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ON GRAPHS WITH JUST THREE DISTINCT EIGENVALUES

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Abstract

Let G be a connected non-bipartite graph with exactly three distinct eigenvalues ρ, μ, λ, where $\rho > \mu > \lambda$. In the case that G has just one non-main eigenvalue, we find necessary and sufficient spectral conditions on a vertex-deleted subgraph of G for G to be the cone over a strongly regular graph. Secondly, we determine the structure of G when just μ is non-main and the minimum degree of G is $1 + \mu - \lambda\mu$: such a graph is a cone over a strongly regular graph, or a graph derived from a symmetric 2-design, or a graph of one further type.

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1 Introduction

Let G be a graph of order n with $(0, 1)$-adjacency matrix A. An eigenvalue σ of A is said to be an eigenvalue of G, and σ is a *main* eigenvalue if the eigenspace $\mathcal{E}_A(\sigma)$ is not orthogonal to the all-1 vector in $\mathbb{R}^n$. Always the largest eigenvalue, or *index*, of G is a main eigenvalue, and it is the only main eigenvalue if and only if G is regular. We say that G is an *integral* graph if every eigenvalue of G is an integer. We use the notation of the monograph [5], where the basic properties of graph spectra can be found in Chapter 1.

Let $\mathcal{C}_1$ be the class of connected graphs with just three distinct eigenvalues, and let $\mathcal{C}_2$ be the class of connected graphs with exactly two main eigenvalues. It is an open problem to determine all the graphs in $\mathcal{C}_1$, and another open problem to determine all the graphs in $\mathcal{C}_2$. Here we investigate graphs in $\mathcal{C}_1 \cap \mathcal{C}_2$. From [6, Propositions 2 and 3] we know that if G is a non-integral graph in $\mathcal{C}_1$ then either G is complete bipartite or the two smaller eigenvalues of G are algebraic conjugates. In the latter case, G has exactly 1 or 3 main eigenvalues, and so a graph in $\mathcal{C}_1 \cap \mathcal{C}_2$ is either integral or complete bipartite.

The class $\mathcal{C}_1$ contains all connected non-complete strongly regular graphs; moreover it is known that if H is a strongly regular graph of order n with eigenvalues $\nu > \mu > \lambda$ then the cone $K_1 \bigtriangledown H$ lies in $\mathcal{C}_1$ if and only if $\lambda(\nu - \lambda) = -n$ (see [8] and Lemma 2.1 below). We shall see in Section 2 that the condition $\lambda(\nu-\lambda) = -n$ is equivalent to the condition $\nu = \mu(1-\lambda)$, and that when this condition is satisfied we have $K_1 \bigtriangledown H \in \mathcal{C}_1 \cap \mathcal{C}_2$. There are infinitely many strongly regular graphs which satisfy the condition (see [8, Proposition 7.1]); examples include the Petersen graph ($\mu = 1,\ \lambda = -2$), the Gewirtz graph ($\mu = 2,\ \lambda = -4$) and the Chang graphs ($\mu = 4,\ \lambda = -2$).

Now let G be a non-bipartite graph in $\mathcal{C}_1 \cap \mathcal{C}_2$ with spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$ where $\rho > \mu > \lambda$. In Section 3, we prove that the following are equivalent: (a) G is the cone over a strongly regular graph, (b) G has a vertex-deleted subgraph with just three distinct eigenvalues, (c) G has a vertex-deleted subgraph with index $\nu = \mu(1-\lambda)$. In particular, for $G \in \mathcal{C}_1 \cap \mathcal{C}_2$, application of the condition $\nu = \mu(1-\lambda)$ is not confined to a strongly regular graph H such that $G = K_1 \bigtriangledown H$.

We note that $\mathcal{C}_1 \cap \mathcal{C}_2$ also contains the graphs constructed by van Dam [6] from a symmetric 2-(q^3-q+1, q^2, q) design $\mathcal{D}$: such a graph is obtained from the incidence graph of $\mathcal{D}$ by adding an edge between each pair of blocks. We refer to such graphs as graphs of *symmetric type*; they exist whenever q is a prime power and there exists a projective plane of order $q-1$ [7]. Their eigenvalues are q^3, $q-1$, $-q$ with multiplicities 1, q^3-q, q^3+1 respectively. These graphs share with the cones described above the properties that μ is non-main and $1 + \mu - \mu\lambda = \delta(G)$, the minimum degree in G. In Section 4, we determine the structure of all graphs in $\mathcal{C}_1 \cap \mathcal{C}_2$ with these properties.

2 Preliminaries

Our first proof begins with a short derivation of the condition $\lambda(\nu-\lambda) = -n$, which was obtained by other means in [8, Proposition 6.1(b)].

Lemma 2.1. *Let H be a strongly regular graph of order n with spectrum $\nu, \mu^{(s)}, \lambda^{(t)}$, where $\nu > \mu > \lambda$. Then $K_1 \bigtriangledown H$ has just three distinct eigenvalues if and only if $\lambda(\nu - \lambda) = -n$, equivalently $\nu = \mu(1-\lambda)$. In this situation, $K_1 \bigtriangledown H$ has spectrum $\rho, \mu^{(s)}, \lambda^{(t+1)}$, where $\rho = \nu - \lambda$, and the main eigenvalues of $K_1 \bigtriangledown H$ are ρ and λ.*

Proof. Note that $\mu \geq 0$ and $\lambda < -1$ (cf. [5, Theorem 3.6.5]). From [5, Eq.(2.23)] we know that the characteristic polynomial of $K_1 \bigtriangledown H$ is given by

$$P_{K_1 \bigtriangledown H}(x) = P_H(x)\left(x - \frac{n}{x-\nu}\right) = (x-\mu)^s(x-\lambda)^t(x^2 - \nu x - n).$$

