advice from Carl Stumpf, who in a letter encouraged him to spell out his ideas in a more accessible manner. See Frege (1976), XL, p. 257.

idea or intuition of it?" (*Grundlagen*, §62). The curious twist in the history of Abstractionism is that Frege rejects the attempt to answer the question by appeal to Hume's Principle.

Frege explores addressing the epistemic challenge he poses at the beginning of §62 by appeal to the *context principle*: only in the context of a sentence do words have meaning.[20] According Frege, then, to account for our knowledge of numbers we have to explain the content of a sentence in which the number word occurs, in particular, we require a general criterion that allows us to recognise some object a as the same again. Frege's proposal is to use Hume's Principle, which fixes the truth-conditions of identity statements involving the concept CARDINAL NUMBER.[21]

It is exactly this basic proposal in which semantic and epistemic Abstractionism take their origin. The story, however, does not quite end here for Frege: in the subsequent sections, he discusses three criticisms against the use of Hume's Principle *qua* abstraction principle to account for our grasp of numbers. He rejects the first two challenges, and then presents what is now known as the *Caesar Problem*. Having considered the abstraction principle for directions, AP_d instead of HP in most of his discussion, Frege presents the following challenge in §66:

> In the proposition,
>
> "the direction of a is identical with the direction of b"
>
> the direction of a appears as object, and our definition affords us a means of recognizing this object as the same again, in case it should happen to appear in some other guise, say as the direction of b. But this means does not suffice for all cases. One cannot, for instance, decide on its basis whether England is the same as the direction of the Earth's axis. Please forgive the example which seems nonsensical! Of course, no one will confuse England with the direction of the Earth's axis; but that is no thanks to our explanation. It says nothing as to whether the proposition
>
> "the direction of a is identical with q"
>
> should be affirmed or denied, unless q is given in the form of 'the direction of b'. What we lack is the concept of direction [. . .]. (Frege, 1884, pp. 77–78)

The challenge is structurally similar to one raised in §56. Here, Frege's example does not involve the direction of the Earth's axis and England, but rather concerns the question whether Julius Caesar is a number—hence the

[20]"*Nur im Zusammenhange eines Satzes bedeuten die Wörter etwas.*" We here follow Austin and use "meaning" to render the German "*Bedeutung*" given that Frege is yet to draw his famous sense-reference distinction.

[21]Frege does not use the term "Hume's Principle", but he does refer to a passage in Hume's *Treatise* (as quoted by Baumann (1869, p. 565), (Frege, 1884, p. 78)). The original reads as follows: "We are in the possest of a precise standard, by which we can judge of the equality and proportion of numbers; and according as they correspond or not to that standard, we determine their relations, without any possibility of error. When two numbers are so combined, as that the one has always an unite answering to every unite of the other, we pronounce them equal'; and 'tis for want of such a standard of equality in extension, that geometry can scarce be esteem'd a perfect and infallible science." (Hume, 1739/1987, book I, part III, section I, p. 71)

label "Caesar Problem".[22] More generally, abstraction principles do not settle truth-conditions of mixed identity statements of the form:

$$\S(\alpha) = t$$

where t is not of the form $\S(\beta)$.

Frege rejects Hume's Principle as an adequate foundation for his logicist project for this reason, the Caesar Problem, and instead turns to an explicit definition of cardinal numbers as *extensions*. On the basis of this explicit definition, he then proceeds to derive Hume's Principle as a theorem, and using it and other presumed laws of logic and further explicit definitions, he offers proof sketches of numerous familiar laws of arithmetic.[23]

Unfortunately, Frege says preciously little about extensions—in fact, he simply presupposes that the reader knows what extensions are and even considers them ultimately superfluous for his logicism. In a tantalizing footnote in §68 he writes:[24]

> I believe that one could say instead of "extension of the concept' simply "concept". [...] I presuppose that one knows what the extension of a concept is.

Given the lack of gapless proofs to establish the laws of number and the lack of support for his notion of *extension*, Frege provides a somewhat cautiously optimistic summary of his achievement in *Grundlagen* in §90:

> I do not claim to have made the analytic character of arithmetical propositions more than probable, since one may still doubt whether the proofs can be conducted solely from purely logical laws, whether somewhere an unacknowledged premise of a different kind is involved. Also, the concern is not sufficiently addressed by the indications I have given of the proofs of some of the propositions [...]. (Frege, 1884, p. 102)

So it was the goal of his *magnum opus*, *Grundgesetze der Arithmetik*, to establish beyond doubt what according to *Grundlagen* is a probable account of arithmetic by providing gapless proofs and by stating explicitly the basic laws and explicit definitions required for these proofs. It is here that we encounter the second twist in the Abstractionist history.

In *Grundgesetze*, Frege offers six basic laws and here presents an abstraction principle as one of the basic laws of his formal system. Having previously shown Hume's Principle to be inadequate as a definition for the concept CARDINAL NUMBER, Basic Law V takes that very same form by fixing the

[22] "[W]e can never, to take a crude example, decide by means of our definitions whether any concept has the number *Julius Caesar* belonging to it, whether this familiar conqueror of Gaul is a number or not." (Frege, 1884, p. 68)

[23] For example, Frege provides definitions of zero, natural numbers, and successor. However, as shown by Boolos and Heck (1998), Frege's sketch in §§82–83 of the existence of the successor, having previously established its uniqueness, is confused and ultimately unsuccessful. This situation is remedied in *Grundgesetze* where Frege offers a correct proof. See also Heck (2011a, ch. 3).

