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EIGENVALUE MULTIPLICITY IN QUARTIC GRAPHS

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Abstract

Let G be a connected quartic graph of order n with μ as an eigenvalue of multiplicity k. We show that if $\mu \notin \{-1, 0\}$ then $k \leq (2n-5)/3$ when $n \leq 22$, and $k \leq (3n-1)/5$ when $n \geq 23$. If $\mu \in \{-1, 0\}$ then $k \leq (2n+2)/3$, with equality if and only if $G = K_5$ (with $\mu = -1$) or $G = K_{4,4}$ (with $\mu = 0$).

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1 Introduction

Let G be a regular graph of order n with μ as an eigenvalue of multiplicity k, and let $t = n - k$. Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix A of G has dimension k and codimension t. From [1, Theorem 3.1], we know that if $\mu \notin \{-1, 0\}$ then $k \leq n - \frac{1}{2}(-1 + \sqrt{8n+9})$, equivalently $k \leq \frac{1}{2}(t+1)(t-2)$. For connected quartic graphs, a bound which is linear in t follows easily from the equation $\mathrm{tr}(A) = 0$. To see this, we suppose that $k \geq \frac{1}{2}n$, i.e. $k \geq t$. Then G is non-bipartite; also μ is an integer, for otherwise it has an algebraic conjugate which is a second eigenvalue of multiplicity k. It follows that if G is a connected quartic graph then $\mu \in \{-3, -2, 1, 2, 3\}$ (see [5, Sections 1.3 and 3.2]). Let d be the mean of the eigenvalues other than 4 and μ, so that $4 + k\mu + (n-k-1)d = 0$. We have $-4 < d < 4$, and so:

(a) if $\mu = -3$ then $k < \frac{4}{7}n$, i.e. $k < \frac{4}{3}t$;
(b) if $\mu = -2$ then $k < \frac{2}{3}n$, i.e. $k < 2t$;
(c) if $\mu = 1$ then $k < \frac{4}{5}n - \frac{8}{5}$, i.e. $k < 4t - 8$;
(d) if $\mu = 2$ then $k < \frac{2}{3}n - \frac{4}{3}$, i.e. $k < 2t - 4$;
(e) if $\mu = 3$ then $k < \frac{4}{7}n - \frac{8}{7}$, i.e. $k < \frac{4}{3}t - \frac{8}{3}$.

We show first that $k \leq 2t - 5$ whenever $\mu \notin \{-1, 0\}$. Then k is at most $\lfloor (2n-5)/3 \rfloor$, a bound which is sharp for $n = 6, 9, 12$. The arguments are somewhat different from those in the paper [8], where a corresponding bound for cubic graphs was established. Section 2 contains the required results on star complements, while Section 3 provides details of the proof. It is quickly established that the bound holds when $t > 9$ or $n > 23$, and subsequently we are able to improve the bound to $(3n-1)/5$ when $n \geq 23$. The large number of quartic graphs of order ≤ 23 justifies our case-by-case analysis when $t \leq 9$: the cases $n > 17$ are relatively easy to deal with, but there are already 86221634 connected quartic graphs of order 17 [7, Sequence A006820]. In Section 4 we show that when $\mu \in \{-1, 0\}$ we have $k \leq (2n+2)/3$, with equality if and only if $G = K_5$ (with $\mu = -1$) or $G = K_{4,4}$ (with $\mu = 0$).

2 Preliminaries

Let G be a graph of order n with μ as an eigenvalue of multiplicity k. A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. The fundamental properties of star sets and star complements are established in [5, Chapter 5]. We shall require the following results, where we write $u \sim v$ to mean that vertices u and v are adjacent. For any $U \subseteq V(G)$, we write G_U for the subgraph of G induced by U, and $\Delta_U(v)$ for the set $\{u \in U : u \sim v\}$. For the subgraph H of G it is convenient to write $\Delta_H(v)$ for $\Delta_{V(H)}(v)$.

Theorem 2.1. (See [5, Theorem 5.1.7].) *Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X.*

(i) *Then* X *is a star set for* μ *in* G *if and only if* μ *is not an eigenvalue of* C *and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \qquad (1)$$

(ii) *If* X *is a star set for* μ *then* $\mathcal{E}(\mu)$ *consists of the vectors* $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in I\!R^k)$.