If $K_1 \bigtriangledown H$ has just three distinct eigenvalues, then $x^2 - \nu x - n$ is either $(x-\rho)(x-\mu)$ or $(x-\rho)(x-\lambda)$, where ρ is the index of $K_1 \bigtriangledown H$. The first possibility cannot arise because then $\rho + (s+1)\mu + t\lambda = 0 = \nu + s\mu + t\lambda$, whence $\rho = \nu - \mu \leq \nu$, contradicting [5, Proposition 1.3.9]. Hence $K_1 \bigtriangledown H$ has spectrum $\rho, \mu^{(s)}, \lambda^{(t+1)}$, where now $\rho = \nu - \lambda$. Since also $\rho\lambda = -n$, we have $\lambda(\nu - \lambda) = -n$ as required. In this situation, $K_1 \bigtriangledown H$ has adjacency matrix $A = \begin{pmatrix} 0 & \mathbf{j}^\top \\ \mathbf{j} & A' \end{pmatrix}$, where $\mathbf{j}$ is the all-1 vector in $I\!R^n$ and A' is the adjacency matrix of H. Now μ is a non-main eigenvalue of H, and so if $\mathbf{x} \in \mathcal{E}_{A'}(\mu)$ then $\begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix} \in \mathcal{E}_A(\mu)$. Since $\mathcal{E}_{A'}(\mu)$ and $\mathcal{E}_A(\mu)$ have the same dimension, it follows that μ is a non-main eigenvalue of $K_1 \bigtriangledown H$. Since $K_1 \bigtriangledown H$ is not regular, the main eigenvalues of $K_1 \bigtriangledown H$ are ρ and λ.

Conversely if $\lambda(\nu - \lambda) = -n$ then $x^2 - \nu x - n = (x - (\nu - \lambda))(x - \lambda)$. Then $\nu - \lambda$ is the index of $K_1 \bigtriangledown H$ and $K_1 \bigtriangledown H$ has just three distinct eigenvalues.

Finally, from [5, Theorem 3.6.4] we have $n = (\nu - \mu)(\nu - \lambda)/(\nu + \mu\lambda)$, and so $\lambda(\nu - \lambda) = -n$ if and only if $\nu(\lambda + 1) + \mu(\lambda^2 - 1) = 0$, equivalently $\nu = \mu(1 - \lambda)$. □

The parameters of a strongly regular graph are expressible in terms of its eigenvalues [5, Theorem 3.6.4]. For future reference we note that the graph H of Lemma 2.1 has parameters (q, r, e, f), where $q = \lambda^2\mu + \lambda^2 - \lambda\mu$, $r = \mu - \lambda\mu$, $e = 2\mu + \lambda$ and $f = \mu$.

Lemma 2.2. *A graph G in $\mathcal{C}_1 \cap \mathcal{C}_2$ has exactly two distinct degrees (say d_1, d_2), and these degrees determine an equitable bipartition of G. Moreover, if G has spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$, where $\rho > \mu > \lambda$, then $d_i = \alpha_i^2 - \lambda\mu$, where $\alpha_i > 0$ $(i = 1, 2)$ and either*
(a) μ is non-main and $\alpha_1\alpha_2 = -\lambda(\mu + 1)$, or
(b) λ is non-main and $\alpha_1\alpha_2 = -\mu(\lambda + 1)$.

Proof. Suppose that G has vertex set $V(G) = \{1, \ldots, n\}$ and adjacency matrix A. Since $G \in \mathcal{C}_1$ we have (cf. [6, Section 4]):

$$(A - \mu I)(A - \lambda I) = \mathbf{a}\mathbf{a}^\top, \tag{1}$$

where $\mathbf{a}$ spans $\mathcal{E}_A(\rho)$ and each entry of $\mathbf{a}$ is positive. Thus if $\mathbf{a} = (a_1, \ldots, a_n)^\top$ then $\deg(i) = a_i^2 - \lambda\mu$ $(i = 1, \ldots, n)$. Since $G \in \mathcal{C}_2$, either (a) μ is non-main and $(A - \rho I)(A - \lambda I)\mathbf{j} = \mathbf{0}$ or (b) λ is non-main and $(A - \rho I)(A - \mu I)\mathbf{j} = \mathbf{0}$ (cf. [9, Proposition 2.1]). In particular, $A^2\mathbf{j} \in \langle \mathbf{d}, \mathbf{j} \rangle$, where $\mathbf{d} = A\mathbf{j}$. Now $\mathbf{a}(\mathbf{a}^\top\mathbf{j}) \in \langle \mathbf{d}, \mathbf{j} \rangle$, and $\mathbf{a}^\top\mathbf{j} \neq 0$. Accordingly we have $\mathbf{a} = r\mathbf{d} + s\mathbf{j}$ for some $r, s \in I\!R$. Note that $r \neq 0$ since G is not regular. It follows that

$$ra_i^2 - a_i - r\lambda\mu + s = 0 \ (i = 1, \ldots, n),$$

and hence that the a_i take just two values, say α_1, α_2. By Eq.(1), G has just two degrees: $d_1 = \alpha_1^2 - \lambda\mu$, $d_2 = \alpha_2^2 - \lambda\mu$. Let V_i be the set of vertices of degree i $(i = 1, 2)$. Since the A-invariant subspace $\langle \mathbf{d}, \mathbf{j} \rangle$ is spanned by the characteristic vectors of V_1 and V_2, $V_1 \dot\cup V_2$ is an equitable bipartition of $V(G)$.