[24] Compare here also §107, where Frege suggests that drawing on *extensions* of concepts is not, ultimately, of great importance for his logicism. These are puzzling remarks that are still debated in current Frege scholarship.

identity-conditions for value-ranges by means of co-extensionality as the relevant equivalence relation. It is clear that Frege did not regard Basic Law V as a definition—he says so explicitly in the second volume of *Grundgesetze* (§146, p. 148, fn. 1)—and so he did not change his mind as to the force of the Caesar Problem; but given this, it is equally clear that the concept *value-range* is either still presupposed as known or based on something different.[25]

What exactly the philosophical role of Basic Law V in *Grundgesetze* is—that is whether Frege himself should be regarded as a semantic or even epistemic abstractionist—is itself part of a lively debate in Frege scholarship. Without doubt, however, Frege was a mathematical abstractionist: abstraction principles play a crucial part in the formal system that Frege presents. Part II of *Grundgesetze* shows step by step, in a manner exemplifying the ideal of gapless proofs, how we can, taking an abstraction principle as a basic law of logic, arrive at arithmetic.

However, as is well known, Frege's logicist project failed: Basic Law V is inconsistent in the system of Frege's *Grundgesetze*. In the Afterword to volume II of *Grundgesetze*, Frege gives an account of the antinomy reformulated in his formal system and then offers a fix—replacing Basic Law V by so-called V′:

$$\dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}g(\varepsilon) \leftrightarrow \forall x\big(x = \dot{\varepsilon}f(\varepsilon) \lor x = \dot{\varepsilon}g(\varepsilon) \lor f(x) = g(x)\big) \qquad (\text{V}')$$

For the special case of extensions, V′ states that the extensions of two concepts are the same if and only if the same objects fall under these concepts, with the possible exception of these extensions themselves (Frege, 1903, vol. II, p. 262).

We may assume that Frege later realised that this would not suffice since he never published further work based on V′. Indeed, V′ is inconsistent with the assumption that there are at least two distinct objects.[26] More intriguingly, Frege seems to think that the paradox does not merely concern Basic Law V but also affects the status of abstraction principles in general. In a letter to Russell, Frege considers other abstraction principles but concludes that "the difficulties here, however, are the same as with the transformation of a generality of an equality in a value-range equality."[27] So, it seems that ultimately

[25] Frege does consider a version of the Caesar Problem in §10 of volume I. How this fits into a broader interpretation of Frege's philosophy is another big issue in Frege scholarship. For an extremely insightful discussion of the role of the Caesar Problem in Frege's mature theory, see Heck (2011a) and (Heck, 2012, part I).

[26] In fact, V′ is arguably inconsistent in the system of *Grundgesetze*, because that system entails the existence of two objects: the True and the False. See the thorough investigation in Cook (forthcoming). Leśniewiski was apparently the first person to discover the inconsistency of V′ with the assumption that there are at least two objects (reported by Sobociński (1949–1950, 1984), §IV). The result was popularised by Quine (1955). Linsky and Schumm (1971) seem to have been the first to explicitly recognise the one element model of V′. See also Geach (1956), Dummett (1973), Linsky and Schumm (1973), Klement (2002, pp. 56–57), Landini (2006), Landini (2012, ch. 6), and Heck (2012, ch. 4).

[27] Frege (1902), p. 224; the English translation in Gabriel et al. (1980, p 141) erroneously inserts a "not" between "are" and "the same".

Frege not only gave up on his logicism but also had more general misgivings about the prospect of Abstractionism as a whole.[28]

1.3 ABSTRACTIONISM AND NEO-FREGEANISM

Wright's defense of neo-Fregeanism in *Frege's Conception of Numbers as Objects* (1983) triggered a revival of Abstractionism. Wright defends versions of semantic, epistemic, and ontological Abstractionism and came to regard Hume's Principle as the main foundational principle for our grasp of the concept CARDINAL NUMBER, our knowledge of arithmetic, and our knowledge of numbers as objects. Shortly after the publication, Wright was joined by Bob Hale (1987) as another proponent of neo-Fregeanism. Neo-Fregeanism so understood adopts Frege's assumption of classical (higher-order) logic. Neil Tennant (1987), on the other hand, provided a detailed formal derivations of the Peano–Dedekind axioms within a free intuitionistic relevant logic, and has since developed a view called *constructive logicism* (see also Tennant (2009)). A further distinct form of neo-logicism inspired by Frege's *Grundgesetze* is Edward Zalta's defense of modal logicism (Zalta, 1999).[29] In general, there are now various position that adopt the label "neo-logicism" and take Hume's Principle (or a suitable version thereof) as a foundation of natural number arithmetic.[30] We are not able to provide a survey of the different forms of neo-logicism here (see however the survey in Tennant (2014)), much less assess them. Rather, we focus mainly on Hale and Wright's version of Neo-Fregeanism which was defended in numerous articles and books since 1983.[31] Naturally, the various philosophical concerns raised in the context of Hale and Wright's conception may also be raised with regards to other forms of logicism. We hope that future research will help to establish how well other forms of neo-logicism fare with respect to them, and how the different positions compare.[32]

At bottom, the possibility of this revival of logicism is based on the discovery of Frege's Theorem by Wright (1983, pp. 158–169). In what follows,

[28] See here also Blanchette (2016). Also note that V′ is itself already problematic if viewed as an abstraction principle: there are occurrences of the value-range operator on the right-hand side of the abstraction principle, and so the *explanandum* appears in the *explanans*.