Let $H = G - X$, where X is a star set for μ. In the notation of Theorem 2.1, C is the adjacency matrix of H, while the columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$. We write $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}^\top(\mu I - C)^{-1}\mathbf{y}$ $(\mathbf{x}, \mathbf{y} \in I\!R^t)$, where $t = n - k$. Eq. (1) shows that

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \begin{cases} \mu \text{ if } u = v \\ -1 \text{ if } u \sim v \\ 0 \text{ otherwise,} \end{cases}$$

and we deduce from Theorem 2.1:

Lemma 2.2. *If* X *is a star set for* μ, *and* $\mu \notin \{-1, 0\}$, *then the neighbourhoods* $\Delta_H(u)$ $(u \in X)$ *are non-empty and distinct.*

We write $\mathbf{j}$ for an all-1 vector, its length determined by context. Recall that μ is a *main* eigenvalue of G if $\mathcal{E}(\mu)$ is not orthogonal to $\mathbf{j}$, and that in an r-regular graph, every eigenvalue other than r is non-main. The next observation follows from Theorem 2.1(ii).

Lemma 2.3. (See [5, Proposition 5.2.4].) *If* X *is a star set for the non-main eigenvalue* μ *then* $\langle \mathbf{b}_u, \mathbf{j} \rangle = -1$ *for all* $u \in X$.

Lemma 2.4. *If* X *is a star set for* μ *in* G *and if* U *is a proper subset of* X *then* $X \setminus U$ *is a star set for* μ *in* $G - U$. *Moreover, if* μ *is a non-main eigenvalue of* G *then it is also a non-main eigenvalue of* $G - U$.

Proof. We repeat the following argument as necessary. If $u \in X$ and $|X| = k$ then μ has multiplicity $k - 1$ in $G - u$, and the first assertion follows. When μ is non-main we take $u = 1$ and observe that if $\begin{pmatrix} 0 \\ \mathbf{y} \end{pmatrix} \in \mathcal{E}(\mu)$ then $\mathbf{y}$ is a μ-eigenvector of $G - u$. It follows that the vectors $\mathbf{y}$ are orthogonal to $\mathbf{j}$ and constitute the $(k-1)$-dimensional eigenspace of μ in $G - u$. □

Lemma 2.5. (See [5, Theorem 5.1.6].) *Let* μ *be an eigenvalue of the graph* G. *If* G *is connected then* G *has a connected star complement for* μ.

For subsets U, V of $V(G)$ we write $E(U, V)$ for the set of edges between U and V. When $H = G - X$ it is convenient to write $\overline{X}$ for $V(H)$. The authors of [2] have determined all the graphs with a star set X for which $E(X, \overline{X})$ is a perfect matching, equivalently all the graphs for which $B = I$ in Eq.(1). Their result is the following.

Theorem 2.6. *Let* G *be a graph with* X *as a star set for the eigenvalue* μ. *If* $E(X, \overline{X})$ *is a perfect matching then one of the following holds:*
(a) $G = K_2$ *and* $\mu = \pm 1$, (b) $G = C_4$ *and* $\mu = 0$, (c) G *is the Petersen graph and* $\mu = 1$.

The spectra of all the connected graphs of order 6 or 7 are listed in [4] and [3] respectively. We say that a graph G is of *type* N or M-N according as G is numbered N in [4] or labelled M-N in [3].

3 The case $\mu \notin \{-1, 0\}$

For the remainder of the paper, G denotes a connected quartic graph of order n with an eigenvalue μ of multiplicity $k = n - t \geq t$. Then μ is an integer, and in this Section $\mu \notin \{-1, 0\}$. By Lemma 2.5 we may take H $(= G - X)$ to be a connected star complement for μ. Let $Q = \{i \in X : |\Delta_H(i)| = 1\}$ and $R = X \setminus Q$. Let Q' be the set of vertices in $\overline{X}$ with a neighbour in Q, and let $R' = \overline{X} \setminus Q'$. By Lemma 2.2, $E(Q, Q')$ is a perfect matching when $Q \neq \emptyset$. Moreover, if $i \in R$ then $|\Delta_H(i)| = 1 + g_i$, where $g_i \geq 1$. Let $q = |Q|$ $(= |Q'|)$ and $g = \Sigma_{i \in R} g_i$. We shall make use of the following four observations.

