Star sets and related aspects of algebraic graph theory

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Abstract

Let μ be an eigenvalue of the graph G with multiplicity k. A star set corresponding to μ in G is a subset of $V(G)$ such that $|X| = k$ and μ is not an eigenvalue of $G - X$. It is always the case that the vertex set of G can be partitioned into star sets corresponding to the distinct eigenvalues of G. Such a partition is called a star partition. We give some examples of star partitions and investigate the dominating properties of the set $V(G) \setminus X$ when $\mu \notin \{-1, 0\}$.

The induced subgraph $H = G - X$ is called a star complement for μ in G. The Reconstruction Theorem states that for a given eigenvalue μ of G, knowledge of a star complement corresponding to μ, together with knowledge of the edge set between X and its complement $\overline{X}$, is sufficient to reconstruct G. Pursuant to this we explore the idea that the adjacencies of pairs of vertices in X is determined by the relationship between the H-neighbourhoods of these vertices. We give some new examples of cubic graphs in this context.

For a given star complement H the range of possible values for the corresponding eigenvalue μ is constrained by the condition that μ must be a simple eigenvalue of some one-vertex extension of H, and a double eigenvalue of some two-vertex extension of H. We apply the Reconstruction Theorem to the generic form of a two-vertex extension of H, thereby obtaining sufficient information to construct a graph containing H as a star complement for one of the possible eigenvalues. We give examples of graph characterizations arising in the case where the star complement is (to within isolated vertices) a complete bipartite graph.

Declaration.

I, Penelope S. Jackson, declare that this thesis is of my own composition and that the results therein are due to my own research except where stated otherwise.

Penelope S. Jackson.

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I dedicate this thesis to my father; sadly he will never see it.

Chapter 1

Introduction.

The subject of this thesis is star sets and related aspects of algebraic graph theory. There is a close link between graph theory and linear algebra and the theory behind star sets exploits this. The first chapter provides the necessary background in linear algebra, together with some basic graph theory, and includes a little design theory which was used to describe and identify some graphs. The second chapter gives a detailed summary of star sets and star partitions, much of which can be found in [CvRoS] where it appears in a slightly different form. The third chapter gives some results on star partitions.

In Chapter 4 we explain the relationship between star sets and the structure of graphs, and in the fifth we explore the notion of constructing or reconstructing a graph from an induced subgraph (called the star complement) and an associated eigenvalue. The work in Chapter 6 was inspired by [Row1] and as an introduction we provide the details for one of the results there. We then go on to find additional examples for one particular case. From Chapter 7 onwards we examine the task of constructing graphs from specified star complements. Some of the preliminary work was done with the aid of the computer package *GRAPH*. For details of this package see [CvLRoS].

In the main, the results in the first five chapters can be found elsewhere in some form or another. The additional examples given in Chapter 6 are the work of the author, as is everything from Chapter 7 onwards except where otherwise stated. Notation is introduced as required and should be regarded as cumulative.

Chronologically, the work on cubic graphs which appears in Chapter 6 was done first. This was followed by the case covered in Chapter 8 where we investigate the construction of graphs from the star complement $H \cong K_{r,s}$ and an associated eigenvalue μ. This work led to many of the definitions and observations expressed in Chapter 5. In dealing with the particular case where $H \cong K_{2,5}$ (Chapter 9) we found our first explicit example, the Schläfli graph which is strongly regular. This in turn led to the search for another strongly regular graph containing a star complement of the form $K_{r,s}$. The MacLaughlin graph was found to contain $K_{1,16} \cup 6K_1$ as a star complement for the eigenvalue 2; hence the investigation into the general case where $H \cong K_{r,s} \cup tK_1$ presented in Chapter 7. In the case where $H \cong K_{r,s}$, the possibility of proving that μ^2 was an integer and that $\mu^2 < rs$ was considered. This was done in the special case where $r = 1$ and the star set X in the constructed graph is such that $|X| \geq 3$. These results are amongst those presented in Chapter 10.

Background.

A graph G is defined by a set of vertices $V(G)$ and a set of edges $E(G)$ where $E(G) = \{(u,v) : u, v \in V(G), u \sim v\}$. One of the fundamental problems of graph theory is to find an efficient way of determining if two graphs are isomorphic. This can always be done but for the general case there is no known algorithm in polynomial time. (See [BaGM] for a polynomial algorithm in the case where the multiplicities of the eigenvalues are bounded.) One approach to this problem is to use algebraic invariants. A function f defined on graphs is a *graph invariant* if the fact that G_1 is isomorphic to G_2 implies that $f(G_1) = f(G_2)$. Thus we can say G_1 is *not* isomorphic to G_2 if $f(G_1) \neq f(G_2)$. We say a set $\mathcal{F}$ of invariants is *complete* when $G_1 \cong G_2$ if and only if $f(G_1) = f(G_2)$ for all $f \in \mathcal{F}$.

Throughout this work we shall restrict ourselves to finite simple graphs, that is to finite graphs which are undirected with no loops or multiple edges. If the vertices of G are labelled $1, \dots, n$ then the corresponding adjacency matrix of G, is defined to be the $(0, 1)$ matrix $A_G = (a_{uv})$ such that $a_{uv} = 1$ if and only if $u \sim v$. Re-labelling the vertices of G is equivalent to re-ordering the rows and columns of A_G, resulting in a similar matrix with identical eigenvalues. The eigenvalues of G are simply the eigenvalues of A_G and, as we have seen, they are independent of labelling. The distinct eigenvalues together with their multiplicities are known as the *spectrum* of G which is denoted by $Sp(G)$. The spectrum of a graph is an algebraic invariant since if $Sp(G_1) \neq Sp(G_2)$ then $G_1 \ncong G_2$. In general the converse of this does not hold, and we have co-spectral non-isomorphic graphs, or PINGS [CvDS, Chap. 6]. In special cases however, knowledge of the spectrum is equivalent to knowledge of the graph; for example the graph K_n is the only graph to have the eigenvalues $n-1$ and -1 with multiplicities 1 and $n-1$ respectively.

The overall aim is to be able to identify uniquely a graph from as small a set of algebraic invariants as possible. Clearly knowledge of A_G is equivalent to knowledge of G, but determining whether or not two matrices are similar is a problem of the same complexity as determining whether or not two graphs are isomorphic. However we will see that it is sometimes possible to identify a graph from part of the adjacency matrix, together with a specific eigenvalue. In order to do this we explore the notion of constructing or reconstructing a graph from a star complement and an associated eigenvalue. Whether this is reconstruction or construction is a moot point since the process of investigation begins with a known graph from within which is chosen a star complement which we know to correspond to one of the eigenvalues of the graph. From this smaller graph and corresponding eigenvalue we attempt to reconstruct the graph we started with, hopefully uniquely, and so we could regard this as a reconstruction process. On the other hand, the results are obtained by starting with a graph and an associated eigenvalue, and then constructing graphs for which the initial graph is a star complement, and so we could regard this as a construction process. In some cases we obtain interesting graphs in addition to the prototype. We shall see that the multiplicity of the eigenvalue increases by one each time we add a vertex so that we are in fact constructing graphs with multiple eigenvalues.

Linear Algebra

We have said that knowledge of a graph G is equivalent to the knowledge of an adjacency matrix A_G and it is because of this connection between graphs and matrices that we are able to utilize results from linear algebra and apply them to graphs. We offer a miscellany of results and definitions, one criterion for inclusion being that they were used in this thesis. For further results we refer the reader to two important source books, [CvDS] and [Biggs].

Definition 1.1 [Hal, Chap. 3] Let $\mathcal{U}$ be a subspace of the finite-dimensional inner product space $\mathcal{V}$, so that $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$ and each $\mathbf{v} \in \mathcal{V}$ is uniquely expressible as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in \mathcal{U}$ and $\mathbf{v}_2 \in \mathcal{U}^\perp$. Then the map $P : \mathbf{v} \mapsto \mathbf{v}_1$ is the *orthogonal projection of* $\mathcal{V}$ *onto* $\mathcal{U}$. Note that $\ker P = \mathcal{U}^\perp$ and that $\ker(I - P) = \mathcal{U}$.

Definition 1.2 [Seid]. Let $\mathcal{U}$ be a subspace of the finite dimensional inner product space $\mathcal{V}$. A *eutactic star* in $\mathcal{U}$ is the orthogonal projection onto $\mathcal{U}$ of vectors in $\mathcal{V}$ which are of the same length and pairwise orthogonal.

The next few remarks are concerned with matrices.

The matrix M_{ij} is obtained from the matrix M by deleting row i and column j. The *adjoint* of M, or adj M, is the $n \times n$ matrix whose (i, j) entry is $(-1)^{i+j} \det M_{ji}$.

Remark 1.3 [Ait, Chap.3] For any square matrix M, $M \operatorname{adj} M = (\det M)I$.

Let M be an $n \times n$ matrix. Fix a subset R of the rows of M with $|R| = k$ and let M_{RX} be a submatrix of M with entries m_{ij} $(i \in R,\ j \in X)$ where X is a subset of the columns of M. Similarly $M_{\overline{RX}}$ is the submatrix with entries m_{ij} $(i \notin R,\ j \notin X)$.

Remark 1.4 If M is a square matrix then with an appropriate choice of signs, $\det M = \sum_X \pm \det M_{RX} \det M_{\overline{RX}}$ where $\sum_X$ denotes summation over all $\binom{n}{k}$ k-element subsets X of the columns of M.

To see why this is so we count numbers of terms. The number of terms in $\det M$ is $n!$ The number of terms in $\det M_{RX}$ is $k!$ and the number of terms in $\det M_{\overline{RX}}$ is $(n-k)!$. The products of the form $\det M_{RX} \det M_{\overline{RX}}$ account for $n!$ terms since

$$\binom{n}{k} k!(n-k)! = \frac{n!k!(n-k)!}{k!(n-k)!} = n!.$$

Each is a term of $\det M$ to within sign. For examples of this see [Ait, Chap. 4]. It follows that if M is invertible, equivalently $\det M$ is non-zero, then for a fixed R there is a subset X such that $\det M_{RX}$ and $\det M_{\overline{RX}}$ are non-zero; equivalently M_{RX} and $M_{\overline{RX}}$ are invertible.

The next remark is taken from Theorem 3.1.1 [Pras].

Remark 1.5 Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block matrix with square matrices A and D. If $\det D \neq O$, then $\det M = \det(A - BD^{-1}C)\ \det D$.