[29] See also Linsky and Zalta (1995), Zalta (2000); moreover, see Anderson and Zalta (2004) for a different approach.

[30] For further examples of broadly (neo-)logicist approaches to arithmetic, see Antonelli and May (2005), Boccuni (2010, 2013), Bostock (1974–79), Demopoulos (1998, 2000), Fine (2002), Heck (1997a, 2011a), Hodes (1984), Linnebo (forthcoming), Rayo (2002, 2005, 2013, ch. 3–4), Urbaniak (2010), and Wehmeier (1999), amongst many others.

[31] Most notably in Wright (1983), Hale (1987), and Hale and Wright (2000).

[32] The beginnings of such a debate may be found in Linsky and Zalta (2006), Ebert and Rossberg (2006), Ebert and Rossberg (2009), and Zalta (2009).

we swiftly outline some of the main results of mathematical Abstractionism to provide a technical background for this volume.[33]

A fully rigorous proof of Frege's Theorem in a classical setting was first presented by Boolos (1990); see also the detailed exposition and discussion in Heck (2011a).[34] Heck (1997a) has since shown that an abstraction principle weaker than Hume's Principle suffices for the foundation of natural number arithmetic: so-called *Finite Hume*, a version of Hume's Principle in which the range of the second-order variables is restricted to finite concepts. *Frege Arithmetic*—i.e., the second-order logic plus Hume's Principle—was shown to be equiconsistent with second-order arithmetic by Boolos (1987)—the model Boolos provides for Hume's Principle was previously hinted at by Geach (1976); see also the (independent) results by Hodes (1984), Burgess (1984), and Hazen (1985).

The success of mathematical Abstractionism is not restricted to capturing natural number arithmetic, however. Much work has gone into investigating the viability of abstractionist foundations for other areas of mathematics. A first step is to extend the abstractionist treatment to real analysis, while a more ambitious goal is providing an abstractionist set theory strong enough to yield Zermelo-Fraenkel set theory.

An approach to analysis inspired by Richard Dedekind's construction of the reals[35] was developed by Stewart Shapiro (2000) (see also Wright (2000)), through step-wise abstraction of integers, rationals, and real numbers. We start with an abstraction principle for *ordered pairs*, $\langle a, b\rangle$:

$$\langle a, b\rangle = \langle c, d\rangle \leftrightarrow (a = c \ \wedge \ b = d)$$

Ordered pairs of natural numbers (provided by Hume's Principle) can then be utilized to define *integers*. This proceeds via an abstraction principle for *differences*:

$$\mathrm{Diff}\langle a, b\rangle = \mathrm{Diff}\langle c, d\rangle \leftrightarrow (a + d = b + c)$$

Integers can be identified with these difference. Since we are working in second-order logic, addition and multiplication for integers can be explicitly defined. The next step is an abstraction principle for *quotients*, using the defined integers and multiplication:

$$\mathrm{Q}\langle m, n\rangle = \mathrm{Q}\langle p, q\rangle \leftrightarrow \big((n = 0 \wedge q = 0) \vee (n \neq 0 \wedge q \neq 0 \wedge m \times q = n \times p)\big)$$

Rational numbers are then identified with quotients $\mathrm{Q}\langle m, n\rangle$, where $n \neq 0$. Defining again addition and multiplication, this time for the rationals, and

[33] Cook (2007) collects a number of important essays on the mathematics of abstraction.

[34] As we noted above Tennant (1987) offers a detailed proof using a free intuitionistic relevant logic. See also Bell (1999) for a discussion of Hume's Principle in a constructive setting. Shapiro and Linnebo (2015) show that Hume's Principle embedded in intuitionistic logic yields Heyting Arithmetic.

[35] Dedekind (1872); regarding the question to what extent Dedekind himself was a logicist see Demopoulos and Clark (2005), Reck (2013), Reck (forthcoming), Yap (forthcoming); see also Tait (1996).

also defining the natural *less than*-relation, we use a final abstraction principle, called *Cut Abstraction*:

$$\mathrm{Cut}(P) = \mathrm{Cut}(Q) \leftrightarrow \forall r(P \leq r \leftrightarrow Q \leq r)$$

‘$P \leq r$’ holds for a concept P (applying to rationals) and a rational r if and only if every rational that is P is less than or equal to r. We can now identify *real numbers* as those cuts Cut(P) where the rationals falling under P are bounded above.

The procedure above yields uncountably many reals that form an ordered field that has the least-upper-bound property, as required for the real numbers. This approach has a decidedly *structural* feel however. A more object-oriented approach was in fact the first proposal for an abstractionist foundation of real analysis. It is due to Bob Hale (2000b), and arguably more Fregean in character (Wright (2000), Hale and Wright (2005, §6)). Hale proposes a first-order abstraction principle that abstracts reals directly from pairs of *quantities*:

$$\mathrm{R}\langle a, b\rangle = \mathrm{R}\langle c, d\rangle \leftrightarrow \mathrm{E}(\langle a, b\rangle, \langle c, d\rangle)$$

where E is an equivalence relation on pairs of quantities. While more Fregean in spirit, this approach has open questions regarding the nature of quantities and the possibility of a purely logical definition of the required equivalence relation E. Moreover, further research needs to establish whether the resulting theory can yield a sufficiently large ontology—that is, a continuum—and can thus interpret real analysis.