In case (a), Eq.(1) yields:

$$\mathbf{a}(\mathbf{a}^\top\mathbf{j}) = (A - \mu I)(A - \lambda I)\mathbf{j} = (\rho - \mu)\mathbf{d} - \lambda(\rho - \mu)\mathbf{j},$$

and so $s = -\lambda r$. Since α_1, α_2 are the roots of $x^2 - r^{-1}x - \lambda\mu + r^{-1}s$, we have $\alpha_1\alpha_2 = -\lambda(\mu + 1)$. We may interchange λ and μ to obtain $\alpha_1\alpha_2 = -\mu(\lambda + 1)$ in case (b). □

A graph with just two degrees is said to be *biregular*. A wider discussion of the biregular graphs in $\mathcal{C}_1$ may be found in the recent paper [3]. Here we shall also make use of the following intermediate result.

Proposition 2.3. *Let G be a connected non-bipartite integral graph with spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$, where $\rho > \mu > \lambda$, and let v be a vertex of G. Then*
(i) $k > 1$, $l > 1$ and λ, μ are eigenvalues of $G - v$,
(ii) $G - v$ has just three distinct eigenvalues if and only if $G - v$ is strongly regular and G is the cone over $G - v$.

Proof. Let $|V(G)| = n$. Note that $\lambda < -1$ and $\rho < n - 1$ because G is not complete. Now $k > 1$ for otherwise

$$2 \leq -\lambda = \frac{\rho + \mu}{n - 2} \leq 1 + \frac{\mu}{n - 2},$$

whence $\mu \geq n - 2 \geq \rho$, a contradiction. Suppose by way of contradiction that $l = 1$. If $\mu > 0$ then $-\lambda > \rho$, contradicting [5, Theorem 1.3.6]. If $\mu = 0$ then $\lambda = -\rho$ and G is bipartite, contrary to assumption (see [5, Theorem 3.2.4]). If $\mu < 0$ then $\rho = (n - 2)(-\mu) - \lambda \geq n$, a contradiction. Hence also $l > 1$, and by interlacing $G - v$ has both λ and μ as eigenvalues.

Let $H = G - v$, with spectrum $\nu, \mu^{(k-1)}, \theta, \lambda^{(l-1)}$, where $\rho \geq \nu \geq \mu \geq \theta \geq \lambda$ by interlacing, and $\rho > \nu$ because G is connected [5, Proposition

1.3.9]. If $\nu = \mu$ then H is not connected; moreover, $\mu > \theta > \lambda$ for otherwise H has just two distinct eigenvalues and $\lambda = -1$. Now some component C of H does not have θ as an eigenvalue. Since C has at most two distinct eigenvalues, C is complete and $\lambda \in \{-1, 0\}$, a contradiction. Hence $\nu > \mu$.

Now suppose that H has just three distinct eigenvalues. Then $\theta \in \{\mu, \lambda\}$. If $\theta = \lambda$ then $\nu + (k-1)\mu + l\lambda = 0 = \rho + k\mu + l\lambda$, whence $\rho = \nu - \mu < \nu$, a contradiction. Hence $\theta = \mu$ and H has spectrum $\nu, \mu^{(k)}, \lambda^{(l-1)}$. As before, H is connected, for otherwise some component does not have ν as an eigenvalue.

Let A' be the adjacency matrix of H. For any eigenvalue σ of H, we write Q_σ for the matrix of the orthogonal projection of $\mathcal{E}_{A'}(\sigma)$ onto $I\!R^{n-1}$ (with respect to the standard orthonormal basis of $I\!R^{n-1}$). Let $\Delta_H(v)$ be the set of vertices in H adjacent to v, and let $\mathbf{r}$ be the characteristic vector of $\Delta_H(v)$ in $I\!R^{n-1}$. From [5, Theorem 2.2.8] we have

$$P_G(x) = P_H(x)\left(x - \frac{\|Q_\nu \mathbf{r}\|^2}{x-\nu} - \frac{\|Q_\mu \mathbf{r}\|^2}{x-\mu} - \frac{\|Q_\lambda \mathbf{r}\|^2}{x-\lambda}\right).$$

Since the multiplicities of λ and μ in G are not less than their multiplicities in H, we have $Q_\lambda \mathbf{r} = \mathbf{0}$ and $Q_\mu \mathbf{r} = \mathbf{0}$. Hence $\mathbf{r} \in (\mathcal{E}_{A'}(\lambda) \oplus \mathcal{E}_{A'}(\mu))^\perp = \mathcal{E}_{A'}(\nu)$. Since H is connected, $\mathcal{E}_{A'}(\nu)$ is spanned by a vector whose entries are all positive. It follows that $\mathbf{r} = \mathbf{j}$ and $\Delta_H(v) = V(H)$. Moreover, H is regular, with just three distinct eigenvalues, and hence is strongly regular. The converse is immediate. □

3 Vertex-deleted subgraphs

Here we take G to be a non-bipartite graph in $\mathcal{C}_1 \cap \mathcal{C}_2$ with spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$ where $\rho > \mu > \lambda$. We noted in Section 1 that G is integral; hence by Proposition 2.3, $k > 1$, $l > 1$ and every vertex-deleted subgraph of G has λ and μ as eigenvalues. Our objective is to prove that if one of these subgraphs has index $\mu(1-\lambda)$ then G is the cone over a strongly regular graph.