The matrix $A - BD^{-1}C$ is called the *Schur complement of D in M*. The next remark is also taken from [Pras] [Theorem 34.2.2]

Remark 1.6 Let $A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}$ be an Hermitian matrix. Let the eigenvalues of A and B form increasing sequences: $\alpha_1 \leq \ldots \leq \alpha_n$, $\beta_1 \leq \ldots \leq \beta_m$. Then

$$\alpha_i \leq \beta_i \leq \alpha_{i+n-m}.$$

Let M be a symmetric matrix with real non-negative entries. Then the matrix M is orthogonally diagonalizable; that is, there exists an orthogonal matrix U such that $U^{-1}MU = D$ where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$. The columns of U consist of orthonormal eigenvectors of M, since the relation $MU = UD$ may be written

$$M[\mathbf{u}_1|\mathbf{u}_2|\ldots|\mathbf{u}_n] = [\mathbf{u}_1|\mathbf{u}_2|\ldots|\mathbf{u}_n] \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}.$$

With appropriate ordering of the columns of U we have

$$M[\mathbf{u}_1|\mathbf{u}_2|\ldots|\mathbf{u}_n] = [\mathbf{u}_1|\mathbf{u}_2|\ldots|\mathbf{u}_n] \begin{pmatrix} \mu_1 I_{k_1} & 0 & \ldots & 0 \\ 0 & \mu_2 I_{k_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_m I_{k_m} \end{pmatrix}.$$

Let $E_1 = \begin{pmatrix} I_{k_1} & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{k_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and so on. Then $\sum_{i=1}^{m} E_i = I$ and since $M = UDU^{-1}$ we have

$$M = \mu_1 P_1 + \mu_2 P_2 + \ldots + \mu_m P_m$$

where $P_i = UE_iU^{-1}$. This expression for M is called the *spectral decomposition* of M.

Remark 1.7 The matrix P_i represents the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu_i)$.

We consider the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu_1)$, with respect to the standard basis of $\mathbb{R}^n$.

Let $\mathbb{R}^n$ have orthonormal basis of eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ with $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k_1}\}$ the basis for $\mathcal{E}(\mu_1)$. Then for each $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$ and the orthogonal projection of $\mathbf{v}$ onto $\mathcal{E}(\mu_1)$ is $\sum_{i=1}^{k_1} \alpha_i \mathbf{u}_i$.

Now consider $P_1 = UE_1U^{-1} = UE_1U^T$. We have

$$\begin{aligned} P_1 &= [\mathbf{u}_1 \ldots \mathbf{u}_n] \begin{pmatrix} I_{k_1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{pmatrix} \\ &= [\mathbf{u}_1 \ldots \mathbf{u}_n] \begin{pmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_{k_1}^T \\ 0 \\ \vdots \end{pmatrix} \end{aligned}$$

$$= \mathbf{u}_1\mathbf{u}_1^T + \ldots + \mathbf{u}_{k_1}\mathbf{u}_{k_1}^T.$$

Now

$$\begin{aligned} P_1\mathbf{v} &= P_1\alpha_1\mathbf{u}_1 + P_1\alpha_2\mathbf{u}_2 + \ldots + P_1\alpha_n\mathbf{u}_n \\ &= \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \ldots + \alpha_{k_1}\mathbf{u}_{k_1} \end{aligned}$$

since $(\mathbf{u}_i\mathbf{u}_i^T)\mathbf{u}_j = \begin{cases} \mathbf{u}_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$

Thus $P_1\mathbf{v} = \sum_{i=1}^{k_1} \alpha_i\mathbf{u}_i$. Hence P_1 represents the orthogonal projection of $I\!R^n$ onto $\mathcal{E}(\mu_1)$. This argument applies for each i and so the general conclusion is that P_i represents the orthogonal projection of $I\!R^n$ onto $\mathcal{E}(\mu_i)$.

The matrix P_i has certain properties which follow from the corresponding properties of the matrices E_i.

Remark 1.8 The matrix P_i is idempotent and symmetric. That is $P_i^2 = P_i = P_i^T$. Moreover $P_iP_j = O$ $(i \neq j)$.

Another property of a diagonalizable matrix is that the algebraic and geometric multiplicities of the eigenvalues are equal. The adjacency matrix of a graph is diagonalizable and consequently we can state unequivocally that μ is an eigenvalue of G with multiplicity k. Furthermore since A_G is orthogonally diagonalizable we can make the following observation.

Remark 1.9 The adjacency matrix of a graph has spectral decomposition

$$A_G = \mu_1P_1 + \mu_2P_2 + \ldots + \mu_mP_m.$$

(See also [GoMo].)

We now note some properties of the eigenvalues and eigenvectors of a symmetric matrix with real non-negative entries.

Remark 1.10 Let μ_i $(i = 1 \ldots m)$ be the distinct eigenvalues of the n-vertex graph G, with multiplicities k_i $(i = 1 \ldots m)$. Then

(*i*) $\sum_{i=1}^{m} k_i = n$.

(*ii*) $\sum_{i=1}^{m} \mu_i k_i = 0$ [Biggs, Result 2h].

Remark 1.11 [CvDS, Thm. 0.2] The spectral radius (or largest eigenvalue) of M is not less that the absolute value of any other eigenvalue of M. The spectral radius of A_G is called the *index* of G.

The following definition is taken from [Gant, Chap. 13].

Definition 1.12 A matrix A is called *reducible* if there is a permutation matrix P such that the matrix $P^{-1}AP$ is of the form $\begin{pmatrix} X & O \\ Y & Z \end{pmatrix}$, where X and Z are square matrices. Otherwise, A is called *irreducible.*

Remark 1.13 The adjacency matrix of a graph is irreducible if and only if the graph is connected. It follows [Gant, Chap. 13] that if G is connected then the index of G has multiplicity 1.

Remark 1.14 [Biggs, Prop 3.1] If the row sums of A_G all equal k then G is k-regular and the index of G is k. Moreover the index has the all-one vector $\mathbf{j}$ as an eigenvector.

Remark 1.15 [Biggs, Result 3a] Let G be an n-vertex, k-regular connected graph with eigenvalues $k > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$. Then the adjacency matrix of the complement of G is $\overline{A}_G = -(A_G - J + I)$ where J is the all-one matrix. The eigenvalues of $\overline{G}$ are $n-1-k \geq -1-\lambda_n \geq -1-\lambda_{n-1} \geq \ldots \geq -1-\lambda_2$. If $\overline{G}$ is connected then $n-1-k > -1-\lambda_n$.

Definition 1.16 The eigenvalue μ is a *main eigenvalue* if $\mathbf{j} \notin \mathcal{E}(\mu)^{\perp}$.

Remark 1.17 In the case where μ is an eigenvalue of a k-regular graph G, if $\mu \neq k$ then μ is a main eigenvalue of G.

Remark 1.18 The eigenvalue μ is an eigenvalue of a graph with adjacency matrix A_G if and only if there exists a non-zero vector $\mathbf{x}$ such that $A_G\mathbf{x} = \mu\mathbf{x}$. Moreover $A_G\mathbf{x} = \mu\mathbf{x}$ if and only if

$$\mu x_i = \sum_j a_{ij} x_j = \sum_{j \sim i} x_j \quad \forall i.$$

The next remark is the Interlacing Theorem. We denote a vertex-deleted subgraph of G by $G - j$.

Remark 1.19 Let A_{G-j} be the adjacency matrix of a vertex-deleted subgraph. Let the eigenvalues of A_G and A_{G-j} form increasing sequences: $\mu_1 \leq \ldots \leq \mu_n, \lambda_1 \leq \ldots \leq \lambda_{n-1}$. Then by Remark 1.6

$$\mu_i \leq \lambda_i \leq \mu_{i+1}.$$

See also [Lanc, Chap. 8. Cor 1.] and [MaMi, p. 119].

Remark 1.20 Let μ be an eigenvalue of G with multiplicity k. The multiplicity of μ as an eigenvalue of $G - j$ differs from k by at most one.

To see why this is so let us denote the eigenvalues of G by μ and those of $G - j$ by λ. Furthermore suppose that μ_i is an eigenvalue of G with multiplicity k and that $\lambda_i = \mu_i$ is an eigenvalue of $G - j$. By 1.19 we have

$$\ldots \mu_h \geq \lambda_h \geq \mu_i = \lambda_i = \ldots = \lambda_i = \mu_i \geq \lambda_j \geq \mu_j \ldots$$

where $\mu_h > \mu_i > \mu_j$. There are three possibilities for the multiplicity of λ_i as an eigenvalue of $G - j$.

If $\lambda_h > \mu_i > \lambda_j$ then neither $\lambda_h = \lambda_i$ nor $\lambda_j = \lambda_i$ and so the multiplicity of λ_i is $k - 1$.

If either $\lambda_h = \mu_i > \lambda_j$ or $\lambda_h > \mu_i = \lambda_j$ then either $\lambda_h = \lambda_i$ or $\lambda_j = \lambda_i$ and so the multiplicity of λ_i is k.

If $\lambda_h = \mu_i = \lambda_j$ then $\lambda_h = \lambda_i = \lambda_j$ and so the multiplicity of λ_i is $k + 1$.

The multiplicity of λ_i cannot be greater than $k+1$ since this would mean that either $\lambda_i \geq \mu_h > \mu_i$ or $\lambda_i \leq \mu_j < \mu_i$.

Definition 1.21 [CvRoS1, Defn. 2.4.4] Given an $s \times s$ matrix $B = (b_{ij})$, suppose that the vertex set of G can be partitioned into (non-empty) subsets $X_1, X_2, \ldots, X_s$ so that for any $i, j = 1, 2, \ldots, s$ each vertex from X_i is adjacent to exactly b_{ij} vertices of X_j. The multidigraph with adjacency matrix B is called a *divisor* of G.

Remark 1.22 [Cvet3, Thm. 3] The spectrum of any divisor H of a graph G includes the main eigenvalues of G.

The next two remarks are results from number theory.

Remark 1.23 [Birk, Chap. 14.8] Eigenvalues are algebraic integers since the characteristic polynomial $\Phi_G(x)$ is monic with integral coefficients; consequently, if an eigenvalue is also rational, it is a rational integer.