As mentioned above, the ambitious aim is an abstractionist foundation of set theory.[36] The abstraction principle that might have seemed promising for a foundation of set theory is, of course, Basic Law V. As explained earlier (see footnote 2 above), if we restrict the range to concepts, rather than all functions, Basic Law V states that two concepts have the same extension if and only if they are co-extensional:

$$\acute{\varepsilon}F\varepsilon = \acute{\varepsilon}G\varepsilon \leftrightarrow \forall x(Fx \leftrightarrow Gx) \qquad \text{(V)}$$

Extensions of concepts would be sufficiently similar to classes[37] to underwrite set theory. Alas, Basic Law V provides a wee bit too much in Frege’s system.

Frege’s first instinct was to restrict Basic Law V in order to avoid the inconsistency, but as already mentioned (§1.2), he did not go about it in the

[36] For assessments of the prospects of an abstractions set theory see, for instance, Clark (1993), Shapiro and Weir (1999), Hale (2000a), Cook (2003a), Shapiro (2003), Linnebo and Uzquiano (2009), Cook (2016).

[37] In the second volume of *Grundgesetze* (1903), §147, Frege concurs: “When logicians have long spoken of the extension of a concept and mathematicians have spoken of sets, classes, and manifolds, then such a conversion forms the basis of this too; for, one may well take it that what mathematicians call a set, etc., is really nothing but the extension of a concept, even if they are not always clearly aware of this.”

right way. George Boolos (1989) also proposes a restriction of Basic Law V, but bases it on the limitation-of-size approach to set theory.[38] The abstraction principle Boolos proposes states that two concepts have the same extension if and only if they are co-extensional *unless* the concepts are “too big”. Concepts are consider to be too big just in case they are the same size as the universe. The latter property is expressible in second-order logic again: as a one-to-one correspondence with the concept of self-identity. If we call this property ‘Big’, Boolos’s abstraction principle for extensions reads like this:

$$\textsc{Ext}(F) = \textsc{Ext}(G) \;\leftrightarrow\; ((\mathrm{Big}(F) \wedge \mathrm{Big}(G)) \vee \forall x(Fx \leftrightarrow Gx)) \qquad \text{(New V)}$$

We may define a set to be the extension of a concepts unless the concept is big:

$$\mathrm{Set}(x) \;=_{\mathrm{df}}\; \exists F(x = \textsc{Ext}(F) \wedge \neg\mathrm{Big}(F))$$

and define membership for extensions as:

$$x \in y \;=_{\mathrm{df}}\; \exists F(Fx \wedge y = \textsc{Ext}(F))$$

Restricted to sets thus defined, the second-order theory containing New V proves the ZF principles of *extensionality*, *empty set*, *pairing*, *separation*, and *replacement*, but not *union*. The extension of Big concepts, call it “Bad”, is still an object in the theory, but Bad cannot be a set on pain of inconsistency. The union of the singleton set containing Bad, however, would have to be a set whose members are all the elements of Bad, which cannot be. New V thus in fact entails the negation of union. We can, however, reformulate union: instead of saying that the union of a set S is the set containing all and only those objects that are members of members of S, we restrict the *definiens* by stipulating that the union of S be the set containing all and only those elements that are members of the *sets* that are members of S (thus excluding the members of Bad, should Bad be a member of S). Formally:

$$\forall x[\mathrm{Set}(x) \rightarrow \exists y(\mathrm{Set}(y) \wedge \forall z[z \in y \leftrightarrow \exists w(\mathrm{Set}(w) \wedge (z \in w \wedge w \in x))])] \qquad \text{(Union}^*\text{)}$$

The occurrence of ‘Set(w)’ expresses the restriction to members that are sets explained above. Note that with this restriction in place, the union* of the singleton of Bad is the empty set. New V entails union*.

New V does not, however, entail *infinity*, *power set*, or *foundation*. Of these, foundation alone can be recaptured if we further restrict the scope of sets taken into account. The obvious restriction is to *pure* sets—those that can

[38] See Hallett (1984) for a careful and thorough study of the limitation-of-size conception of set and a comparison with the iterative conception.

be "build up" from the empty set, as it were. Let a concept F be *closed* exactly when all sets, all of whose members are F, are also F:

$$\text{Closed}(F) \leftrightarrow_{\text{df}} \forall y((\text{Set}(y) \wedge \forall z(z \in y \rightarrow Fz)) \rightarrow Fy)$$

We can define the pure sets as those objects that fall under all closed concepts:[39]

$$\text{Pure}(x) \leftrightarrow_{\text{df}} \forall F(\text{Closed}(F) \rightarrow Fx)$$

New V proves that foundation holds for pure sets.[40]

While the set theory gained from New V certainly is to be booked as a considerable success, it still falls short of full ZF set theory. Roy Cook (2003b) sets out to remedy the shortcoming. Cook aims for a set theory based on abstraction principles that captures the iterative conception of set. His starting point is an abstraction principle for ordinals, the *Size-Restricted Ordinal Abstraction Principle*, SOAP. To get there, consider the obvious, but owing to the the Burali-Forti Paradox inconsistent, Order-Type Abstraction Principle:

$$\text{O}{\small\text{T}}(R) = \text{O}{\small\text{T}}(S) \leftrightarrow R \cong S \tag{OAP}$$

where '≅' denotes the second-order definable relation of *being isomorphic*. Introducing a restriction to relations that are well-ordered[41] (WO) and whose field is not Big (analogous to the way 'Big' is defined above), we may arrive at the consistent *Size-Restricted Ordinal Abstraction Principle*:

$$\begin{aligned} &\text{O}{\small\text{RD}}(R) = \text{O}{\small\text{RD}}(S) \leftrightarrow \\ &[((\neg\text{WO}(R) \vee \text{Big}(R)) \wedge (\neg\text{WO}(S) \vee \text{Big}(S))) \vee \\ &(\text{WO}(R) \wedge \text{WO}(S) \wedge \neg\text{Big}(R) \wedge \neg\text{Big}(S) \wedge R \cong S)] \end{aligned} \tag{SOAP}$$

The abstracta provided by SOAP are used to enumerate the stages in the "construction" of the iterative hierarchy. Moreover, SOAP is satisfiable on all and only infinite domains.