Lemma 3.1. *If $j \in R$ then $\mathbf{b}_j$ is not a linear combination of the vectors $\mathbf{b}_i$ $(i \in Q)$.*

Proof. Suppose by way of contradiction that $\mathbf{b}_j = \Sigma_{i \in Q} a_i \mathbf{b}_i$ $(j \in R)$. (Here $Q \neq \emptyset$ because $\mathbf{b}_j \neq \mathbf{0}$ by Lemma 2.2.) Now $\Sigma_{i \in Q} a_i \geq 2$, and by Lemma 2.3 we have $-1 = \langle \mathbf{b}_j, \mathbf{j} \rangle = \Sigma_{i \in Q} a_i \langle \mathbf{b}_i . \mathbf{j} \rangle = -\Sigma_{i \in Q} a_i \leq -2$, a contradiction. □

Lemma 3.2. *We have $q \leq t - 1$, and if $q = t - 1$ then the vertex of H not adjacent to a vertex in Q is adjacent to every vertex in R.*

Proof Clearly, $q \leq t$ because the singleton neighbourhoods $\Delta(i)$ $(i \in Q)$ are distinct. If $q = t$ then $G - R$ has a perfect matching between H and the star set Q. By Lemma 2.4 and Theorem 2.6, $G - R$ is the Petersen graph, $\mu = 1$ and H is a 5-cycle. No graph of order > 10 has a 5-cycle as a star complement for the eigenvalue 1 [5, Example 5.2.3], and so $R = \emptyset$, a contradiction.

The assertion in the case $q = t - 1$ follows from Lemma 3.1. □.

We write $\{\mathbf{f}_1, \ldots, \mathbf{f}_t\}$ for the standard orthonormal basis of $I\!R^t$. We take $X = \{1, \ldots, k\}$, $\overline{X} = \{1', \ldots, t'\}$ with $Q' = \{1', \ldots, q'\}$ and $i \sim i'$ $(i = 1, \ldots, q)$. Without loss of generality $\mathbf{b}_i = \mathbf{f}_i$ for each $i \in Q$.

Lemma 3.3. *Suppose that S is a proper subset of $\overline{X}$ such that*
(i) $\langle \mathbf{f}_i, \mathbf{j} \rangle = -1$ when $i' \in S$, $\langle \mathbf{f}_i, \mathbf{j} \rangle = 0$ when $i' \notin S$, and
(ii) each vertex in S has a neighbour in X.
Then $\mu = 1$ and G_S is 2-regular.

Proof. With suitable labelling we have $(\mu I - C)^{-1}\mathbf{j} = (-1, \ldots, -1, 0 \ldots 0)^\top$. Thus if $C = (c_{ij})$ then for $i \in S$ we have $1 + \mu = \Sigma_{j \in S} c_{ij}$. It follows that G_S is regular of degree $1 + \mu$, where necessarily $\mu \in \{1, 2\}$. Finally, condition (ii) ensures that $\mu \neq 2$ because H is connected. □

By Lemma 2.3 the set S above necessarily contains Q'. We shall make repeated use of the following application of Lemma 3.3 in the case that $S = Q'$.

Lemma 3.4. *Suppose that for each $v' \in R'$ there exist vertices $u' \in Q'$ and $w \in R$ such that $\Delta_H(w) = \{u', v'\}$. Then $\mu = 1$ and Q' induces a 2-regular subgraph.*

Proof. By Lemma 2.3, we have $\langle \mathbf{b}_w, \mathbf{j} \rangle = -1 = \langle \mathbf{f}_u, \mathbf{j} \rangle$. We have a matching between R' and a subset of R, and so we may take $\mathbf{b}_w = \mathbf{f}_u + \mathbf{f}_v$. Then $\langle \mathbf{f}_v, \mathbf{j} \rangle = 0$. It follows that

$$\langle \mathbf{f}_i, \mathbf{j} \rangle = \begin{cases} -1 & \text{if } i = 1, \ldots, q, \\ 0 & \text{if } i = q + 1, \ldots, t, \end{cases}$$

and the result follows from Lemma 3.3. □

Our objective is to show that if G is a quartic connected graph of order n with μ ($\neq -1, 0$) as an eigenvalue of multiplicity k then $k \leq \lfloor (2n-5)/3 \rfloor$. One can check directly that this inequality holds when $n \leq 7$, since the quartic graphs of order < 8 are K_5, $\overline{3K_2}$, $\overline{C_7}$ and $\overline{C_3 \dot{\cup} C_4}$. Accordingly we suppose that $n \geq 8$. Since $k \leq \frac{1}{2}(t+1)(t-2)$, we have $t \geq 4$.

Suppose that $k \geq 2t - a$, where $0 \leq a \leq t$. For $j \in \overline{X}$, let $d_j = |\Delta_H(j)|$, $e_j = |\Delta_X(j)|$. Then

$$2t - a + g \leq k + g = |E(X, \overline{X})| = \Sigma_{j \in \overline{X}} e_j = 4t - \Sigma_{j \in \overline{X}} d_j = 4t - 2|E(H)|.$$

Since $|E(H)| \geq t - 1$ we deduce that $|E(X, \overline{X})| \leq 2t + 2$. Note also that $|E(X, \overline{X})|$ is even. Since $g \leq a + 2$, we have $q \geq k - g \geq k - a - 2$. Also, $q \leq t - 1$ by Lemma 3.2 and so

$$2t - a \leq k \leq t + a + 1, \quad t \leq 2a + 1. \tag{2}$$

Note that $k < 2t - 1$ for otherwise $t \leq 3$. Accordingly we suppose by way of contradiction that $k = 2t - a$ where $a \in \{2, 3, 4\}$. Note that $t \leq 9$, equivalently $n \leq 23$.