Remark 1.24 [Hun, Thm. 53] The set of all integral solutions (x, y, z) of the diophantine equation $x^2 + y^2 = z^2$ is given by

$$\begin{aligned} (x,y,z) &= (t(a^2-b^2), 2tab, t(a^2+b^2)) \quad \text{when} \quad y \quad \text{is even,} \\ \text{and} \quad (x,y,z) &= (2tab, t(a^2-b^2), t(a^2+b^2)) \quad \text{when} \quad x \quad \text{is even} \end{aligned}$$

where t, a, b are integral parameters such that $(a, b) = 1$ and a, b are of opposite parity. Note that we can also include the trivial solution obtained when $a = b$.

We conclude this chapter by demonstrating different ways of describing graphs, taking as examples the MacLaughlin graph and the Schläfli graph. Both of these graphs are strongly regular, that is, connected with parameters (n, k, e, f) where n is the number of vertices, k is the degree of regularity, e is the number of common neighbours of every adjacent pair of vertices and f is the number of common neighbours of every non-adjacent pair. The eigenvalues of strongly regular graphs can be obtained from their parameters. Since the graphs are connected and regular, one eigenvalue is k with multiplicity 1. The other eigenvalues, say μ_1, μ_2, are the roots of the equation $\mu^2 + (f - e)\mu + (f - k) = 0$ [CvL, Eqn. 2.15. Pg 37]. We give the parameters of the MacLaughlin and Schläfli graphs in Table 1.1.

Graph	n	k	e	f	μ_1	μ_2
McLaughlin	275	112	30	56	2	−28
Schläfli	27	10	1	5	1	−5

Table 1.1

For a more extensive survey of strongly regular and other associated graphs see [Hub].

Knowledge of any graph is equivalent to the knowledge of its complement. In the literature both the graph and its complement are sometimes referred to by the same name. Since the graphs we name are regular we can usually avoid any confusion by subscripting the name with the degree of regularity. Thus the Schläfli graph with 27 vertices of degree ten is denoted Sch_{10}. It should be noted that the complement of a strongly regular graph is itself strongly regular. [Biggs]

The first example is taken from [BrCoNe, Section 3.11] and illustrates the representation of a graph by a root system.

Example 1.25 The root system E_8 has 240 roots, each of which represents a vertex of the graph of this system, with adjacency if and only if their inner product is one. The graph Sch_{16} is an induced subgraph of this graph with vertices represented by the vectors $\mathbf{e}_i + \mathbf{e}_7$, $\mathbf{e}_i + \mathbf{e}_8$ $(i \leq 6)$, $\frac{1}{2}\mathbf{j} - \mathbf{e}_i - \mathbf{e}_j$ $(i < j \leq 6)$ where we follow the usual convention that $\mathbf{e}_i$ is the $(0,1)$ vector with 1 in the i^{th} position and $\mathbf{j}$ is the all-one vector. Thus the vector $\frac{1}{2}\mathbf{j} - \mathbf{e}_i - \mathbf{e}_j$ would have $-\frac{1}{2}$ in the i^{th} and j^{th} positions and $\frac{1}{2}$ elsewhere. A natural labelling for these vertices would be the twelve pairs (i,j) $(1 \leq i \leq 6,\ j = 7,8)$ and the fifteen pairs $(i,j)'$ $(1 \leq i < j \leq 6)$. These labels have at most one common digit. For the adjacencies we have $(i_1,j_1) \sim (i_2,j_2)$ if and only if the two pairs have a single digit in common; $(i_1,j_1)' \sim (i_2,j_2)'$, again, if and only if the two pairs have a single digit in common, and $(i_1, j_1) \sim (i_2, j_2)'$ if and only if the two pairs have no digit in common. Thus the vertices $(i, 7)$ induce K_6 as do the vertices $(i, 8)$. The vertices $(i, j)'$ induce a graph which is regular of degree eight whilst ten of these vertices, say (i, j) $(2 \leq i < j \leq 6)$ induce a graph which is regular of degree 6 and is in fact the complement of the Petersen graph.

Taking the complement of Sch_{16} and simplifying the labelling somewhat so that i replaces $(i, 7)$; i' replaces $(i, 8)$ and ij replaces $(i, j)'$ we have a graph where $a \sim b'$ if and only if $a \neq b$; $\{1, \ldots, 6\}$ is an independent set (that is the vertices are pairwise non-adjacent); $\{1', \ldots, 6'\}$ is also an independent set; the vertex a or a' is adjacent to ij if and only if $a \in \{i, j\}$ and $ij \sim ab$ if and only if the two labels have no digit in common. This graph is Sch_{10}. The Petersen graph is the subgraph induced by ij $(2 \leq i < j \leq 6)$; the set $\{1, 1'\} \cup \{1j : j = 2, 3, 4, 5, 6\}$ induces the subgraph $K_{2,5}$ and the remaining vertices comprise two independent sets, namely $\{2, 3, 4, 5, 6\}$ and $\{2', 3', 4', 5', 6'\}$. This is the precise description of the graph discussed in Chapter 9. (For more on root systems and graphs see [CvDS]).

There is a strong link between graph theory and design theory as our second example illustrates. It is not our intention to elaborate upon design theory but we will give a brief summary of the relevant results. But first we give an explanation of *switching.*

Let G be a graph and let Y be a subset of $V(G)$. Suppose we switch with respect to Y: then the subgraphs induced by Y and its complement remain the same but any edge between Y and $\overline{Y}$ becomes a non-edge, and any non-edge becomes an edge. In particular, switching with respect to the neighbourhood of a vertex has the effect of isolating that vertex. Another interesting aspect of switching is that if switching is performed on certain strongly regular graphs it is possible to construct other strongly regular graphs; see [Seid1] for details.

A t-design with parameters (v, k, λ) is a collection of k-subsets (called *blocks*) of a v-set V such that every t-subset of V is contained in precisely λ blocks.

Let b be the number of blocks in a given t-design and consider an ordered pair whose first element is a block, and whose second element is a t-set contained in that block. There are ${}^kC_t$ t-sets in each block, and since there are b blocks we have $b\,{}^kC_t$ such pairs altogether. Counting these pairs in another way, there are ${}^vC_t$ ways of choosing a t-set, each appearing in λ blocks and so we have

$$b = \lambda \frac{{}^vC_t}{{}^kC_t}.$$

It is sometimes useful to know how many blocks contain an s-set where s is a number of points $1 \leq s \leq t$. Let b_s be the number of blocks containing an s-set. Consider an ordered pair whose first element is a block, and whose second element is an s-set contained within that block. Counting these pairs in two ways as before we get

$$b_s = b \frac{{}^{k}C_s}{{}^{v}C_s}.$$

For a t-design with $\lambda = 1$ we get a *Steiner* system $S(t, k, v)$; v is the size of the point set, k is the block size, and t is the t-set size, each t-set appearing in exactly one block.

Another point-block design is the two-graph. This design is not itself a graph but there is a direct connection between graphs and two-graph designs. Let G be a graph with vertex set X. Let $\mathcal{B}$ be the set of 3-sets of X such that each 3-set induces either K_3 or $K_1 \cup K_2$. Then $(X, \mathcal{B})$ is a two-graph design. We say that the graph G *yields* the design $(X, \mathcal{B})$. [CvL, pg 58]. A two-graph design is regular when it is a 2-$(n, 3, \lambda)$ design.

From a given graph we are able to construct, by switching, others which yield the same two-graph. [CvL, pg 59]. Such graphs are said to belong to the switching class of the two-graph. [BrCoNe, pg 15], [CvL, Thm. 4.5]. Within this switching class lies a unique graph which has one isolated vertex. [BrCoNe, pg 373], [CvL, pg 62]. In the case where the two-graph is regular, deletion of this isolated vertex will result in either a complete graph, or a strongly regular graph with parameters $(n, 2f, e, f)$. [BrCoNe, Prop 1.5.1], [CvL, Thm. 4.11]. Both the Schläfli and the McLaughlin graph arise in this way. It follows from this that if we wish to obtain a picture of the McLaughlin graph we may do so by examining the appropriate regular two-graph. The McLaughlin graph has 275 vertices arising from Higman's regular two-graph on 276 vertices. (See also [GoeSei] and [Taylor].)

Example 1.26 We follow the construction of Higman's regular two-graph given in [CvL, Ex 4.15], (see also [BrCoNe, Chap. 11, H]).

We start with the Steiner system $S(4, 7, 23)$ which has point set V and block set $\mathcal{B}$. The number of blocks is $\frac{^{23}C_4}{^{7}C_4} = 253$ and the cardinality of V is 23. It can be shown that any two blocks are incident in either one or three points.

Let G be a graph with vertex set $V \cup \mathcal{B}$ so that $|V(G)| = 276$ and define the edges of G as follows:

(i) all points in V are pairwise adjacent so that the subgraph induced by V is the complete graph K_{23}.

(ii) a point $u \in V$ is adjacent to a block B if and only if $u \in B$.

(iii) two blocks B_1 and B_2 are adjacent if and only if they have three points in common.

Then G yields Higman's regular two-graph design on 276 points.

The graph G is switching equivalent to the graph McL $\cup K_1$, and so to obtain the McLaughlin we must switch with repect to the neighbourhood of a vertex, thus isolating it, and then delete that vertex.

The vertices of G are the 23 points and 253 blocks of $S(4, 7, 23)$. We fix a vertex u in V and denote the neighbourhood of u by $\Delta(u)$. The number of points in $\Delta(u)$ is 22. The number of blocks in $\Delta(u)$ is b_1, the number of blocks having one point in common. In $S(4, 7, 23)$,

$$b_s = 253 \frac{{}^7C_s}{{}^{23}C_s}$$

and so $b_1 = 77$, leaving 176 blocks outside $\Delta(u)$.

Let x be another vertex in V. Then x is adjacent to u and is adjacent to 21 points in $\Delta(u)$. The number of blocks adjacent to both u and x is b_2, that is 21, leaving x adjacent to 56 blocks outside $\Delta(u)$.

Let B be any block. For a given 3-set the number of adjacent blocks is b_3. The number of ways of choosing such a 3-set is ${}^7C_3$ and so the total number of blocks adjacent to B is $(b_3 - 1)^7C_3 = 140$.

Let B_u be a block within $\Delta(u)$. Then B_u is adjacent to the point u and to 6 other points in $\Delta(u)$. The number of blocks which are adjacent to B_u and also within $\Delta(u)$ must have the point u in common and so the number of ways of choosing the remaining two elements of the 3-set is ${}^6C_2$.