It follows the definition of 'being at the stage of (ordinal) α'. The formal definition is omitted here for the sake of brevity, but, roughly, there will be a base stage, which consists of elements of a chosen basis (if any), and each subsequent stage will contain that basis as well as all those extensions that only contain objects from prior stages. Let 'Bad' be true of a concept if there is no well-ordered ordinal α such that all object falling under that concept are

[39] The rationale, roughly, is that the empty set, not having any members, has to fall under every closed concept; hence, so does its singleton; hence, so do the sets that contain only one or both of these, and so on. See also Boolos (1989), Theorems 1 and 2, for the adequacy of the definition.

[40] New V also proves that foundation holds for *hereditary* sets (in the usual sense). All pure sets (in the sense defined above) are hereditary, but not *vice versa*. For an extensive study of non-pure hereditary sets in the context of New V set theory see Jané and Uzquiano (2004).

[41] Well-ordered in the usual sense, which is second-order definable; see Shapiro (1991, p. 106), Cook (2003b, §4).

at the stage of α. With these notions in hand, we can formulate the abstraction principle *Newer V*:

$$\text{Ext}(F) = \text{Ext}(G) \;\leftrightarrow\; [\forall x(Fx \leftrightarrow Gx) \vee (\text{Bad}(F) \wedge \text{Bad}(G))] \quad \text{(Newer V)}$$

Even in this informal presentation, the reader will have noticed a circularity: Newer V introduces extensions, using “being at the stage of α”, on the right-hand-side of the abstraction principle; but extensions feature in the *definiens* of “being at the stage of α”. Cook (2003b, §5) shows that there are equivalent recursive formulations available, which, if metaphors are permissible, also highlight the mutual “seesawing up” of ordinal stages and extensions in an imagined step-wise construction: the ordinal stages form a spine for the hierarchy of extensions, but more extensions also allow adding more ordinal stages, which, in turn, allows the hierarchy of extensions to be built up further, and so on, indefinitely.

‘Set’ and ‘$\in$’ can be defined as for New V above. Restricting the relevant quantifiers to sets, Newer V entails *extensionality*, *empty set*, *pairing*, *separation*, *union** (but not *union*), and *powerset*. Restricting further to *pure* sets, we can prove *foundation*, just like in the case of New V; also *union* holds for pure sets.[42]

SOAP plus Newer V does not entail *replacement*, and despite SOAP only having infinite models, SOAP plus Newer V does not in general entail *infinity* (i.e., the existence of an infinite set). They do, however, together with an “axiom of infinity”, i.e., the claim that the basis (as mentioned above) contains all finite well-ordered ordinals.[43]

Neither New V (expressing the abstractionist limitation-of-size conception of set), nor SOAP plus Newer V (expressing the abstractionist iterative conception of set) thus provides a set theory as strong as (second-order) ZF. The situation changes, if we adopt both New V and Newer V in tandem. First, however, note the divergence from the original abstractionist claim, according to which terms for abstract are implicitly defined by a single abstraction principle. We might be able to finess the situation regarding New V and Newer V. If Finite Hume and Hume’s Principle are about the same abstract objects (as one might not implausibly argue), then perhaps New V and Newer V can govern the extension-operator in concert. (See the discussions by Fine (2002, esp. p. 49) and Cook (2003b, §9).)

Setting this discussion aside, the technical results are as follows: the second-order theory containing all three abstraction principles, New V, Newer V, and SOAP,[44] is consistent, and with the definitions of ‘Set’ and ‘$\in$’ as above,

[42] Union also holds for hereditary sets (see fn. 40 above), but foundation does not.

[43] Let us call the non-sets *urelemente*. It is tempting to think of the basis as the collection of *urelemente*, but there is in fact no guarantee that it contains all or even only *urelemente*.

[44] Interestingly, in this setting, SOAP can be dispensed with: instead of the ordinals provided by SOAP, the stages can be ordered according to the more “conventional” ordinals (transitive pure sets, well-ordered by ‘$\in$’) supplied by New V, but some complicating adjustments in New V and Newer V are required; see Cook (2003b, p. 90, n. 30).

it proves that 'Big' and 'Bad' are co-extensional and that all non-sets (or *urelemente*) are in the basis.[45] Moreover, we capture all ZF axioms except *foundation* and *infinity*. If we assume, in addition, that there are infinitely many non-sets we can prove *infinity* (thanks to the fact that all *urelemente* are now in the basis). Note that obtaining infinitely many non-sets may be straightforward for the abstractionist, if the cardinal numbers governed by, say, Finite Hume,[46] are not identical to extensions—that is, if we can suppose a favorable solution regarding the question of the identity of abstracta governed by different abstraction principles (Cook (2003b, §10); see also Fine (2002), Cook and Ebert (2005), Mancosu (2015b)). Moreover, once again, *foundation* holds for pure sets. With theses caveats, full second-order ZF is recaptured by way of abstraction.