The case $k = 2t - 2$. Taking $a = 2$, we have $t \leq 5$. If $t = 5$ then $k = 8$ and $8 + g = |E(X, \overline{X})| \leq 12$. Now $q \leq 4$ and $g \geq k - q \geq 4$. Hence $g = 4$, $q = 4$ and the vertex v of H not adjacent to a vertex of Q is adjacent to each of the four vertices in R. Thus v is isolated in H, a contradiction.

If $t = 4$ then $k = 6$ and so $n = 10$. But $n \leq 9$ by [1, Theorem 3.1], a contradiction.

The case $k = 2t - 3$. Taking $a = 3$, we have $t \leq 7$ and $k + g = |E(X, \overline{X})| \leq 18$. If $t = 7$ then $q \leq 6$, $k = 11$ and $11 + g = |E(X, \overline{X})| \leq 16$. Now $g \geq k - q \geq 5$ and so $g = 5$, $q = 6$; then the vertex v of H not adjacent to a vertex of Q is adjacent to each of the five vertices in R, contradicting 4-regularity.

If $t = 6$ then $k = 9$ and $9 + g = |E(X, \overline{X})| \leq 14$. Now $g \geq k - q \geq 4$ and so $g = 5$. Then $q \in \{4, 5\}$; and if $q = 5$ then H has an isolated vertex. Hence $q = 4$, H is a tree and each vertex in R is adjacent to exactly two vertices of H. By Lemma 3.1, each vertex of R is adjacent to $5'$ or $6'$ (or both). On the other hand, at most one vertex of R is adjacent to both $5'$ and $6'$, while each of $5', 6'$ is adjacent to at most 3 vertices of R. It follows that there exist vertices $i \in R$, $j' \in Q'$ such that $\Delta_H(i) = \{j', 5'\}$. We have $\mathbf{b}_h = \mathbf{f}_h$ ($h = 1, 2, 3, 4$) and (without loss of generality) $\mathbf{b}_i = \mathbf{b}_j + \mathbf{f}_5$. Since $\langle \mathbf{b}_i, \mathbf{j} \rangle = -1 = \langle \mathbf{b}_j, \mathbf{j} \rangle$, we have $\langle \mathbf{f}_5, \mathbf{j} \rangle = 0$. Again there exist vertices $u \in R$, $v' \in Q'$ such that $\Delta_H(u) = \{v', 6'\}$, and we deduce similarly that $\langle \mathbf{f}_6, \mathbf{j} \rangle = 0$. It follows that no vertex of R is adjacent to both $5'$ and $6'$. Hence there are just two possibilities for the degree sequence of the tree H, namely (a) 112222 and (b) 111223. In case (a), $H \cong P_6$ and there exists $w \in Q$ such that $H + v \cong P_7$. But P_7 has no integer eigenvalues. In case (b) there exists $w \in Q$ such that $H + w$ has degree sequence 1111233. Now among the trees of order 7 only those of type 6-4, 6-5 and 6-8 have an integer eigenvalue $\neq -1, 0$. It follows that $H + w$ is of type 6-5 and H is of type 111. Then there exists $z \in Q$ such that $H + z$ is of type 6-3, a contradiction. The possibility $t = 6$ is therefore eliminated.

If $t = 5$ then $k = 7$ and we find that $g \in \{3, 5\}$. If $g = 3$ then by Lemma 3.2, $q = 4$ and H has degree sequence 12223. In this case, there are two possibilities for the unicyclic graph H, but always there exists $w \in R$ such that $H + w$ is a bicyclic graph of type 89 or 93; but these graphs have no integer eigenvalues $\neq -1, 0$. If $g = 5$ then $q \geq 2$, and we consider the three possibilities for the tree H. If H is $K_{1,4}$ or P_5 then there exists $w \in Q$ such that $H + w$ is a graph of type 108, 111or 112; but none of these has an integer eigenvalue $\neq -1, 0$. Hence H is the tree with degree sequence 11123. If $w \in Q$ then $H + w$ is a tree of type 108, 109, 110 or 111. The first and last of these have no integer eigenvalue $\neq -1, 0$. Since $q \geq 2$ it follows that w may be chosen so that $H + w$ is the tree of type 110. Then $\mu = 1$ and we obtain a contradiction as follows. Let $\mu_1, \mu_2, \mu_3, \mu_4$ (with mean d) be the eigenvalues of G different from 4, 1. Then $d = -11/4$ and $\Sigma_{i=1}^{4}\mu_i^2 = 4n - 16 - 7 = 25$. Hence $\Sigma_{i=1}^{4}(\mu_i - d)^2 = \Sigma_{i=1}^{4}\mu_i^2 - 4d^2 < 0$, which is impossible.