Therefore B_u is adjacent to $(b_3 - 1)^6C_2 = 60$ blocks within $\Delta(u)$ and 80 blocks outside $\Delta(u)$.

Let $B_{\bar{u}}$ be a block outside $\Delta(u)$. Then $B_{\bar{u}}$ is adjacent to 7 points in $\Delta(u)$. Counting the edges between the blocks within $\Delta(u)$ and those outside $\Delta(u)$ we have $77 \times 80 = 176 \times k$ where k is the number of blocks within $\Delta(u)$ which are adjacent to $B_{\bar{u}}$. Now $k = 35$ and so $B_{\bar{u}}$ is adjacent to 105 outside $\Delta(u)$. This information is summarised in Figure 1.1.

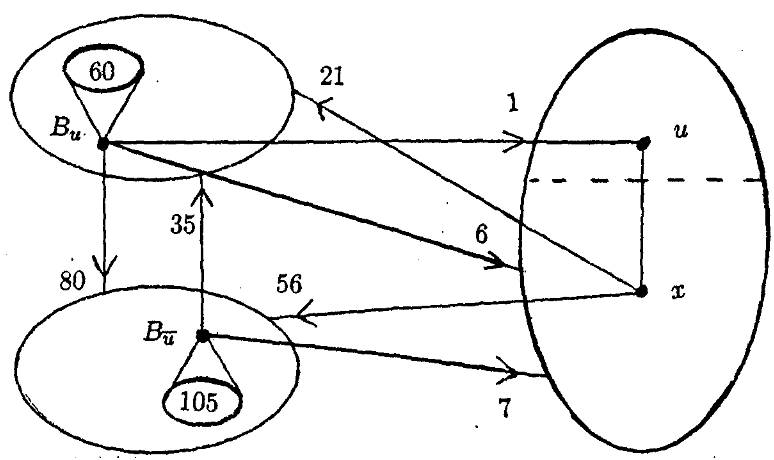


Figure 1.1: Before switching.

We now switch with respect to $\Delta(u)$. The block B_u is still adjacent to 60 blocks in $\Delta(u)$ but is now adjacent to $176 - 80 = 96$ other blocks; it is adjacent to 6 points and so has degree 162. The block $B_{\bar{u}}$ is now adjacent to $77-35=42$ blocks in $\Delta(u)$ but still adjacent to 105 blocks outside $\Delta(u)$; it is adjacent to $22-7=15$ points and so has degree 162. The point x is still adjacent to 21 blocks in $\Delta(u)$ but now it is adjacent to $176-56=120$ blocks outside $\Delta(u)$; it is adjacent to 21 points and so has degree 162. If we now delete the vertex u we have McL_{162}. Taking the complement of this we get McL_{112}. The block B_u is then adjacent to $22-6=16$ points and so the subgraph induced by $B_u \cup V$ is $K_{1,16} \cup 6K_1$. The relevance of this construction will become apparent in Chapter 7.

Chapter 2

Star sets and star partitions.

Many of the results in this chapter can be found in a slightly different form in [CvRoS, Chapter 7] where a star set is regarded as one star cell of a star partition (we define these terms later). Here we focus on a single star set corresponding to a particular eigenvalue of G.

There are several results arising from the spectral decomposition of A_G. We have already established that the matrix P_i represents the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu_i)$ and we shall see now the relevance of this in the derivation of the name *star set.*

Let μ be any distinct eigenvalue of G with multiplicity $m_\mu(G)$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard orthonormal basis for $\mathbb{R}^n$: then the vectors $P\mathbf{e}_1, \ldots, P\mathbf{e}_n$ form a eutactic star (hence the name) in $\mathcal{E}(\mu)$. In particular,

$\mathcal{E}(\mu)$ is spanned by these vectors since the map $P : I\!R^n \mapsto \mathcal{E}(\mu)$ is surjective, and so we can always find a basis for $\mathcal{E}(\mu)$ of the form $P\mathbf{e}_j$ $(j \in X)$ where X is a subset of the vertex set of G with cardinality $m_\mu(G)$. This subset X is called a *star set* for the eigenvalue μ. In general we find that more than one subset of $V(G)$ can be a star set for μ. Before expounding on the subject of star sets we return to the spectral decomposition of A_G, in particular the vectors $P_i\mathbf{e}_j$. The relevance of this will become apparent later.

The vectors $P_i\mathbf{e}_j$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ are pertinent to the search for algebraic invariants. A graph angle is the cosine of the acute angle between a co-ordinate axis and an eigenspace. Thus the angles of a graph are simply the numbers $||P_i\mathbf{e}_j||$ $\quad (i = 1, \ldots, m, \quad j = 1, \ldots, n)$. The corresponding angle matrix is the $m \times n$ matrix with (i, j) entry $\alpha_{ij} = ||P_i\mathbf{e}_j||$. A natural ordering for the rows and columns of this matrix is obtained by ordering the m distinct eigenvalues so that $\mu_1 > \mu_2 > \ldots > \mu_m$ for the rows; the columns are ordered lexicographically. The matrix is then an algebraic invariant. Graph angles were first introduced in [Cvet2]; additional material on graph angles may be found in [CvRo].

Graph angles can also be used in determining the characteristic polynomial of a vertex-deleted subgraph $G - j$; indeed knowledge of the spectrum of G and of the angles $\alpha_{1j}, \ldots, \alpha_{mj}$ is equivalent to the knowledge of the spectra of G and of $G-j$. This is because the characteristic polynomial of a vertex-deleted subgraph is

$$\Phi_{G-j}(x) = \Phi_G(x) \sum_{i=1}^{m} \frac{\alpha_{ij}^2}{(x-\mu_i)}$$

where j is the deleted vertex.

To see how this expression is obtained we consider the spectral decomposition of A_G. We have $A_G = \mu_1 P_1 + \mu_2 P_2 + \ldots + \mu_m P_m$ and $I = P_1 + P_2 + \ldots + P_m$. Thus $xI - A_G = (x-\mu_1)P_1 + \ldots + (x-\mu_m)P_m$. Now $P_i^2 = P_i = P_i^T$ and $P_i P_j = O$ $(i \neq j)$ (see Remark 1.8) and so $(xI - A_G)^{-1} = (x-\mu_1)^{-1}P_1 + \ldots + (x-\mu_m)^{-1}P_m$. From Remark 1.3 we have $\operatorname{adj}(xI - A_G) = \det(xI - A_G) \times (xI - A_G)^{-1}$, that is

$$\operatorname{adj}(xI - A_G) = \Phi_G(x) \times \sum_{i=1}^{m} (x-\mu_i)^{-1} P_i. \tag{2.1}$$

Now the (j,j) entry of P_i is

$$\begin{aligned} \mathbf{e}_j^T P_i \mathbf{e}_j &= \mathbf{e}_j^T P_i P_i \mathbf{e}_j \\ &= (P_i \mathbf{e}_j)^T P_i \mathbf{e}_j \\ &= ||P_i \mathbf{e}_j||^2. \end{aligned}$$

and so the (j,j) entry of Equation (2.1) is

$$(-1)^{j+j} \det(xI - A_{G-j}) = \Phi_G(x) \times \sum_{i=1}^{m} \frac{||P_i \mathbf{e}_j||^2}{(x-\mu_i)},$$

that is

$$\begin{aligned}\Phi_{G-j}(x) &= (x-\mu_1)^{k_1}(x-\mu_2)^{k_2}\dots(x-\mu_m)^{k_m}\times \\ &\left(\frac{\alpha_{1j}^2}{(x-\mu_1)}+\frac{\alpha_{2j}^2}{(x-\mu_2)}+\dots+\frac{\alpha_{mj}^2}{(x-\mu_m)}\right).\end{aligned}$$

If $\alpha_{ij} = 0$ for some j then $(x-\mu_i)^{k_i}$ is a factor of $\Phi_{G-j}(x)$ and so μ_i has multiplicity $\geq k_i$ in $G-j$. For example, the graph in Figure 2.1 has eigenvalues $3.3234, 0.3579, -1, -1, -1.6813$.

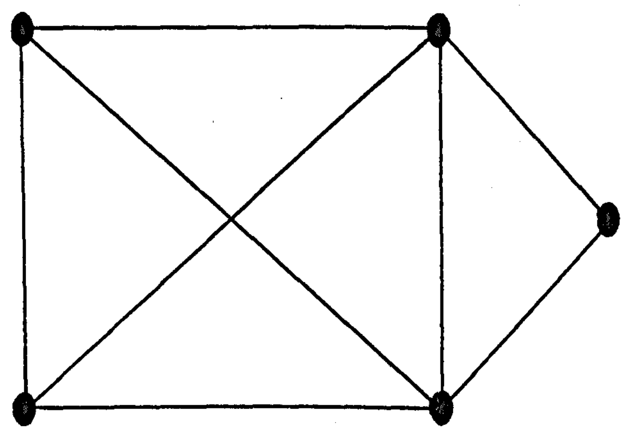


Figure 2.1

Deletion of the vertex of degree two results in the complete graph K_4 which has eigenvalues $3, -1, -1, -1$. Thus in this case the multiplicity of the eigenvalue -1 has been increased by one on deletion of a single vertex.

By Remark 1.20 we know that if μ is an eigenvalue of G then $m_\mu(G-j)$ differs from $m_\mu(G)$ by at most one and so we have $m_{\mu_i}(G-j) \leq k_i+1$. Most importantly, the matrix P_i is non-zero and so we have $\alpha_{ij} \neq 0$ for some j, with the result that $(x-\mu_i)^{k_i}$ is not a factor of $\Phi_{G-j}(x)$ for this j. Thus, having chosen a particular eigenvalue μ with multiplicity k, we can always find a vertex j such that the multiplicity of μ is reduced by one. Deleting an appropriate succession of k vertices we will eventually obtain a subgraph which does not have μ as an eigenvalue. The subset of k deleted vertices is called a *polynomial set* for μ. A formal definition follows.

Definition 2.1 Let μ be an eigenvalue of G with $m_\mu(G) = m$, and let X be a subset of $V(G)$. Then X is a *polynomial set* for μ if and only if $|X| = m$ and $G - X$ does not have μ as an eigenvalue.

For a given graph G with n vertices it is possible to find a polynomial set for a specific eigenvalue μ with $m_\mu(G) = m$ by deleting successive vertices chosen in such a way that with each successive graph the multiplicity of μ is reduced by one. Another approach would be to find an induced subgraph of G with $n-m$ vertices which did not have μ as an eigenvalue and to delete these vertices. The remaining vertices would then be a polynomial set for μ.