Mathematical Abstractionism has been developed in a variety of other directions, exploring further aspects of abstractionist mathematics and extending its reach. For instance (with no claim to completeness), Graham Leach-Krouse (2015) investigates structural abstraction principles; Shay Logan (2015, 2016) presents abstractionist foundations for category theory; Morgan Thomas (ms) proposes a single third-order abstraction principle on isomorphisms as an approach to set theory; Stewart Shapiro and Geoffrey Hellman (2016) investigate an abstraction principle for points in a point-free geometry; James Studd (forthcoming) proposes a dynamic approach to abstraction.

These advances in the mathematics of abstraction, of course, do not by themselves answer the philosophical questions that arise for Abstractionism, and indeed they raise further philosophical problems. In the following two sections, we turn our attention to philosophical and mathematical challenges for Abstractionism—and with that to the contents of this volume.

1.4 PHILOSOPHICAL ABSTRACTIONISM: CHALLENGES

This volume is structured to reflect the main themes of Abstractionism. Part II deals with the semantic and ontological issues surrounding Abstractionism, while Part III focuses mainly on the epistemic aspects of Abstractionism.

As discussed above, Frege briefly considered Hume's Principle as an explanation of number-terms, but rejected it because of the *Caesar Problem*. An Abstractionist account of arithmetic that goes back to taking Hume's Principle as a fundamental principle meets the problem again. In the first essay of Part II, "Caesar and Circularity", William Stirton presents a critical investigation of the solution to the Caesar Problem proposed in Hale and Wright (2001b).

Another challenge Abstractionism faces has been dubbed the *Proliferation Problem* (Heck, 2000). Abstractionism appears to make it too easy to refer to

[45] Compare footnote 43 above.

[46] Alas, full Hume's Principle is inconsistent with this abstractionist set theory, see Cook (2003b, p. 90, n. 33).

abstract objects. Any equivalence relation could in principle do as the right-hand side of an abstraction principle, no matter how gerrymandered it may be, or how unusual the resulting *abstracta* are. Critics object that this leads to an undue proliferation of abstract objects. Richard Heck develops this criticism in "The Existence (and Non-Existence) of Abstract Objects" and presents a solution that gives rise to an account of what it is for abstract objects to exist.

A debate concerning the metaontology of abstractionist accounts of mathematics has recently arisen in light of different interpretations of what Wright (1983) calls the *syntactic priority thesis*: singular terms occurring in true atomic sentences are guaranteed to have a referent—i.e., an object is guaranteed to exist for the term to pick out. On that basis then, number-terms, assuming they are singular terms, will be guaranteed to pick out numbers as objects provided that Hume's Principle is true. This, however, seems somewhat too easy to establish platonism, and critics have argued that abstraction principles appear to stipulate abstract objects into existence, and that Abstractionism hence is incompatible with a broadly platonist philosophy of mathematics.[47]

Matti Eklund subjects the ontology of Abstractionism to such a metaontological investigation in "Hale and Wright on the Metaontology of Neo-Fregeanism". He contrasts *maximalism*, which he argues follows from Hale and Wright's reliance the syntactic priority thesis and which, roughly, holds that everything that can exist, does exist—a "maximally promiscuous ontology"—with other metaontological doctrines including a *minimalism* endorsed by Hale and Wright themselves.

Fraser MacBride, in "Neo-Fregean Ontology: Just Don't Ask Too Many Questions", investigates the syntactic priority thesis directly. He presents a dilemma for the abstractionist: if reality is crystalline, possessing a language-independent structure, then the thesis is "hostage to cosmological fortune"; if, on the other hand, reality has plasticity, this seems to be in tension with the thought that statements about these objects are true of an independent reality. MacBride argues that ontological quietism is the only option for neo-Fregeans to meet the challenge: that is, neo-Fregeans need to argue that the question how language harmonises with reality can be rejected.

In the last chapter of Part II, "The Number of Planets, a Number-Referring Term?", Friederike Moltmann challenges the Fregean view that number terms, like 'the number of planets', refer to numbers as abstract object. Moltmann argues that on the contrary such natural-language expressions refer to number tropes, that is, properties instantiated in the plurality of objects that a given statement of number is about.

Part III opens with Philip Ebert's "A Framework for Implicit Definitions and the *A Priori*". According to the view labeled *traditional connection*, which

[47] The origins of this debate can be found in Dummett (1981a,b, 1991); for a discussion of some of his criticisms, see for example Wright (1998a,b), Hale (1994b), MacBride (2003), and Ebert (2015). For the more recent metaontological debate see Eklund (2006), Hawley (2007), Sider (2007), and Hale and Wright (2009).

at least for some time was held by Hale and Wright, abstraction principles are regarded as a special kind of *implicit definitions* of the mathematical terms. Hume's Principle, for instance, would be seen as an implicit definition of the concept CARDINAL NUMBER and offers an *a priori* foundation for classical mathematics. Ebert present a general framework for implicit definitions, identifies the main tenets for this view, and highlights the main challenges it faces.

A new approach to the understanding of the epistemic foundation of abstraction principles employs Wright's notion of *entitlement*, a type of defeasible, non-evidential warrant for presuppositions of particular cognitive projects. In "Abstraction and Epistemic Entitlement: On the Epistemological Status of Hume's Principle", Crispin Wright argues that Hume's Principle enjoys the status of an entitlement. Wright argues that the *a priori* status of our arithmetical knowledge can be secured in this way.