We use the notation of Lemma 2.2 and Proposition 2.3. We assume that $d_1 > d_2$, and we take H to be a vertex-deleted graph with index $\nu = \mu(1-\lambda)$. Let $H = G - v$ and suppose by way of contradiction that H has four distinct eigenvalues. By interlacing H has spectrum $\nu, \mu^{(k-1)}, \theta, \lambda^{(l-1)}$, where $\rho > \nu > \mu > \theta > \lambda$. Note that since ν is an integer, so too is θ. If $\mathbf{r}$ is the characteristic vector of $\Delta_H(v)$ then

$$P_G(x) = P_H(x)\left(x - \frac{\|Q_\nu \mathbf{r}\|^2}{x-\nu} - \frac{\|Q_\mu \mathbf{r}\|^2}{x-\mu} - \frac{\|Q_\theta \mathbf{r}\|^2}{x-\theta} - \frac{\|Q_\lambda \mathbf{r}\|^2}{x-\lambda}\right), \qquad (2)$$

where again $Q_\lambda \mathbf{r} = \mathbf{0}$ and $Q_\mu \mathbf{r} = \mathbf{0}$. Let $c = \|Q_\nu \mathbf{r}\|$, $d = \|Q_\theta \mathbf{r}\|$. Then Eq.(2) yields

$$(x-\rho)(x-\mu)(x-\lambda) = x(x-\nu)(x-\theta) - c^2(x-\theta) - d^2(x-\nu). \qquad (3)$$

Equating coefficients of x^2 and coefficients of x in Eq.(3) we find:

$$\rho + \lambda + \mu = \nu + \theta, \quad \rho\lambda + \rho\mu + \lambda\mu = \nu\theta - c^2 - d^2.$$

Suppose that $v \in V_h$ ($h \in \{1, 2\}$). Since $c^2 + d^2 = \|\mathbf{r}\|^2 = \deg(v) = \alpha_h^2 - \lambda\mu$ we have:

$$\nu + \theta = \rho + \lambda + \mu, \quad \nu\theta = \rho(\lambda + \mu) + \alpha_h^2. \tag{4}$$

Since $\rho = \theta - \lambda - \lambda\mu$, we have

$$\alpha_h^2 = \mu(1 - \lambda)\theta - (\theta - \lambda - \lambda\mu)(\lambda + \mu) = -\lambda(\mu + 1)(-\lambda - (\mu - \theta)).$$

Note that $-\lambda > \mu - \theta$ because $\mu > 0$ and $\alpha_h \neq 0$.

We deal first with the case in which μ is non-main. Then we have $\alpha_1\alpha_2 = -\lambda(\mu + 1)$ by Lemma 2.2. If $h = 1$ then

$$\alpha_2^2 = \frac{-\lambda(\mu + 1)}{-\lambda - (\mu - \theta)} \geq \frac{-\lambda(\mu + 1)}{-\lambda - 1} > \mu + 1.$$

But $\alpha_2^2 - \lambda\mu - 1 = d_2 - 1 \leq \delta(H) \leq \nu = \mu - \lambda\mu$, and so $\alpha_2^2 \leq \mu + 1$, a contradiction. If $h = 2$ then

$$\alpha_1^2 = \frac{-\lambda(\mu + 1)}{-\lambda - (\mu - \theta)}, \quad \alpha_2^2 = -\lambda(\mu + 1)(-\lambda - (\mu - \theta)).$$

Since $d_2 < d_1$ we have $\alpha_2^2 < \alpha_1^2$, and so $|-\lambda - (\mu - \theta)| < 1$. This is a contradiction because $-\lambda - (\mu - \theta)$ is a positive integer.

Secondly we consider the case in which λ is non-main. Then $\alpha_1\alpha_2 = -\mu(\lambda + 1)$ by Lemma 2.2. If $h = 2$ then

$$\alpha_1^2 = \frac{(-\lambda - 1)^2\mu^2}{-\lambda(\mu + 1)(-\lambda - (\mu - \theta))}, \quad \alpha_2^2 = -\lambda(\mu + 1)(-\lambda - (\mu - \theta)).$$

Since $\alpha_2^2 < \alpha_1^2$ we have

$$-\lambda - (\mu - \theta) < \frac{(-\lambda - 1)\mu}{-\lambda(\mu + 1)} < 1,$$

a contradiction as before. Now suppose that $h = 1$, and let $\alpha = \mu - \theta$. We have $-\lambda > \alpha > 0$ and

$$\alpha_1^2 = -\lambda(\mu + 1)(-\lambda - \alpha), \quad \alpha_2^2 = \frac{(-\lambda - 1)^2\mu^2}{-\lambda(\mu + 1)(-\lambda - \alpha)}.$$

Note that

$$\frac{(-\lambda - 1)^2\mu^2}{-\lambda(\mu + 1)} - (\mu - 1)(-\lambda - 2) = \frac{\mu^2 - 1 + (\lambda + 1)^2}{-\lambda(\mu + 1)} > 0.$$

Hence

$$\alpha_2^2 > \frac{(\mu - 1)(-\lambda - 2)}{-\lambda - \alpha}.$$

If $\alpha = 1$ then $\alpha_2^2 = \frac{(-\lambda-1)\mu^2}{-\lambda(\mu+1)}$. In this case, we consider a prime p which divides $-\lambda$. Note that p divides μ and hence also ν. But $\nu + (k-1)\mu + \theta + (l-1)\lambda = 0$, and so p divides α, a contradiction. Hence $\alpha \geq 2$ and $\alpha_2^2 \geq \mu$.