If $t = 4$ then $k = 5 = \frac{1}{2}(t + 1)(t - 2)$. By [1, Theorem 3.1], G is strongly regular; but there is no strongly regular graph with eigenvalue multiplicities 1,3,5.

The case $k = 2t - 4$. Taking $a = 4$ in Eq.(2) we see that $t \leq 9$, while $k - q \leq g \leq 6$ and $q \geq 2t - 10$. The cases $t = 9$, $(t, q) = (8, 7)$ are ruled out by Lemma 3.2. Thus if $t = 8$ then $k = 12$, $q = 6$ and $g = 6$. Necessarily $g_i = 1$ for all $i \in R$. Since H is connected Lemma 3.1 ensures that $\Delta_R(7')$ and $\Delta_R(8')$ are disjoint 3-sets in R; moreover, $7' \not\sim 8'$. Now Lemma 3.4 applies and we deduce that Q' induces a 2-regular graph. This is a contradiction because H is a tree.

If $t = 7$ then $k = 10$ and $q \in \{4, 5, 6\}$. If $q = 6$ then by Lemma 3.2 the vertex $7'$ is adjacent to all four vertices in R, hence is isolated in H, a contradiction. If $q = 5$ then $g = 6$ (since $g \geq 5$ and g is even). The neighbourhoods $\Delta_R(6'), \Delta_R(7')$ are either (a) a 3-set and a disjoint 2-set or (b) 3-sets with just one common vertex. Now the summands g_i $(i \in R)$ are 1,1,1,1,2. It follows that in either case there exist vertices $i, j \in R$ such that $\Delta_H(i) = \{6', u\}$ for some $u \in Q'$ and $\Delta_H(j) = \{7', v\}$ for some $v \in Q'$. By Lemma 3.4, Q' induces a 5-cycle, a contradiction because H is a tree. Now suppose that $q = 4$. Then $g = 6$ and so $g_i = 1$ for all $i \in R$; moreover H is a tree. If each vertex in R' is adjacent to a vertex in R then examination of the possibilities for $E(R, \overline{X})$ shows that $\langle \mathbf{f}_i, \mathbf{j} \rangle = -1$ or 0 for each $i' \in R'$. By Lemma 3.3, H contains a cycle, a contradiction. Taking $7'$ to be non-adjacent to R, we see that the neighbourhoods $\Delta_R(5'), \Delta_R(6')$ are disjoint 3-sets. By Lemma 2.2 the tree H has degree sequence 4311111 or 4221111. There are three possible trees, but in all cases we can choose $v \in Q$ such that $H + v$ is one of the trees shown in Fig.1. Then $H + v$ has no integer eigenvalue μ such that $\mu \neq -1, 0$ and μ is not an eigenvalue of H. This disposes of the case $t = 7$.

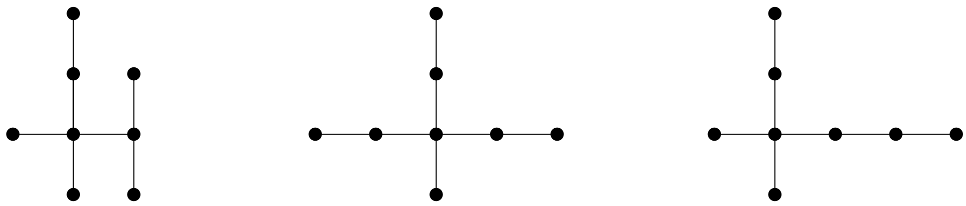

Figure 1: A choice of graphs $H+v$ $(v \in Q)$.

It is convenient to introduce one further lemma before proceeding with the case $t = 6$.

Lemma 3.5. *If $q \geq 2$ and H is a tree of order 6 then $\mu = 1$, H is the graph with degree sequence 421111, and there are just three possibilities for the neighbour in Q' of a vertex in Q.*

Proof. If $v \in Q$ then $H+v$ is a tree with an integer eigenvalue $\mu \neq -1, 0$, hence one of the trees of type 6-4, 6-5 or 6-8. Since μ is not an eigenvalue of H, H is determined uniquely up to isomorphism in each case: H is of type 110, 111 or 108, with $\mu = \pm 2, \pm 2, 1$ respectively.