In the next chapter, "Hume's Principle and Entitlement", Nikolaj Pedersen takes up the same topic. He investigates the notion of entitlement within the setting of Abstractionism and asks, *inter alia*, whether entitlements are indeed defeasible, as Wright suggests. Further, Pedersen raises a concern he labels "Generosity Problem": entitlements, he suggests, may seem too easy to come by, so that a wide range of irrational and bizarre projects would appear to have entitlements as their basis. He suggests that a proper assessment of the Generosity Problem highlights the inherent relativity of entitlements.

Part III closes with Agustín Rayo's "Neo-Fregeanism Reconsidered". Rayo presents a platonist account of mathematics that employs a primitive relation: the *just is* relation. *Just is* statements exhibits a tight connection to statements of metaphysically necessary equivalence. According to the account, for the number of the planets to be eight *just is* for there to be eight planets, for instance. Rayo argues that abstraction principles are best understood as *just is* statements of the kind he introduces in his more general account.

1.5 MATHEMATICAL ABSTRACTIONISM: CHALLENGES

Part IV of this collection focuses on aspects that chiefly concern mathematical Abstractionism. We start with the so-called *Bad Company* objection.[48] All abstraction principles are biconditionals featuring statements of identity of *abstracta* of a certain type on one side, and the specification of an equivalence relation on the other. Hume's Principle is the abstractionists' poster child. It gives rise to arithmetic. On the other end, we have Basic Law V—which gives rise to Russell's Paradox. But how do we tell the "good" abstraction principles, like Hume's Principle, from the "bad" ones, like Basic Law V, when their logical form is fundamentally the same? Requiring that an abstraction principle

[48]Bad Company concerns were first raised by Neil Tennant (1987, p. 236) and George Boolos (1987, p. 184); see the discussion in Tennant (2014, §1.2.3). The more recent discussion takes wing from Dummett (1991, pp. 188–189), Wright (1998a), Dummett (1998), and Wright (1998b). See also the articles in the special issue on Bad Company edited by Linnebo (2009).

has to be consistent may seem *ad hoc* and unilluminating; moveover, importantly, it is not even sufficient.

First, there is the question of the underlying logic in which the consistency question is to be decided. Terence Parsons (1987) provides a consistency proof for the first-order fragment of Frege's system in *Grundgesetze*, including Basic Law V. Richard Heck (1996) shows that Basic Law V is consistent in predicative second-order logic. As Kai Wehmeier (1999) and Fernando Ferreira and Wehmeier (2002) show, Basic Law V is indeed consistent in a second-order logic with Δ^1_1-comprehension.[49]

Second, even the restriction to abstraction principles that are individually consistent or perhaps satisfiable in full second-order logic will not suffice to demarcate the "good" from the "bad" abstraction principles. There are abstraction principles that are individually consistent, but not consistent with one another. Hume's Principle, for instance, is not jointly satisfiable with George Boolos's *Parity Principle* or Wright's *Nuisance Principle*.[50] Both of these abstraction principles require the domain to be finite, whereas Hume's Principle is only satisfiable on an infinite domain. Wright (1997) proposes that acceptable abstraction principles have to be *conservative*, in the sense that they should not put any constraints on any objects other than those abstracts that it introduces. Hume's Principle passes this test, but the Nuisance Principle does not: it does not only require that there are only finitely many nuisances, but indeed that there are only finitely many object in total—a violation of conservativeness. The Parity Principle is disqualified for the same reason. While conservativeness might be a necessary condition, it does not appear to be sufficient: Alan Weir (2003) produces a pair of abstraction principles each of which is conservative; however, they are not jointly satisfiable.

Roy Cook's contribution to this volume, "Conservativeness, Cardinality, and Bad Company", systematizes the discussion. Cook rigorously formulates

[49]Wehmeier (1999) and Ferreira and Wehmeier (2002) investigate different Δ^1_1 theories, as explained in the latter publication. Restrictions on the second-order comprehension schema in effect regulate how much impredicativity is allowed in the definition of predicates. This is achieved by considering formulae in the pre-fix normal form. If, and only if, the formula in question is logically equivalent to a formula that features only pre-fix universal second-order quantifiers, and no other second-order quantifiers, we call the formula Π^1_1. Analogously, we call a formula is Σ^1_1 if, and only if, it is equivalent to a formula that features only pre-fix existential second-order quantifiers, and no others. (Note that the block of pre-fixed quantifiers must not be interrupted by negations.) A formula that is equivalent to both a Π^1_1 and a Σ^1_1 formula is called Δ^1_1. Accordingly, in second-order logic with Δ^1_1-comprehension the comprehension schema is restricted to instances where the defining open sentence is Δ^1_1.

Both Σ^1_1- and Π^1_1-comprehension are inconsistent with Basic Law V, so, in that sense, Δ^1_1 is the highest complexity we can allow in the presence of Basic Law V.

See also Fine (2002), Linnebo (2004), Burgess (2005), Visser (2009), Antonelli (2010), Heck (2011a, ch. 12), Heck (2011b, 2014), Walsh (2012, 2014, 2015, 2016), and Walsh and Ebels-Duggan (2015), for further technical results in this area.