Now $d_2 - 1 \leq \overline{d} \leq \nu$, where $\overline{d}$ is the mean degree in H. If $d_2 - 1 = \nu$ then H is regular of degree $d_2 - 1$; in this case, $V_1 = \{v\}$, v is adjacent to every vertex in V_2, and (since θ is a non-main eigenvalue of H), θ is an eigenvalue of G. This contradiction shows that $d_2 \leq \nu$, that is, $\alpha_2^2 - \lambda\mu \leq \mu(1-\lambda)$, and we deduce that $\alpha_2^2 = \mu \neq 0$. We have

$$\mu(-\lambda - 1)^2 = -\lambda(\mu + 1)(-\lambda - \alpha),$$

and so $\mu = t(-\lambda)$ for some positive integer t. It follows that $-\lambda - \alpha = -\lambda t(\alpha - 2) + t$ and hence that $\alpha = 2$. Then $\rho = \nu + \theta - \lambda - \mu = \mu(1-\lambda) - \lambda - 2$. Since $\rho + k\mu + l\lambda = 0$, we see that $-\lambda$ is a divisor of 2. Hence $-\lambda = 2 = \alpha$, a final contradiction. We have proved that if a graph $G \in \mathcal{C}_1 \cap \mathcal{C}_2$ has a vertex-deleted subgraph H with index $\mu(1-\lambda)$ then H has just three distinct eigenvalues. By Proposition 2.3, H is strongly regular, and $G = K_1 \bigtriangledown H$. We may summarize most of our results as follows.

Theorem 3.1. *Let G be a connected non-bipartite graph with exactly three distinct eigenvalues, just one of them non-main. If G has spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$, where $\rho > \mu > \lambda$, then $k > 1$, $l > 1$ and the following are equivalent:*
(a) G is the cone over a strongly regular graph,
(b) G has a vertex-deleted subgraph with just three distinct eigenvalues,
(c) G has a vertex-deleted subgraph with index $\mu(1-\lambda)$.

In addition, it follows from Lemma 2.1 that if H is a strongly regular graph such that $K_1 \bigtriangledown H$ has spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$, where $\rho > \mu > \lambda$, then H has spectrum $\nu_1, \mu^{(k)}, \lambda^{(l-1)}$, where $\rho + \lambda = \nu_1 = \mu(1-\lambda)$ and μ is the sole non-main eigenvalue of $K_1 \bigtriangledown H$. In this situation, let $G = K_1 \bigtriangledown H$ and let $v \in V(H)$. Then $G - v$ has four distinct eigenvalues because G is not the cone over $G - v$. Thus $G - v$ has spectrum $\nu_2, \mu^{(k-1)}, \theta_2, \lambda^{(l-1)}$, where $\nu_2 > \mu > \theta_2 > \lambda$. By Eq,(4), we have $\rho + \lambda + \mu = \nu_2 + \theta_2$, and we deduce that $\nu_2 > \nu_1$. In particular, the index of any vertex-deleted subgraph of G is at least $\mu(1-\lambda)$. More generally we have the following.

Corollary 3.2. *Let G be a connected non-bipartite graph with spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$, where $\rho > \mu > \lambda$, and let H be a vertex-deleted subgraph of G. If μ is the only non-main eigenvalue of G then the index of H is at least $\mu(1-\lambda)$, with equality if and only if H is strongly regular and G is the cone over H.*

Proof. Let H be a vertex-deleted subgraph with index ν. Since G is connected, G has an edge ij with $i \in V_1$ and $j \in V_2$. The (i,j)-entry of A^2 is at most $\deg(j) - 1$, and so $\alpha_1\alpha_2 + \lambda + \mu \leq d_2 - 1$. By Lemma 2.2, we have $\alpha_1\alpha_2 = -\lambda(\mu+1)$, while $d_2 - 1 \leq \nu$ as before. It follows that $\nu \geq \mu(1-\lambda)$. If $\nu = \mu(1-\lambda)$, then we see from the proof of Theorem 3.1 that H is strongly regular and G is the cone over H. Conversely, if H is strongly regular and $G = K_1 \bigtriangledown H$ then (as noted above) H has index $\mu(1-\lambda)$. □

4 The minimum degree

Again we take G to be a non-bipartite graph in $\mathcal{C}_1 \cap \mathcal{C}_2$ with spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$ where $\rho > \mu > \lambda$. Recall from Section 1 that ρ, μ and λ are integers. It is straightforward to check that if G is the cone over a strongly regular graph then $\delta(G) = 1+\mu-\lambda\mu$; moreover we saw in Section 2 that μ is a non-main eigenvalue. If G is of symmetric type then again $\delta(G) = 1+\mu-\lambda\mu$, while μ is non-main because the degrees determine an equitable bipartition with a divisor matrix whose trace is $\rho + \lambda$ (cf. [5, Theorem 3.9.5]). Now we suppose conversely that $\delta(G) = 1 + \mu - \lambda\mu$ and μ is non-main; in this situation we can determine the structure of G.

We retain previous notation and write $u \sim v$ to mean that the vertices u and v are adjacent. We let $\Delta(v) = \{u \in V(G) : u \sim v\}$, $A^2 = (a^{(2)}_{ij})$, $|V_1| = n_1$, $|V_2| = n_2$, $G_1 = G - V_2$, $G_2 = G - V_1$. Also, let $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ be the divisor matrix determined by the equitable bipartition $V_1 \dot\cup V_2$. Note that $n_1 r_{12} = n_2 r_{21}$. Since $A\mathbf{a} = \rho\mathbf{a}$, we have $r_{11}\alpha_1 + r_{12}\alpha_2 = \rho\alpha_1$. Since also $r_{11} + r_{12} = d_1 > d_2$, we have (cf. [3, Theorem 4.3(i)]:

$$r_{12} = \frac{\alpha_1(d_1-\rho)}{\alpha_1-\alpha_2}, \quad \text{and similarly} \quad r_{21} = \frac{\alpha_2(d_2-\rho)}{\alpha_2-\alpha_1}. \tag{5}$$