Since $q \geq 2$ there exists $w \in Q$ with $w' \neq v'$. The graph $H+w$ has μ as an eigenvalue, and it follows that the only possibility is that $H+v$, $H+w$ are both of type 6-8. (When H is of type 110, $H+w$ is of type 6-3, 6-6 or 6-8; when H is of type 111, $H+w$ is of type 6-2, 6-3, 6-6 or 6-7; and when H is of type 108, $H+w$ is of type 6-7, 6-8 or 6-9.) Then H is the graph of type 110 (described in the lemma) and the only vertices of H at which we may attach a pendant edge are are the three endvertices adjacent to the vertex of degree 4. $\square$

If $t = 6$ then $k = 8$, $q \in \{2, 3, 4, 5\}$ and $g \in \{4, 6\}$. If $(q, g) = (5, 4)$ then we may take $g_6 = g_7 = 1$, $g_8 = 2$. It follows that $\langle \mathbf{f}_6, \mathbf{j} \rangle = 0$ and hence that $\langle \mathbf{b}_8, \mathbf{j} \rangle = -2$, a contradiction. Next suppose that $(q, g) = (5, 6)$. Without loss of generality, either $g_6 = 1$, $g_7 = 2$, $g_8 = 3$ or $g_6 = g_7 = g_8 = 2$. In the first case, $\langle \mathbf{f}_6, \mathbf{j} \rangle = 0$ and we obtain the contradiction $\langle \mathbf{b}_7, \mathbf{j} \rangle = -2$. In the second case we adapt the argument of Lemma 3.3 as follows. We may take $\mathbf{b}_6 = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_6$. Then $\langle \mathbf{f}_6, \mathbf{j} \rangle = 1$ and $\mathbf{j} = (\mu I - C)(-1. -1, -1, -1, -1, 1)^\top$, whence $1 = \mu + \Sigma_{j=1}^{5} c_{6j}$. Now the vertex $6'$ is adjacent to all three vertices in R, hence to just one vertex in Q', and so $\mu = 0$, contrary to assumption. If $(q, g) = (4, 4)$ then $g_i = 1$ for each $i \in R$. If a vertex $i \in R$ is adjacent to both vertices in R' then by Lemma 2.2, there exist vertices $u, v \in R$ such that $\Delta_{R'}(u) = \{5'\}$ and $\Delta_{R'}(v) = \{6'\}$ (for otherwise H has an isolated vertex). Then $\langle \mathbf{f}_5, \mathbf{j} \rangle = \langle \mathbf{f}_6, \mathbf{j} \rangle = 0$, contradicting $\langle \mathbf{b}_i, \mathbf{j} \rangle = -1$. Hence $|\Delta_{R'}(i)| = 1$ for all $i \in R$, and by Lemma 3.4, Q' induces a 4-cycle. Since H is unicyclic, we deduce that $|E(R, R')| \geq 5$, a contradiction. The case $(q, g) = (4, 6)$ is eliminated by Lemma 3.5. Hence $q \leq 3$ and H is the tree described in Lemma 3.5. If $q = 3$ then the vertices $1', 2', 3'$ are the endvertices of H adjacent to the vertex of degree 4 in H (say $4'$). Without loss of generality, $|\Delta_X(5')| = 2$ and $|\Delta_X(6')| = 3$. Since the numbers g_i $(i \in R)$ are 1,1,1,1,2, each of $5', 6'$ is adjacent to a vertex in R with just one neighbour in Q'. It follows that $\langle \mathbf{f}_5, \mathbf{j} \rangle = \langle \mathbf{f}_6, \mathbf{j} \rangle = 0$. Hence if $g_i = 2$ then $\langle \mathbf{b}_i, \mathbf{j} \rangle = -2$, a contradiction.

Hence $q = 2$, $g = 6$ and H is a tree. By Lemma 3.5 we have $\mu = 1$ and we may take $\Delta_H(3') = \{1', 2', 4', 5'\}$, $\Delta_H(5') = \{3', 6'\}$. Some vertex

$w \in \Delta_X(4')$ is adjacent to a vertex in Q' and then $H + w$ is the unicyclic graph of type 7-22, but this graph does not have 1 as an eigenvalue, a contradiction.