[50]Boolos (1990), Wright (1997); given modest assumptions about infinite concepts, the Nuisance Principle can indeed be shown to be (proof-theoretically) inconsistent with Hume's Principle; see Ebels-Duggan (2015).

several proposed criteria for acceptable abstraction principles and presents an ordering of these criteria by relative strength.[51] Moreover, he argues that the correct criterion for acceptable abstraction principles is *strong stability*.[52]

Regarding the inconsistency of Frege's logic presented in his *Basic Laws of Arithmetic*, we have mentioned already that the principle that is usually considered to be the culprit, Basic Law V, is in fact consistent in weak fragments of second-order logic. Indeed, Dummett (1991) suggests that the impredicativity of the second-order quantifiers is to blame for the inconsistency, rather than Basic Law V. Øystein Linnebo's "Impredicativity in the Neo-Fregean Programme" revisits the debate about this question that takes centre-stage in the exchange between Boolos (1993) and Dummett (1994). Linnebo examines the role of impredicative reasoning required for neo-Fregean programme (both technically and philosophically), and distinguishes two types of impredicativity that are conflated by Dummett and others. Linnebo suggests that some restrictions of impredicativity may lead to fruitful abstractionist theories.

In "Abstraction Grounded", Hannes Leitgeb argues against the very approach to distinguish acceptable from unacceptable abstraction principles. Instead, he develops a groundedness condition for impredicative abstraction principles in analogy to Saul Kripke's theory of truth (Kripke, 1975). The proposal is that any abstraction principle with impredicative second-order variables is to be restricted to those of its instances that satisfy certain groundedness requirements.

As mentioned above, set theory is a big issue for Abstractionism. Much progress has been made, but there currently does not appear to be a contender for a single abstraction principle that on its own, embedded in some higher-order logic, gives rise to a theory that is capable of interpreting Zermelo–Fraenkel set theory without presupposing prior knowledge of set theory, in some sense.[53] A single abstraction principle, viewed as an implicit definition of *set*, would be the abstractionists' ideal.

The assessment of the prospects for an abstractionist foundation for set theory is the topic that Stewart Shapiro and Gabriel Uzquiano tackle in their "Ineffability Within the Limits of Abstraction Alone". They present an abstraction-based set theory that allows capturing all of ordinary mathematics, except for Zermelo-Fraenkel set theory itself. In particular, they discuss the extent to which the thought that the iterative hierarchy, which is underlying standard set theory, is indefinitely extensible, or "ineffable" in some sense, is in tension with an abstractionist foundation.

Part V contains three contributions regarding Frege's application constraint,

[51] Cook draws on previous work in this area by Øystein Linnebo (2010) and himself, Cook (2012).

[52] An abstraction principle A is *strongly stable* if and only if there is a cardinal α such that, for any cardinal κ, A is satisfiable by a model with cardinality κ iff $\kappa \geq \alpha$.

[53] Compare Cook (2003b, p. 91 n. 41), regarding the "distraction" principles of Shapiro and Weir (1999) and Weir (2003).

sometimes just called "Frege's Constraint".[54] Frege held that a successful account of arithmetic, analysis, and so forth, should "present the ways in which arithmetic [etc.] is applied, even though the application itself is not its subject matter" (Frege, 2013, vol. II, §159). The application of a mathematical theory should be in some way build into the abstraction principle that provides the foundation of this part of mathematics: counting all kinds of objects for natural-number arithmetic; measuring different kinds of magnitude for real analysis; etc.

In "On Frege's Application Constraint", Paul McCallion investigates the motivation for the Frege's Constraint and relates his results to Benacerraf-type concerns for Abstractionism. Paul Benacerraf (1965) submitted that the existence of competing but equally successful reduction of the natural numbers—the prime example being Zermelo's versus von Neumann's reduction of cardinal numbers to sets—spells doom for a reductionist programme. McCallion takes this lead and asks whether an analogous problem of alternative, but on the face of it equally attractive abstractionists account of a given mathematical theory, may indeed be solved by appeal to the application constraint.

Peter Simons addresses the topic we dodged above by our casual use of "etc." after listing the relevant applications for arithmetic and real analysis. In "Applications of Complex Numbers and Quaternions", Simons investigates whether obvious applications for mathematical theories other than natural number arithmetic and real analysis can be identified. Simons looks at the cases of complex numbers and quaternions and their physical applications.

The application constraint was important for Frege, but the question may be raised whether a successful philosophy of mathematics must obey it, or indeed can do so in full generality. Bob Hale's contribution, "Definitions of Numbers and Their Applications", discusses the issue of this tight connection between definitions or explanations of fundamental mathematical notions and their applications that Frege demanded. He compares different definitions of natural and real numbers, some of which obey Frege's constraint and some of which do not. His discussion investigates whether Frege himself offered a satisfactory motivation for his constraint and whether there is any other justification of it available. Hale concludes with suggesting an alternative approach.

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In 2000, Crispin Wright founded the Arché Research Centre and under its aegis Bob Hale and Crispin Wright led a dedicated research project into

[54]See Frege (1903, §159) for the original formulation of the requirement, and Dummett (1991, pp. 272ff) and Wright (2000) for discussion.

the philosophical and mathematical prospects of Neo-Fregeanism. The centre brought together numerous researchers in the philosophy of mathematics many of whom have contributed to this volume. We would thus like to thank Crispin and Bob without whom there would not have been this flourishing and exciting research. We would also like to thank the AHRC, who provided much of the funding for the research centre. In particular, Philip Ebert would like to acknowledge a grant from the UK Arts and Humanities Research Council (AH/J00233X/1) which he held while editing of this volume.

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