By Lemma 2.2, we have $\alpha_1\alpha_2 = -\lambda(\mu+1)$. Also, $1 + \mu - \lambda\mu = d_2 = \alpha_2^2 - \lambda\mu$, whence $\alpha_2^2 = \mu + 1$ and $\alpha_1^2 = \lambda^2(\mu+1)$. It follows from Eq.(5) that

$$r_{12} = \frac{-\lambda(d_1-\rho)}{-\lambda-1}, \quad r_{21} = \frac{\rho-d_2}{-\lambda-1}. \tag{6}$$

We shall make implicit use of the following consequence of Eq.(1):

$$a^{(2)}_{ij} = \begin{cases} a_i^2 - \lambda\mu & \text{if } i = j \\ a_i a_j + \lambda + \mu & \text{if } i \sim j \\ a_i a_j & \text{if } i \not\sim j. \end{cases}$$

In particular, $d_1 = \lambda^2(\mu+1) - \lambda\mu$.

Lemma 4.1. *If $r_{22} \neq 0$ then G is the cone over a strongly regular graph.*

Proof. Let $i \in V_1$. Since G is connected, we have $r_{12} \neq 0$, and so V_2 contains a vertex j adjacent to i. Now $a^{(2)}_{ij} = \alpha_1\alpha_2 + \lambda + \mu = \mu - \lambda\mu = \deg(j) - 1$, and so $\Delta(j) \subseteq \Delta(i) \dot\cup \{i\}$. If $j' \in \Delta(j) \cap V_2$ then $j' \sim i$, and so i is adjacent to every vertex in the component $C(j)$ of G_2 containing j. If $i' \in \Delta(j) \cap V_1$ then similarly i' is adjacent to every vertex j' in $C(j)$; moreover $\Delta(j') \cap V_1 = \Delta(j) \cap V_1$ (of size r_{21}). Thus if $X = \Delta(j) \cap V_1$ and $Y = V(C(j))$ then we have a complete bipartite subgraph on $X \dot\cup Y$.

If $C(j)$ is complete then (since $r_{22} \neq 0$) $C(j)$ contains two vertices with the same closed neighbourhood in G, and then we obtain the contradiction $\lambda = -1$ from [5, Theorem 5.1.4]. Accordingly, let j, j' be two non-adjacent vertices in $C(j)$. Since $j \sim i' \sim j'$ for all $i' \in X$, we have $r_{21} \leq a^{(2)}_{jj'} = \alpha_2^2$. If v is a vertex in V_2 outside $C(j)$ then all v-j paths of length 2 pass through $\Delta(j) \cap V_1$ and so $\alpha_2^2 = a^2_{vj} \leq r_{21}$. Thus $a^{(2)}_{vj} = r_{21}$ and v is adjacent to every vertex in X. In particular, i is adjacent to every vertex in V_2. The argument applies to each vertex $i \in V_1$ and so we have a complete bipartite subgraph on $V_1 \dot\cup V_2$.

From Eq.(1), we have $n_1\alpha_1^2 + n_2\alpha_2^2 = \|\mathbf{a}\|^2 = (\rho - \lambda)(\rho - \mu)$. Since $n_1 = r_{21}$ and $n_2 = r_{12}$, Eq.(6) yields:

$$\frac{\rho - d_2}{-\lambda - 1}\lambda^2(\mu + 1) + \frac{\lambda(d_1 - \rho)}{\lambda + 1}(\mu + 1) = (\rho - \lambda)(\rho - \mu),$$

equivalently

$$-\lambda(\mu + 1)[\rho(-\lambda - 1) + d_1 + \lambda d_2] = (\rho - \lambda)(\rho - \mu)(-\lambda - 1).$$

Since $d_1 + \lambda d_2 = -\lambda(-\lambda - 1)$, we deduce that $-\lambda(\mu + 1) = \rho - \mu$, whence $\rho - d_2 = -\lambda - 1$ and $r_{21} = 1$. Thus $n_1 = 1$, say $V_1 = \{u\}$, and G is the cone over $G - u$. Now $G - u$ is a regular graph in which the number of common neighbours of distinct vertices i, j is $\alpha_2^2 - 1$ if $i \not\sim j$ and $\alpha_2^2 + \lambda + \mu - 1$ if $i \sim j$. Therefore $G - u$ is strongly regular, and the lemma is proved. □

In view of Lemma 4.1, we suppose now that $r_{22} = 0$ (equivalently, V_2 is an independent set). In this case, we can express $r_{11}, r_{12}, r_{21}, n_1, n_2, n, k, l$ in terms of λ and μ. Note first that $r_{22} = d_2 - r_{21}$, and so by Eq.(6) we have $\rho = -\lambda d_2 = -\lambda(1 + \mu - \mu\lambda)$. Eq.(6) shows also that $r_{11} = d_1 - r_{12} = (-d_1 - \rho\lambda)/(-\lambda - 1) = \mu\lambda^2 - \mu\lambda$, while $r_{12} = \lambda^2$.