If $t = 5$ then $k = 6$, $n = 11$ and $q \leq 5$. We note first that $\mu \neq 1$ for otherwise G has spectrum $4, 1^{(6)}, \mu_1, \mu_2, \mu_3, \mu_4$, with $\Sigma_{i=1}^{4}(\mu_i - d)^2 < 0$. If $q = 0$ then $g_i = 1$ $(i = 1, \dots, 6)$ and H is a tree. For each $v \in X$, $H + v$ is a unicyclic graph with μ as a non-main integer eigenvalue $\neq -1, 0, 1$. Hence $H + v$ is a 6-cycle and $\mu = -2$. Then H is a path and the neighbours of v in H are the endvertices of H. Now Lemma 2.2 affords a contradiction. If $q > 0$ then each tree $H + v$ $(v \in Q)$ is of type 109, and $\mu = -2$. Then H is the tree with degree sequence 32111, and it has only one vertex at which we can add a pendant edge to obtain a graph with eigenvalue -2. Hence $q = 1$, $g = 6$ and the summands g_i are 1,1,1,1,2. In particular, R contains a vertex w such that the graph $H + w$ is unicyclic with -2 as an eigenvalue. The $H + w$ is a 6-cycle, a contradiction.

If $t = 4$ then $\overline{G}$ is a cubic graph of order 8 with an eigenvalue $-1 - \mu \notin \{-1, 0\}$ of multiplicity 4. But there is no such graph, by [8, Theorem 1.1].

We have shown that $k \leq 2t - 5$, equivalently $3k \leq 2n - 5$, when $\mu \notin \{-1, 0\}$. Since k is an integer, we have $k \leq \lfloor (2n - 5)/3 \rfloor$. This bound is attained in $\overline{3K_2}$ (with $\mu = -2$), in the Paley graph of order 9 (with $\mu = 1$ and $\mu = -2$), and in the graph of order 12 labelled $I_{12,1}$ in [6] (with $\mu = 1$). In these examples, k is at most $\frac{1}{2}n$, and it remains to be seen whether there exists a constant $c \geq 0$ such that $k \leq t + c$. Meanwhile we go on to show that always $k \leq (3n - 1)/5$ when $\mu \notin \{-1, 0\}$: this bound is superior to $(2n - 5)/3$ precisely when $n \geq 23$.

Lemma 3.6. *Let G be a connected quartic graph of order n with μ as an eigenvalue of multiplicity k. If $\mu \notin \{-1, 0\}$ then $k \leq (3n - 1)/5$.*

Proof. In view of the remarks above, we may assume that $n \geq 23$. We have $k = |E(X, \overline{X})| - g \leq 2t + 2 - g$ and $t - 1 \geq q \geq k - g$, whence $k \leq 2t + 2 + (t - 1 - k)$ and $2k \leq 3t + 1$. If $2k = 3t + 1$ then $q = t - 1$, and in this case the vertex in R' is adjacent to all vertices in R by Lemma 3.2. It follows that $k \leq t + 2$ and hence that $n \leq 8$, contrary to assumption. If $2k = 3t$ then $\frac{3}{2}t = k \leq 2t + 2 - g \leq 2t + 2 - k + q \leq \frac{3}{2}t + 1$, and so $q \in \{t - 1, t - 2\}$. If $q = t - 1$ then $k \leq t + 2$ as before, and we have the contradiction $n \leq 10$. If $q = t - 2$ then $|R| \leq 6$ because each vertex in R' is adjacent to at most 3 vertices in R; then $k \leq t + 4$ and we have the contradiction $n \leq 20$. It follows that $2k \leq 3t - 1$, equivalently $5k \leq 3n - 1$. □

We combine the results of this Section as follows:

Theorem 3.7. *Let G be a connected quartic graph of order n with μ as an eigenvalue of multiplicity k. If $\mu \notin \{-1, 0\}$ then $k \leq (2n - 5)/3$ when $n \leq 22$, and $k \leq (3n - 1)/5$ when $n \geq 23$.*

4 The case $\mu \in \{-1, 0\}$

Here again G denotes a connected quartic graph of order n with μ as an eigenvalue of multiplicity k. As before we let $t = n - k$ and we take H

($= G - X$) to be a star complement for μ. In the case that $\mu \in \{-1, 0\}$ we shall require the following observation.