We will now consider a formal definition of a star set.

Definition 2.2 Let μ be an eigenvalue of G with $m_\mu(G) = m$, and let X be a subset of $V(G)$. Let P be the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$. Then X is a *star set* for μ if and only if $\{P\mathbf{e}_j : j \in X\}$ is a basis for $\mathcal{E}(\mu)$.

A method for finding a star set is not immediately apparent from this definition. Fortunately further consideration of this problem is unnecessary since the following theorem will show us that a star set is precisely a polynomial set.

Theorem 2.3 [Row7, Thm 6.2] *Let μ be an eigenvalue of the n-vertex graph G with multiplicity $m_\mu(G)$ and let X be a subset of the vertex set of G. Then the following are equivalent:*

(i) $\{P\mathbf{e}_j : j \in X\}$ *is a basis for* $\mathcal{E}(\mu)$,

(ii) $\mathbb{R}^n = \mathcal{E}(\mu) \oplus \mathcal{V}$ *where* $\mathcal{V} = \langle \mathbf{e}_j : j \notin X \rangle$,

(iii) $|X| = m_\mu(G)$ *and μ is not an eigenvalue of $G - X$.*

Proof. *(i)* $\Rightarrow$ *(ii)*. To show that $\mathcal{E}(\mu) \oplus \mathcal{V} = \mathbb{R}^n$ it is sufficient to show that if $\mathbf{x} \in \mathcal{E}(\mu) \cap \mathcal{V}$ then $\mathbf{x} = \mathbf{0}$. Suppose $\mathbf{x} \in \mathcal{E}(\mu) \cap \mathcal{V}$. Then $\mathbf{x} = P\mathbf{x}$ since $\mathbf{x} \in \mathcal{E}(\mu)$. Also, since $\mathbf{x} \in \mathcal{V}$ we have $\mathbf{x}^T\mathbf{e}_j = 0$ for all $j \in X$. Hence $(P\mathbf{x})^T\mathbf{e}_j = 0$ for all $j \in X$; that is $\mathbf{x}^T P\mathbf{e}_j = 0$ for all $j \in X$. Now $\{P\mathbf{e}_j : j \in X\}$ is a basis for $\mathcal{E}(\mu)$ and so $\mathbf{x} \in \mathcal{E}(\mu)^\perp$. Therefore $\mathbf{x} = \mathbf{0}$.

(ii) $\Rightarrow$ *(iii)*. Firstly $|X| = m_\mu(G)$ since dim $\mathcal{V} = n - m_\mu(G)$. Secondly,

suppose that μ is an eigenvalue of $G - X$. Then there exists a non-zero $\mathbf{x}$ such that $C\mathbf{x} = \mu\mathbf{x}$ where C is the adjacency matrix of $G - X$. Let $\mathbf{x}' = \begin{pmatrix} \mathbf{0} \\ \mathbf{x} \end{pmatrix} \in I\!R^n$, and write $A_G = \begin{pmatrix} * & * \\ * & C \end{pmatrix}$. Then

$$A_G\mathbf{x}' = \begin{pmatrix} * & * \\ * & C \end{pmatrix}\begin{pmatrix} \mathbf{0} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} * \\ C\mathbf{x} \end{pmatrix} = \begin{pmatrix} * \\ \mu\mathbf{x} \end{pmatrix}.$$

Now $\mathcal{V} = \langle \mathbf{e}_j : j \notin X \rangle$ and so for every $\mathbf{v} \in \mathcal{V}$ we have

$$\mathbf{v}^T A_G\mathbf{x}' = \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}^T \begin{pmatrix} * & * \\ * & C \end{pmatrix}\begin{pmatrix} \mathbf{0} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}^T \begin{pmatrix} * \\ \mu\mathbf{x} \end{pmatrix}.$$

Thus $\mathbf{v}^T A_G\mathbf{x}' = \mathbf{w}^T\mu\mathbf{x}$. Now $\mathbf{v}^T\mu\mathbf{x}' = \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}^T \mu \begin{pmatrix} \mathbf{0} \\ \mathbf{x} \end{pmatrix} = \mathbf{w}^T\mu\mathbf{x}$ and so $\mathbf{v}^T A_G\mathbf{x}' = \mathbf{v}^T\mu\mathbf{x}'$. Hence $\mathbf{v}^T(A_G - \mu I)\mathbf{x}' = \mathbf{0}$ for all $\mathbf{v} \in \mathcal{V}$ and so $(A_G - \mu I)\mathbf{x}' \in \mathcal{V}^\perp$.

Similarly, if $\mathbf{u} \in \mathcal{E}(\mu)$ we have $\mathbf{u}^T A_G\mathbf{x}' = (A_G\mathbf{u})^T\mathbf{x}' = (\mu\mathbf{u})^T\mathbf{x}'$, since $A_G\mathbf{u} = \mu\mathbf{u}$. Hence $\mathbf{u}^T A_G\mathbf{x}' = \mathbf{u}^T\mu\mathbf{x}'$ and we have $\mathbf{u}^T(A_G - \mu I)\mathbf{x}' = \mathbf{0}$. It follows that $(A_G - \mu I)\mathbf{x}' \in \mathcal{E}(\mu)^\perp$ and so $(A_G - \mu I)\mathbf{x}' \in \mathcal{V}^\perp \cap \mathcal{E}(\mu)^\perp = (\mathcal{E}(\mu) \oplus \mathcal{V})^\perp$. But $(\mathcal{E}(\mu) \oplus \mathcal{V}) = I\!R^n$. Hence $\mathbf{x}' \in \mathcal{E}(\mu) \cap \mathcal{V} = \{\mathbf{0}\}$. Therefore $\mathbf{x}' = \mathbf{0}$ and so $\mathbf{x} = \mathbf{0}$. Thus μ is not an eigenvalue of $G - X$.

(iii) $\Rightarrow$ *(i)*. Suppose by way of contradiction that $\{P\mathbf{e}_j : j \in X\}$ is not a basis for $\mathcal{E}(\mu)$. Then $\langle P\mathbf{e}_j : j \in X \rangle \subset \mathcal{E}(\mu)$ and there exists a non-zero $\mathbf{x} \in \mathcal{E}(\mu) \cap \langle P\mathbf{e}_j : j \in X \rangle^\perp$. Since $\mathbf{x} \in \langle P\mathbf{e}_j : j \in X \rangle^\perp$ then $\mathbf{x}^T P\mathbf{e}_j = 0$ for all $j \in X$. Since $\mathbf{x} \in \mathcal{E}(\mu)$ we have $\mathbf{x} = P\mathbf{x}$ and so $\mathbf{x}^T P\mathbf{e}_j = (P\mathbf{x})^T\mathbf{e}_j = \mathbf{x}^T\mathbf{e}_j = 0$ for all $j \in X$. It follows that $\mathbf{x} \in \mathcal{V}$ where $\mathcal{V} = \langle \mathbf{e}_j : j \notin X \rangle$ and so $\mathbf{x}$ is of

the form $\begin{pmatrix} \mathbf{0} \\ \mathbf{x}' \end{pmatrix}$. Furthermore $A_G\mathbf{x} = \mu\mathbf{x}$ and so $\begin{pmatrix} * & * \\ * & C \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{x}' \end{pmatrix} = \mu \begin{pmatrix} \mathbf{0} \\ \mathbf{x}' \end{pmatrix}$, that is, $C\mathbf{x}' = \mu\mathbf{x}'$. But μ is not an eigenvalue of $G - X$ and so $\mathbf{x}' = \mathbf{0}$ which implies that $\mathbf{x} = \mathbf{0}$, a contradiction.

□

The equivalence of a polynomial set and a star set is of much practical use, as we demonstrate in the following remark.

Remark 2.4 Let X be a star set for μ in G with $m_\mu(G) = m$. Deletion of k vertices from the star set X will result in a graph which has μ as an eigenvalue of multiplicity $m - k$.

This follows directly from the construction of a polynomial set.

Theorem 2.3 is an amalgamation of [CvRoS, Thms 7.2.2, 7.2.3 and 7.2.6] where it is couched in terms of star partitions. Later chapters deal exclusively with star sets but for the remainder of this chapter and the following chapter we will pursue the idea of star partitions, concluding with some examples.

Definition 2.5 Let μ_i $(i = 1 \ldots m)$ be the distinct eigenvalues of G. A *star partition* of a graph G is a partition of $V(G)$ such that $V(G) = X_1 \dot{\cup} X_2 \dot{\cup} \ldots \dot{\cup} X_m$, where X_i is a star set for μ_i $(i = 1, \ldots, m)$.

The subsets of a star partition are sometimes called star cells, each star cell being a star set for a particular eigenvalue.

We have shown that we can always find a star set for an eigenvalue μ of G. Theorem 2.7 will show us that a star set can always be extended to a star partition. But first we must show that every graph has a star partition. The proof of this uses the Laplacian expansion of a determinant (Remark 1.4.)

Theorem 2.6 [CvRoS1] *Every graph has a star partition.*

Proof. Let G be a graph with adjacency matrix A_G. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard orthonormal basis for $\mathbb{R}^n$ and let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for $\mathbb{R}^n$ composed of orthonormal eigenvectors of A_G. Let $\mathbf{e}_j = \sum_{h=1}^{n} t_{hj}\mathbf{x}_h$, so that (t_{hj}) is the transition matrix from $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ to $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. Let R_i be the fixed subset of the rows of (t_{hj}) such that $\{\mathbf{x}_h : h \in R_i\}$ is an orthonormal basis for $\mathcal{E}(\mu_i)$. Then, since P_i is the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu_i)$ we have

$$P_i\mathbf{e}_j = \sum_{h \in R_i} t_{hj}\mathbf{x}_h.$$

Note that $|R_i| = m_{\mu_i}(G)$. Since (t_{hj}) is invertible we can choose a subset X_i of the columns in such a way that the matrices (t_{hj}) $(h \in R_i,\ j \in X_i)$ and (t_{hj}) $(h \notin R_i,\ j \notin X_i)$ are both invertible. Thus the vectors $P_i\mathbf{e}_j$ $(j \in X_i)$ are linearly independent and so form a basis for $\mathcal{E}(\mu_i)$. Therefore X_i is a star set for μ_i.