Next observe that if j, j' are distinct vertices in V_2 then $|\Delta(j) \cap \Delta(j')| = a^{(2)}_{jj'} = \alpha_2^2 = \mu + 1$. Counting in two ways the paths jij' ($j, j' \in V_2,\ j \neq j'$) we have

$$n_2(n_2 - 1)(\mu + 1) = n_1\lambda^2(\lambda^2 - 1).$$

Since $n_1\lambda^2 = n_2(1 + \mu - \lambda\mu)$, we deduce that

$$n_1 = \frac{(1 + \mu - \lambda\mu)(\lambda + \lambda\mu - \lambda^2\mu + \mu)}{\lambda(\mu + 1)}, \quad n_2 = \frac{\lambda(\lambda + \lambda\mu - \lambda^2\mu + \mu)}{\mu + 1}. \tag{7}$$

Hence

$$n = n_1 + n_2 = \frac{(\lambda + \lambda\mu - \lambda^2\mu + \mu)(1 + \mu - \lambda\mu + \lambda^2)}{\lambda(\mu + 1)}. \tag{8}$$

(Equations (7) and (8) are special cases of [3, Theorem 4.3(iv)].) Now we can find k and l from the equations $\rho + k\mu + l\lambda = 0$, $1 + k + l = n$. We obtain

$$k = \frac{(\lambda^2 - 1)(1 + \mu - \lambda\mu)}{\mu + 1}, \quad l = \frac{(1 + \mu - \lambda\mu)(\lambda + \lambda\mu - \lambda^2\mu + \mu)}{\lambda(\mu + 1)}. \tag{9}$$

Since all structural constants of G are expressible in terms of λ and μ we say that G is of *parametric type*, with parameters λ, μ. To investigate G further, we observe again that if $j \in V_2$ and $i \in \Delta(j)$ then $a_{ij}^{(2)} = \deg(j) - 1$ and so i is adjacent to every other vertex in $\Delta(j)$. We deduce that $\Delta(j)$ induces a clique; in particular, if h, h' are non-adjacent vertices in V_1 then h, h' have no common neighbours in V_2. We refer to the V_1-neighbourhoods $\Delta(j)$ $(j \in V_2)$ as the blocks in V_1, and to the V_2-neighbourhoods $\Delta(i) \cap V_2$ $(i \in V_1)$ as the blocks in V_2.

We note next that $\lambda + \mu \geq -1$. To see this, let j, j' be distinct vertices in V_2, and consider a vertex $i \in \Delta(j) \setminus \Delta(j')$. We have $a_{ij'}^{(2)} \leq |\Delta(j')|$ and so $\alpha_1\alpha_2 \leq d_2$, equivalently $-\lambda(\mu+1) \leq 1 + \mu - \lambda\mu$. The inequality follows, and we deduce that $\lambda^2 \leq 1 + \mu - \lambda\mu$, equivalently $n_1 \geq n_2$.

¿From Eqs.(7) and (9) we see that $n_1 = l$ and so the co-clique on V_2 is a star complement for λ. Let $A = \begin{pmatrix} A_1 & B^\top \\ B & O \end{pmatrix}$, partitioned in accordance with $V_1 \dot{\cup} V_2$. By [5, Theorem 5.1.7] we have $\lambda^2 I - \lambda A_1 = B^\top B$. It follows that for $i, i' \in V_1$:

$$|\Delta(i) \cap \Delta(i') \cap V_2| = \begin{cases} \lambda^2 & \text{if} \;\; i = i', \\ -\lambda & \text{if} \;\; i \sim i', \\ 0 & \text{if} \;\; i \not\sim i'. \end{cases}$$

We say that the blocks $\Delta(i) \cap V_2$ $(i \in V_1)$, of size λ^2, have intersection numbers $-\lambda$ and 0. Now $B^\top B$ and $BB^\top$ share the same non-zero eigenvalues, and $BB^\top = d_2 I + (\mu+1)(J - I)$, where J is the all-1 matrix of size $n_2 \times n_2$. Thus $BB^\top = -\lambda\mu I + (\mu+1)J$, with eigenvalues $-\lambda\mu + (\mu+1)n_2$ (of multiplicity 1) and $-\lambda\mu$ (of multiplicity $n_2 - 1$). The relation between the eigenvalues ν^* of A_1 and the eigenvalues ν of $B^\top B$ is given by

$$\lambda^2 - \lambda\nu^* = \nu.$$

If $\nu = -\lambda\mu + (\mu+1)n_2$ then $\nu^* = \lambda^2\mu - \lambda\mu$; if $\nu = -\lambda\mu$ then $\nu^* = \lambda + \mu$; and if $\nu = 0$ then $\nu^* = \lambda$. Thus the eigenvalues of A_1 are $\lambda^2\mu - \lambda\mu$ $(= r_{11})$, $\lambda + \mu$ (of multiplicity $n_2 - 1$) and λ (of multiplicity $n_1 - n_2$). Note that if $n_1 = n_2$ then $\lambda^2 = 1 + \mu - \lambda\mu$, equivalently $\lambda + \mu = -1$. Thus there are two possibilities: (1) $n_1 = n_2$, $\lambda + \mu = -1$ and G_1 is complete, or (2) $n_1 > n_2$, $\lambda + \mu \geq 0$ and G_1 is strongly regular with parameters (n_1, r_{11}, e, f), where n_1 is given by Eq.(7), $r_{11} = \lambda^2\mu - \lambda\mu$, $e = \alpha_1^2 + 2\lambda + \mu = \lambda^2(\mu+1) + 2\lambda + \mu$ and $f = \lambda^2(\mu+1)$.

In case (1), we have $n_1 = n_2 = -\lambda^3 + \lambda + 1$ by Eq.(7); moreover the blocks in V_2 constitute a symmetric 2-$(q^3 - q + 1, q^2, q)$ design, where $q = -\lambda$. Thus in case (1) G is of symmetric type. We summarize our observations as follows.