Lemma 4.1. (See [8, Lemma 4.1].) *Let G be graph with X as a star set for the eigenvalue μ, and let $H = G - X$. Suppose that u, v are distinct vertices in X such that $\Delta_H(u) = \Delta_H(v)$.*
(i) *If $\mu = -1$ then $\Delta_X(u) \dot\cup \{u\} = \Delta_X(v) \dot\cup \{v\}$ (and so u, v are co-duplicate vertices).*
(ii) *If $\mu = 0$ then $\Delta_X(u) = \Delta_X(v)$ (and so u, v are duplicate vertices).*

When $\mu \in \{-1, 0\}$ the neighbourhoods $\Delta_H(u)$ ($u \in X$) are non-empty [8, Lemma 2.4], but not necessarily distinct. In particular, $k \leq |E(X, \overline{X})| = 4t - 2|E(H)| \leq 2t + 2$. In this section we determine the graphs for which $\mu \in \{-1, 0\}$ and $k = 2t + 2$ (equivalently $3k = 2n + 2$). Note that in this situation, $|\Delta_H(u)| = 1$ for all $u \in X$, and H is a tree; moreover, if $t = 1$ then $G = K_5$ and $\mu = -1$.

Now suppose that $t > 1$ and $\mu = -1$. Since H is connected there exists a vertex $v' \in \overline{X}$ with exactly 3 neighbours in X, say $v' = 1'$ and $\Delta_X(1') = \{1, 2, 3\}$. By Lemma 4.1 the vertices $1, 2, 3$ are co-duplicate, and so the vertices $1, 2, 3$ are pairwise adjacent. The fourth neighbour of 1 lies in X and so we take $\Delta_G(1) = \{1', 2, 3, 4\}$; then $\Delta_G(i) \dot\cup \{i\} = \{1, 2, 3, 4\}$ ($i = 1, 2, 3$). The fourth neighbour of 4 lies in $\overline{X} \setminus \{1'\}$ and so we take $\Delta_G(4) = \{1, 2, 2', 3\}$. Note that $\Delta_H(2') = \{4\}$ by Lemma 4.1. By Theorem 2.1(ii), $\mathcal{E}(-1)$ contains a vector $\mathbf{x} = (x(u) : u \in V(G))$ such that $x(1) = 1$ and $x(i) = 0$ ($i = 2, \ldots, k$). We calculate the remaining entries of $\mathbf{x}$ by means of the relation $-x(u) = \Sigma_{v \sim u} x(v)$. We find $x(1') = x(2') = -1$; moreover $x(i') = 0$ whenever $i > 2$ and i' is adjacent to X. If j' is a vertex of degree 4 in H then consider the components T_1, T_2, T_3, T_4 of $H - j'$. Without loss of generality, we may take $V(T_1) \subseteq \overline{X} \setminus \{1', 2'\}$. Note that $\Delta_X(i') \subseteq X \setminus \{1, 2, 3, 4\}$ for all $i' \in V(T_1)$. Now $x(i') = 0$ for all $i' \in V(T_1)$ by induction on the distance of i' from an endvertex of H in T_1. It follows that $x(j') = 0$, and so the only non-zero entries of $\mathbf{x}$ are $1, -1, -1$. This is a contradiction because -1 is a non-main eigenvalue. Our conclusion is the following.

Proposition 4.2. *Let G be a connected quartic graph of order n with -1 as an eigenvalue of multiplicity k. Then $k \leq \frac{2}{3}(n + 1)$, with equality if and only if $G = K_5$.*

Now suppose that $\mu = 0$ and $k = 2t + 2$. Necessarily $t > 1$ $|\Delta_H(u)| = 1$ for all $u \in X$, and H is a tree. Again $\overline{X}$ has a vertex $1'$ adjacent to exactly 3 vertices in X, say 1,2,3. By Lemma 4.1, these vertices are duplicate vertices and so there exist vertices 4,5,6 in X such that 1,2,3 and 4,5,6 induce a complete bipartite graph $K_{3,3}$. If i' is a vertex of $\overline{X}$ adjacent to $\{4, 5, 6\}$ then $\Delta_X(i') \subseteq \{4, 5, 6\}$ by Lemma 4.1. By Theorem 2.1(ii), G has a 0-eigenvector $\mathbf{x} = (x(u) : u \in V(G))$ such that $x(1) = 1$ and $x(i) = 0$ ($i = 2, \ldots, k$). Then $x(1') = 0$ and $x(i') = -1$ whenever $i' \sim \{4, 5, 6\}$. Arguing as in the case $\mu = -1$, we find that $x(i') = 0$ for all remaining $i' \in \overline{X}$. Since $\mathbf{x}$ is orthogonal to the all-1 vector, necessarily 4,5,6 have a common neighbour in $\overline{X}$, say $2'$. Necessarily $1' \sim 2'$, and so we have:

Proposition 4.3. *Let G be a connected quartic graph of order n with 0 as an eigenvalue of multiplicity k. Then $k \leq \frac{2}{3}(n+1)$, with equality if and only if $G = K_{4,4}$.*

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