Since X_i has been chosen so that the matrix (t_{hj}) $(h \notin R_i,\ j \notin X_i)$ is invertible we can repeat the procedure on *this* invertible matrix. Hence, for fixed $R_1 \ldots R_m$, we can choose $X_1 \ldots X_m$ appropriately so that $\{P_i \mathbf{e}_j : j \in X_i\}$ is a basis for $\mathcal{E}(\mu_i)$ $(i = 1 \ldots m)$, and we have a star partition for G.

□

Theorem 2.7 [Row1, Thm 1.2] *Let X be a star set for μ_i in G. Then there exists a star partition $X_1 \dot\cup X_2 \dot\cup \ldots \dot\cup X_m$ such that $X_i = X$.*

Proof. To prove this, it is sufficient to show that, for fixed R, if X is a star set for $\mu = \mu_i$ then the matrix (t_{hj}) $(\ j \notin X,\ h \notin R)$ is always invertible. Let $m_\mu(G) = m$ so that $|X| = m$.

Let $\mathcal{V} = \langle \mathbf{e}_j : j \notin X \rangle$ and consider the map

$$\begin{aligned}(I-P)|_{\mathcal{V}}\ :\ \mathbf{e}_j &\mapsto \mathbf{e}_j - P\mathbf{e}_j \quad (j \notin X)\\ &= \sum_{h=1}^{n} t_{hj}\mathbf{x}_h - \sum_{h \in R} t_{hj}\mathbf{x}_h \quad (j \notin X)\\ &= \sum_{h \notin R} t_{hj}\mathbf{x}_h \quad (j \notin X)\end{aligned}$$

which is a map from $\mathcal{V}$ to $\mathcal{E}(\mu)^\perp$. Thus (t_{hj}) $(h \notin R,\ j \notin X)$ is the matrix of this map, having size $n - m \times n - m$.

The kernel of this map is :

$$\ker((I-P)|_{\mathcal{V}}) = \ker(I-P) \cap \mathcal{V} = \mathcal{E}(\mu) \cap \mathcal{V}.$$

Let $\mathbf{y} \in \ker((I - P)|_{\mathcal{V}})$. Then $\mathbf{y} = P\mathbf{y}$ since $\mathbf{y} \in \mathcal{E}(\mu)$ and $\mathbf{y}^T\mathbf{e}_j = \mathbf{0}$ for all $j \in X$ since $\mathbf{y} \in \mathcal{V}$. Thus $\mathbf{y}^T P\mathbf{e}_j = \mathbf{0}$ for all $j \in X$.

Therefore $\mathbf{y} \in \mathcal{E}(\mu)^\perp$ since the vectors $P\mathbf{e}_j$ $(j \in X)$ span the eigenspace $\mathcal{E}(\mu)$. However we also have $\mathbf{y} \in \mathcal{E}(\mu)$, and so $\mathbf{y} = \mathbf{0}$. Hence this map has nullity zero, and so the $n-m \times n-m$ matrix which represents it has rank $n-m$. Thus, for fixed R, and given that X is a star set for μ the matrix (t_{hj}) $(j \notin X,\ \ h \notin R)$ is always invertible.

□

It would be appropriate now to consider whether or not, given two disjoint star sets for distinct eigenvalues, we can always extend these to a star partition. We find that this is not the case.

Remark 2.8 If X_1 and X_2 are disjoint star sets for the distinct eigenvalues μ_1 and μ_2 then there is not necessarily a star partition of which both X_1 and X_2 are star cells.

To see why this is so it is sufficient to show that, for fixed R_1 and R_2, if X_1 and X_2 are star sets for μ_1 and μ_2, then the matrix (t_{hj}) $(j \notin X_1 \cup X_2$, $h \notin R_1 \cup R_2)$ is not necessarily invertible.

This matrix represents the map defined by:

$$(I - (P_1 + P_2))|_{\mathcal{V}} : \mathbf{e}_j \mapsto \sum_{h \notin R_1 \cup R_2} t_{hj}\mathbf{x}_h \quad (j \notin X_1 \cup X_2)$$

where $\mathcal{V} = \langle \mathbf{e}_j : j \notin X_1 \cup X_2 \rangle$.

The kernel of this map is $(\mathcal{E}(\mu_1) \oplus \mathcal{E}(\mu_2)) \cap \mathcal{V}$. However, in this case, $\mathbf{y} \in \ker(I-(P_1+P_2))|_{\mathcal{V}}$ does not imply that $\mathbf{y} \in (\mathcal{E}(\mu_1) \oplus \mathcal{E}(\mu_2))^{\perp}$, since the vectors $(P_1+P_2)\mathbf{e}_j \quad (j \in X_1 \cup X_2)$ do not necessarily span $(\mathcal{E}(\mu_1) \oplus \mathcal{E}(\mu_2))$. Thus the matrix in question is invertible if and only if

$$\langle (P_1 + P_2)\mathbf{e}_j : j \in X_1 \cup X_2 \rangle = \mathcal{E}(\mu_1) \oplus \mathcal{E}(\mu_2).$$

For an explicit example see Example 3.3.

One approach to the isomorphism problem led to the development of the theory of star partitions, the idea being to define graphs with n vertices recursively in terms of graphs with fewer than n vertices by way of *canonical star bases*. This is explored in some detail in [CvRoS1]. We give a brief outline of the idea here.

Canonical star bases. Let $X_1 \dot{\cup} X_2 \dot{\cup} \dots \dot{\cup} X_m$ be a star partition for G. The vectors $P_i\mathbf{e}_j \ (j \in X_i)$ form a basis $\mathcal{B}_i$ for $\mathcal{E}(\mu_i)$. Such a basis is known as a *star basis for* $\mathcal{E}(\mu_i)$.

It is always the case that $\bigcup_{i=1}^{m} \mathcal{B}_i$ forms a basis for $I\!R^n$ since $I\!R^n$ is the direct sum of the eigenspaces $\mathcal{E}(\mu_i) \ (i = 1 \dots m)$. Such a basis is known as a *star basis for* $I\!R^n$ *associated with* G.

In general, there are infinitely many bases for $I\!R^n$. However there are only finitely many *star* bases for $\mathbb{R}^n$ since there are only finitely many ways of labelling a graph, and only finitely many star partitions for a given graph. These star bases can be totally ordered in some way [Cvet1] and then an extremal star basis is taken as the *canonical star basis for $\mathbb{R}^n$ associated with G*. It is denoted $\mathcal{B}(G)$.

We state without proof the following theorem from [Cvet1].

Theorem 2.9 *Two graphs G and H are isomorphic if and only if $\mathcal{B}(G) = \mathcal{B}(H)$, and G and H have the same eigenvalues.*

Note that $\mathcal{B}(G)$ is not known to be computable in polynomial time.

Chapter 3

Some examples involving star partitions.

Before we leave the subject of star partitions we give a practical though not very efficient method of finding a star partition of a graph and include some examples including Theorem 3.9 by the author.

Let $\mathcal{X} = X_1 \cup X_2 \cup \ldots \cup X_m$ be a partition of $V(G)$. For $\mathcal{X}$ to be a star partition the following conditions must be met for each subset X_i of the partition.

(i) $|X_i| = k_i$ for some eigenvalue μ_i of multiplicity k_i.

(ii) The eigenvalue μ_i is not an eigenvalue of the graph $G - X_i$.

In order to construct a star partition for a graph G we must first find the spectrum of G, $Sp(G)$. Choose any eigenvalue, say μ_1 with multiplicity k_1 and select one subset X_1 of $V(G)$ such that $|X_1| = k_1$.

Delete these vertices and the edges containing them, and find the spectrum of the resultant graph $G - X_1$. If $\mu_1 \notin Sp(G - X_1)$ then X_1 is a star set for μ_1. By Theorem 2.7 we know that this star set can be extended to a star partition.

Choose another eigenvalue, say μ_2 with multiplicity k_2 and select a subset X_2 from $V(G) \setminus X_1$ such that $|X_2| = k_2$. Again, if $\mu_2 \notin Sp(G - X_2)$ then X_2 is a star set for μ_2.

We now choose a third eigenvalue, say μ_3 with multiplicity k_3 and select a subset X_3 from $V(G) \setminus (X_1 \cup X_2)$ such that $|X_3| = k_3$. If $\mu_3 \notin Sp(G - X_3)$ then X_3 is a star set for μ_3 and we can proceed to the next eigenvalue. However if we are unable to find a star set from the available vertices we must backtrack and make another choice for the second star set.

We perform this procedure for each distinct eigenvalue, backtracking where necessary, until we have a star partition.

Note that if $X_1 \cup \ldots \cup X_m$ and if $\pi \in \mathrm{Aut}(G)$ then $\pi(X_1) \cup \ldots \cup \pi(X_m)$ is also a star partition for G. Star partitions related in this way are called *isomorphic*.

In practice, it may be more convenient to find several star sets for each distinct eigenvalue, and then piece together a star partition. For more efficient methods of finding star partitions see [CvRoS2].

Remark 3.1 In any connected graph G, any vertex v constitutes a star set for the index since the index of $G - v$ is less than the index of G.

Notation 3.2 We sometimes give the spectrum of a graph in the following form:

$$\left\{ \begin{array}{cccc} \mu_1 & \mu_2 & \ldots & \mu_m \\ k_1 & k_2 & \ldots & k_m \end{array} \right\},$$

where μ_i is a distinct eigenvalue of G with multiplicity k_i.

In the following examples a star set corresponding to μ is denoted by $X(\mu)$.

Example 3.3 We find a star partition for the Petersen graph, see (Figure 3.1).

The spectrum for this graph is $\left\{ \begin{array}{ccc} 3, & 1, & -2 \\ 1, & 5, & 4 \end{array} \right\}$. Since the graph is connected, by Remark 3.1, each single vertex is a star set for the eigenvalue 3.