Theorem 4.2. *Let G be a connected non-bipartite non-regular graph with spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$, where $\rho > \mu > \lambda$ and μ is non-main. Then G has two degrees, say d_1 and d_2 where $d_1 > d_2$. For $i = 1, 2$, let V_i be the set of vertices of degree d_i, and let G_i be the subgraph of G induced by V_i. Then $V_1 \dot{\cup} V_2$ is an equitable partition of G; moreover, if $d_2 = 1 + \mu - \mu\lambda$ then one of the following holds:*

(a) G_1 is trivial and G is the cone over G_2 where G_2 is strongly regular with parameters (q, r, e, f), where $q = \lambda^2\mu + \lambda^2 - \lambda\mu$, $r = \mu - \lambda\mu$, $e = 2\mu + \lambda$ and $f = \mu$;

(b) G_1 is complete, G_2 is a co-clique and G is of symmetric type, derived from a symmetric 2-$(q^3 - q + 1, q^2, q)$ design with $q = -\lambda = \mu + 1$;

(c) G_2 is a co-clique and G_1 is strongly regular with parameters (q, r, e, f), where $q = (1 + \mu - \mu\lambda)(\lambda + \lambda\mu - \lambda^2\mu + \mu)/\lambda(\mu + 1)$, $r = \lambda^2\mu - \lambda\mu$, $e = \lambda^2(\mu + 1) + 2\lambda + \mu$, $f = \lambda^2(\mu + 1)$ and $\lambda + \mu > -1$.

In case (c) the blocks $\Delta(j)$ $(j \in V_2)$ induce cliques of order $1 + \mu - \mu\lambda$, and any two such blocks intersect in $1 + \mu$ vertices; moreover the blocks $\Delta(i) \cap V_2$ $(i \in V_1)$ are of size λ^2 with intersection numbers $-\lambda$ and 0.

Example 4.3. As an example of case (c) in Theorem 4.2 we have the unique smallest maximal exceptional graph, labelled G001 in [4, Chapter 6]. This graph, first identified in [1], has order 22 and spectrum $14, 2^{(7)}, -2^{(14)}$. A representation in the root system E_8 is given in [4, Section 6.4]; see also [6, pp.112-113]. A different construction is given in [5, Example 5.2.6(c)]. For this graph we have $n_1 = 14$, $n_2 = 8$, $d_1 = 16$ and $d_2 = 7$. We find that $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 12 & 4 \\ 7 & 0 \end{pmatrix}$, with trace equal to $\rho + \lambda$, and so μ is a non-main eigenvalue. Since $r_{11} = 12$ we have $G_1 \cong \overline{7K_2}$. □

The following result narrows the search for further examples.

Proposition 4.4. *If G is of parametric type, with coprime parameters λ, μ, then G is of symmetric type.*

Proof. Suppose that G has coprime parameters λ, μ. We see from Eq.(7) that λ divides $\mu(\mu + 1)$, and so $\mu = -\lambda\beta - 1$ for some positive integer β. From Eq.(7), we have

$$n_1 = \frac{(\beta\lambda - \beta + 1)(\beta\lambda^3 - \beta\lambda^2 + \lambda^2 - \beta\lambda - 1)}{-\lambda\beta},$$

whence $-\lambda$ divides $\beta - 1$. Suppose by way of contradiction that $\beta > 1$. Then $\beta \geq 1 - \lambda$ and $\mu + 1 \geq -\lambda(1 - \lambda)$.

Since $\lambda + \mu \neq -1$ the graph G_1 is not complete. Now consider the complementary graph $\overline{G_1}$, which is strongly regular with parameters $(n_1, n_1 - r_{11} - 1, \overline{e}, \overline{f})$, where $\overline{e} = n_1 - 2r_{11} - 2 + f$ and $\overline{f} = n_1 - 2r_{11} + e$. Then

$$\overline{e} = \frac{(1+\mu-\mu\lambda)(\lambda+\lambda\mu-\lambda^2\mu+\mu)}{\lambda(\mu+1)} - 2(\lambda^2 - \lambda\mu + 1) + \lambda^2(\mu + 1).$$

Hence $\lambda(\mu+1)\overline{e} = (\mu+1)^2-(\mu+1)+\lambda^3-\lambda$. Since $\mu+1 \geq -\lambda(1-\lambda)$, we deduce that $\lambda(\mu+1)\overline{e} \geq \lambda^4 - \lambda^3$. This is a contradiction because $\lambda(\mu+1)\overline{e} \leq 0$. while $\lambda^4 - \lambda^3 > 0$ We deduce that $\beta = 1$. Hence $\lambda + \mu = -1$, and so (as before) G is of symmetric type. □

In view of Proposition 4.4 we say that λ, μ are *feasible* parameters for a graph of parametric non-symmetric type if (i) λ and μ are not coprime, (ii) $\lambda + \mu \geq 0$, and (iii) λ and μ satisfy the integrality conditions imposed by Eqs.(7) and (9). It is clear from Eq.(8) that when $\lambda + \mu = 0$, the graph G001 is the smallest that can arise. When $\lambda + \mu > 0$, the values of feasible parameters with smallest $\mu - \lambda$ are $\mu = 9$, $\lambda = -6$. Then $n_1 = 400$, $n_2 = 225$, $d_1 = 414$, $d_2 = 64$ and G has spectrum $384, 9^{(224)}, -6^{(400)}$. In this case, the graph G_1 in Theorem 4.2(c) is strongly regular with parameters $(400, 378, 357, 360)$. The complement $\overline{G_1}$ has the more appealing parameters $(400, 21, 2, 1)$. According to [2], the existence of such a graph remains an open question, and it is here that we pause our own investigation.

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