Let $X(3) = \{1\}$. By Theorem 2.7 we know that we can extend this star set to a star partition. Let $Y_1 = \{3,4,5,7\}$. Then Y_1 is a star set for -2 since $-2 \notin Sp(G - Y_1)$. Let the remaining vertices constitute the set Y_2 so that $Y_2 = \{2,6,8,9,10\}$. The graph $G - Y_2$ is a path of length

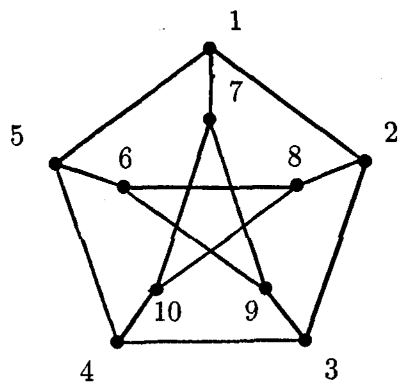


Figure 3.1: The Petersen graph

four which has 1 as an eigenvalue. Therefore Y_2 is not a star set for the eigenvalue 1, and we must consider an alternative star set for the eigenvalue -2. Thus we see explicitly that it is not always possible to extend two star sets to a star partition (see Remark 2.8). Further investigation reveals that $X(3) = \{1\}$, $X(-2) = \{2,3,4,5\}$, $X(1) = \{6,7,8,9,10\}$ is a star partition for the Petersen graph. This demonstrates another property of star partitions; namely that they afford a one-to-one correspondence (in general not unique) between eigenvalues and vertices as illustrated in Figure 3.2.

The Petersen graph has 10 non-isomorphic star partitions, 750 star par-

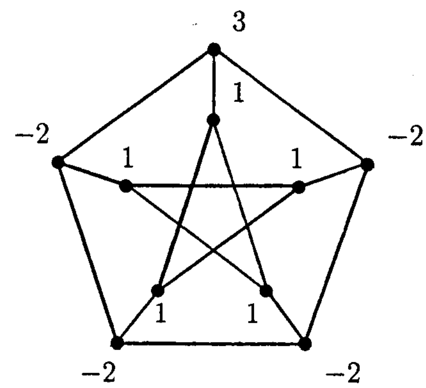


Figure 3.2: A star partition for the Petersen graph

titions altogether.

[CvRoS, Section 7.7].

Example 3.4 Star partitions of a complete graph.

The eigenvalues of the complete graph K_n are $(n-1)$ and -1 with multiplicities 1 and $(n-1)$ respectively. Since K_n is connected, any one vertex can be taken as the star set for the eigenvalue $(n-1)$ This leaves the other $(n-1)$ vertices for the star set corresponding to the eigenvalue -1. See Figure 3.3.

Example 3.5 Star partitions for the complete bipartite graphs $K_{r,s}$.

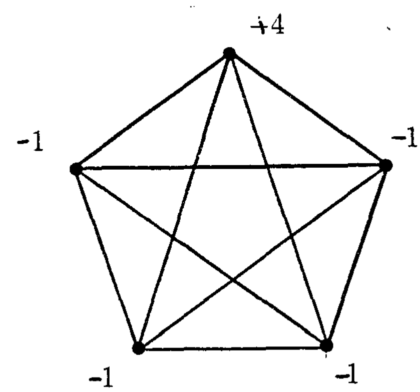


Figure 3.3: A star partition for K_5

This example also illustrates the result given in Remark 3.1. Note that the graph $K_{r,s}$ has $r+s$ vertices: r vertices of degree s and s vertices of degree r. The number of edges is rs. The spectrum of $K_{r,s}$ is

$$\left\{ \begin{array}{ccc} \sqrt{rs} & 0 & -\sqrt{rs} \\ 1 & r+s-2 & 1 \end{array} \right\}.$$

Deletion of any one vertex will result in either $K_{r-1,s}$ or $K_{r,s-1}$ with spectra

$$\left\{ \begin{array}{cc} \pm\sqrt{(r-1)s} & 0 \\ 1 & r+s-3 \end{array} \right\} \text{ and } \left\{ \begin{array}{cc} \pm\sqrt{r(s-1)} & 0 \\ 1 & r+s-3 \end{array} \right\}$$

respectively. Hence any one vertex is a star set for the eigenvalues $\pm\sqrt{rs}$ since they are not in either spectrum.

Note that a graph with no edges has 0 as an eigenvalue. Thus for X to be a star set for the eigenvalue 0, $\overline{X}$ must consist of two adjacent vertices. Hence the star partitions of $K_{r,s}$ are precisely: $X(\sqrt{rs}) = \{v\}$, $X(-\sqrt{rs}) = \{u\}$, $X(0) = V(K_{r,s}) \setminus \{u, v\}$, where $u \sim v$. Figure 3.4 shows such a star partition for $K_{3,5}$.

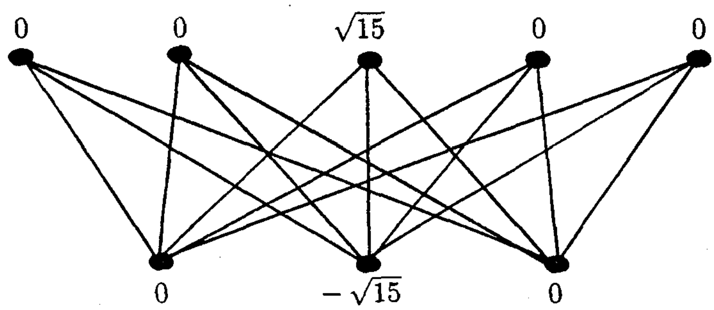


Figure 3.4: A star partition for $K_{3,5}$

Example 3.6 Star partitions of a path.

The eigenvalues of the path P_n are the numbers

$$2\cos\frac{\pi}{n+1}i \quad (i = 1, \ldots .n), \qquad \text{[CvDS, 2.6.7]}$$

all distinct, all of multiplicity one, so all n star sets have but one element.

Deletion of any one vertex j leaves the graph $G - j$ of which each component is a path of length less than n.

Theorem 3.7 [CvRoS2] *The number of star partitions of a graph P_n is $n!$ if and only if $n+1$ is a prime number.*

Proof. Note that the number of ways of arranging n objects is $n!$. Thus the number of star partitions is $n!$ if and only if any vertex is a star set for any eigenvalue; equivalently if and only if no eigenvalue of P_n is also an eigenvalue of P_m where $m < n$. This is the case if and only if $2\cos\frac{\pi}{n+1}i \neq 2\cos\frac{\pi}{m+1}j$ for all i, j, m $(1 \leq m < n;\ i = 1 \ldots n;\ j = 1 \ldots m)$. This is possible only when $(n+1)$ is prime.

□

Since $|\mathrm{Aut}(P_n)| = 2$ if $n > 1$ it follows from Theorem 3.7 that if $n > 1$ then the number of non-isomorphic star partitions of P_n is $\frac{1}{2}(n!)$ [CvRoS, Remark 7.7.4].

Example 3.8 Star partitions of a cycle.

The eigenvalues of a cycle C_n are the numbers $2\cos\frac{2\pi}{n}i$ $(i = 1 \ldots n)$. [CvRoS, Section 2.1] Note that C_n always has 2 as an eigenvalue with multiplicity 1. Further note that if n is even then C_n also has -2 as a simple eigenvalue . All other eigenvalues have multiplicity 2. Deletion of any one or two vertices leaves a subgraph whose components are paths P_m where $m < n$. Thus any single vertex can be a star set for the eigenvalues 2 or -2

since $\pm 2 \notin Sp(P_m)$. For example, the graph C_6 has spectrum

$$\left\{ \begin{array}{cccc} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{array} \right\}.$$

A star partition for this graph is given in Figure 3.5.

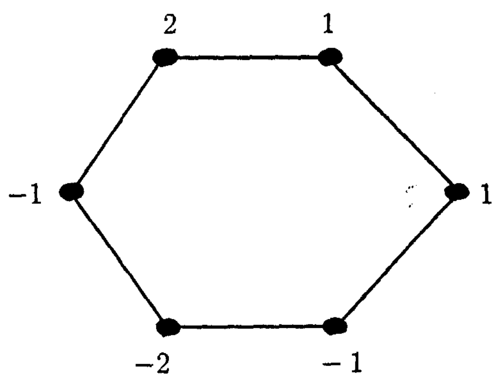


Figure 3.5: A star partition for C_6

As with paths, it is possible to evaluate the number of star partitions for a cycle C_n provided n is prime. In order to do this we need to know the number of ways of arranging $\frac{(n-1)}{2}$ pairs of objects together with one single object in a circle and for that we are indebted to P. Rowlinson. Let $n = 2k+1$. Suppose that the single object is placed at a point S on the circle. Then the required number is $\frac{1}{2}\{\frac{(2k)!}{2^k k!} - f(k)\} + f(k)$ where $f(k)$ is the number of pairings symmetric about the diameter through S. Here $\sum_k \frac{f(k)}{k!} x^k = e^{x+x^2}$, which follows from the recurrence relation:

$f(k) = f(k-1) + 2(k-1)f(k-2)$, where $f(1) = 1$ and $f(2) = 3$.

Theorem 3.9 *Let $n = 2k + 1$. Then the number of non-isomorphic star partitions of a cycle C_n is*

$$\frac{1}{2}\left\{\frac{(2k)!}{2^k} + k!f(k)\right\}$$

if and only if n is prime.

Proof. We have already shown that any vertex is a star set for the eigenvalue 2. The next step is to show that any two vertices constitute a star set for any eigenvalue of C_n other than 2, if and only if n is prime.

Note that these eigenvalues are the numbers

$$2\cos\frac{2\pi}{n}i \quad (i = 1\ldots n-1).$$

Deletion of any two vertices leaves a subgraph whose components are paths of length m where $m < n-1$, and so the eigenvalues of $C_n - X$ where X is a star set such that $|X| = 2$ are of the form

$$2\cos\frac{\pi}{m+1}j \quad (1 \leq m < n-1,\ j = 1\ldots m)$$

Hence it is sufficient to show that n is prime if and only if $2\cos\frac{2\pi}{n}i \neq 2\cos\frac{\pi}{m+1}j$ for all i, j, m $\quad (1 \leq m < n-1;\ i = 1\ldots n-1;\ j = 1\ldots m)$.

Suppose by way of contradiction that n is prime and that the equality holds. Then

$$2i(m+1) = nj \quad (1 \leq m < n-1; i < n; j < m+1).$$

Since n is prime we have $n \mid i$ or $n \mid (m+1)$. Now n does not divide i since $i < n$; therefore $n \mid (m+1)$. However $m < n-1$; a contradiction. If n is not prime then the equation holds for some i, j, m; for example when $n = 4$ the equation holds for $i = j = m = 1$.

Therefore, we may conclude that no eigenvalue of C_n of multiplicity 2 is an eigenvalue of any $C_n - X$ ($|X| = 2$) if and only if n is prime. Hence any 2-set taken from $V(C_n)$ is a star set for any eigenvalue of C_n other than 2, if and only if n is prime.

Now write $n = 2k+1$. The number of feasible partitions of C_{2k+1} is $\frac{1}{2}\{\frac{(2k)!}{2^k k!} + f(k)\}$. For each of these partitions there are $k!$ ways of assigning the k double eigenvalues to the 2-sets, and so the number of non-isomorphic star partitions of C_{2k+1} is

$$k! \times \frac{1}{2}\left\{\frac{(2k)!}{2^k k!} + f(k)\right\} = \frac{1}{2}\left\{\frac{(2k)!}{2^k} + k! f(k)\right\}.$$

□

Example 3.10 Star partitions for C_7.

Here $2k+1 = 7$ and so $k = 3$. From the recurrence relation we have $f(3) = f(2) + 4f(1) = 7$ and so the number of non-isomorphic star partitions is $\frac{1}{2}\{\frac{6!}{2^3} + 3! f(3)\} = 66$. The corresponding partitions of C_7 are shown in Figure 3.6; the vertices in the star sets are joined by dotted lines.

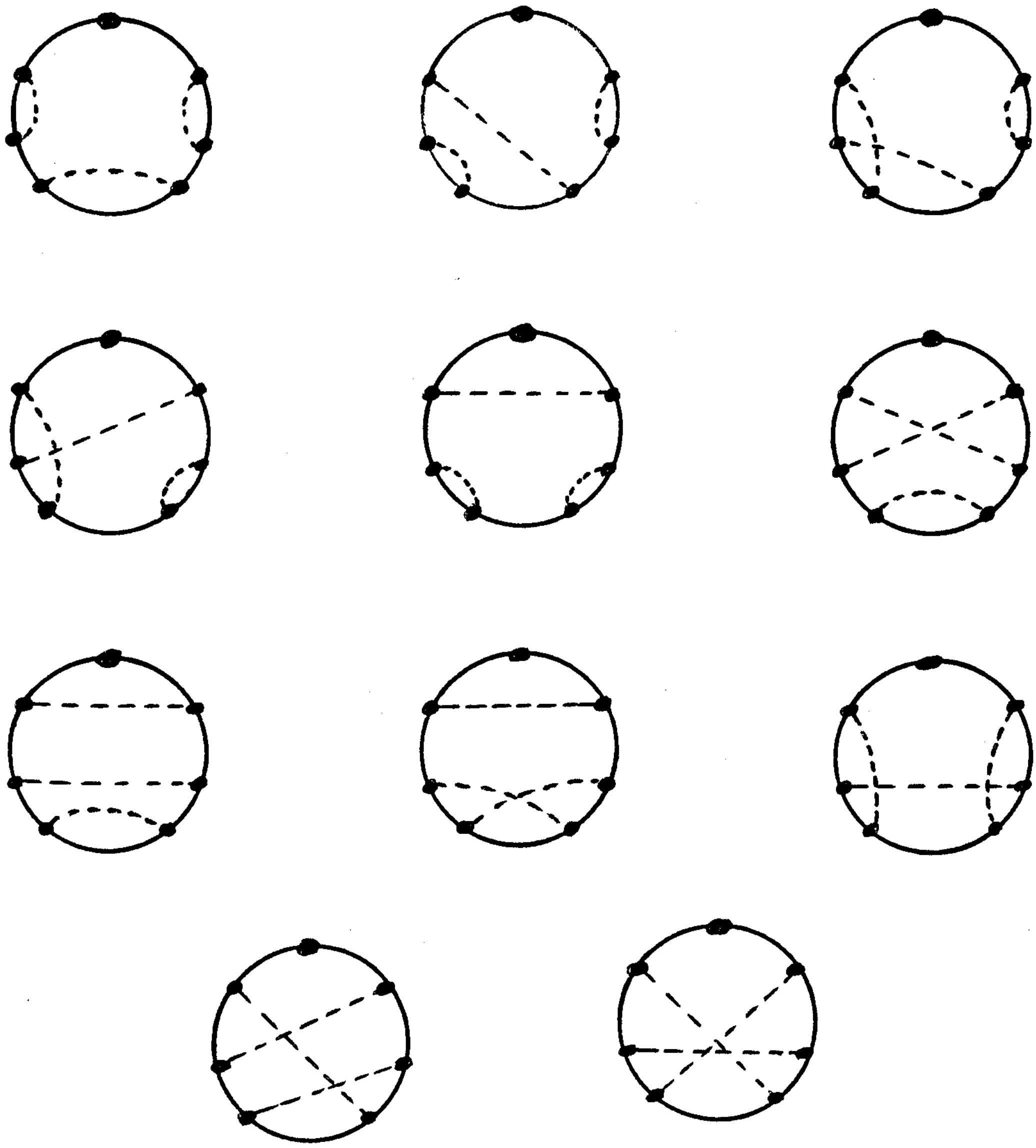


Figure 3.6: The types of star partition for C_7.

The number of non-isomorphic star partitions for C_{11} is

$$\frac{1}{2}\{\frac{10!}{2^5} + 5!f(5)\} = 61560.$$

We make no attempt to list the 513 corresponding partitions of C_{11}.

For further results on the number of star partitions for particular graphs see [CvRoS, Chap. 7].

So far we have given results concerning specific graphs. The final result is more general in that it pertains to almost all regular graphs.

Theorem 3.11 [CvRoS, Thm 7.5.3] *Let G be regular connected graph such that $\overline{G}$ is also connected. Given that $X_1\dot{\cup}X_2\dot{\cup}\ldots\dot{\cup}X_m$ is a star partition for G, it is also a star partition for $\overline{G}$.*

Proof. The theorem follows from the observation that if A_G has spectral decomposition $A_G = \mu_1 P_1 + \mu_2 P_2 + \ldots + \mu_m P_m$ $(\mu_1 > \ldots > \mu_m)$ then by Remarks 1.14 and 1.15, $\overline{A}_G$ has spectral decomposition

$$J - A_G - I = (n - \mu_1 - 1)P_1 + (-\mu_2 - 1)P_2 + \ldots + (-\mu_m - 1)P_m.$$

$\square$

An important consequence of this proof is that in the situation of Theorem 3.11, X is a star set for μ in G if and only if it is a star set for $-\mu-1$ in $\overline{G}$. For example a 7-element subset X of vertices in Sch_{16} is a star set for -2

if and only if it is a star set for 1 in Sch_{10}. One possible set induces $K_2 \cup K_5$ in Sch_{16}, hence $K_{2,5}$ in Sch_{10}. In showing how to construct all the graphs (including the Schläfli graph) with the prescribed star complement H we chose $H \cong K_{2,5}$ as our example in Chapter 9 on the grounds that a connected graph always has a connected star complement for each eigenvalue. (This last result is ascribed to S. Penrice in [Ell].)

Chapter 4

Star sets and the structure of graphs.

Star sets were first investigated as star cells of a star partition but it was soon realised that they were of interest in their own right. In this chapter we explore the relationship between a star set X (for some eigenvalue μ) and the structure of graphs. In particular we exploit the linear independence of the vectors $P\mathbf{e}_j$ ($j \in X$), but first a few preliminaries.

Notation 4.1 Let X be a star set for the eigenvalue μ of the graph G. Given a vertex u of G, the set of vertices adjacent to u is called the neighbourhood of u, and this is denoted by $\Delta(u)$. We define the *closed neighourhood* of u to be the set of vertices $\Delta(u) \cup \{u\}$.

The set $\Delta(u) \cap X$ is known as the X-neighbourhood of u and is denoted by $\Gamma(u)$. Recall that $\overline{X}$ denotes the complement of X in $V(G)$. The $\overline{X}$-neighbourhood of u is given by $\Delta(u) \cap \overline{X}$ and is denoted by $\overline{\Gamma}(u)$.

The edge set between two vertex sets X and Y is denoted by $E(X, Y)$. Since it is often useful to count the edges in two ways, we will take the order to correspond to direction. Clearly $E(X, Y) = E(Y, X)$.

As before, a vertex set is said to be independent if all the vertices therein are pairwise non-adjacent.

Definition 4.2 [Ore] Let Y be a subset of the vertex set $V(G)$. Then Y is a *dominating set* if each vertex outwith Y is adjacent to a vertex within Y.

Definition 4.3 [Slat] Let Y be a dominating set in G. Then Y is a *location-dominating set* if different vertices outwith Y have different neighbourhoods within Y.

We shall now state certain properties of the vectors $P\mathbf{e}_j$ $(j \in X)$ which are useful in the following proofs.

Remark 4.4 Let X be a star set for μ. By definition the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathcal{E}(\mu)$ and so are linearly independent. Furthermore, since $A_G\mathbf{e}_j = \sum_{k\sim j} \mathbf{e}_k$ we have $\mu P\mathbf{e}_j = \sum_{k\sim j} P\mathbf{e}_k$. Lastly, if μ is not a main eigenvalue (Remark 1.16) then $P\mathbf{j} = \mathbf{0}$ where $\mathbf{j}$ is the all-one vector and so we may write $\sum_{u \in X} P\mathbf{e}_u + \sum_{v \in \overline{X}} P\mathbf{e}_v = \mathbf{0}$.

Theorem 4.5 [Row5, Prop 4.1] and [Row6]. *Let G be a graph without isolated vertices. Let X be a star set for the eigenvalue μ. Then $\overline{X}$ is a dominating set.*

Proof. Let X be a star set for $\mu \neq 0$ and suppose by way of contradiction that $\overline{X}$ is not a dominating set. Then there exists a vertex $u \in X$ such that $\overline{\Gamma}(u) = \emptyset$. Hence the vectors $P\mathbf{e}_u$ and $P\mathbf{e}_k$ $(k \in \Delta(u))$ are linearly independent. Moreover, by Remark 4.4,

$$\mu P\mathbf{e}_u - \sum_{k \in \Delta(u)} P\mathbf{e}_k = \mathbf{0}.$$

This equation has the solution $\mu = 0$ and $\Delta(u) = \emptyset$, a contradiction. Hence $\overline{X}$ is a dominating set. $\square$

We give an example of an application of this theorem to cubic graphs.

Corollary 4.6 *Let G be a cubic graph and let X be a star set for μ. Then the subgraph induced by X is a union of paths and cycles.*

Proof. The graph G is regular of degree three; in particular it has no isolated vertices. By Theorem 4.5 each vertex in X has at least one neighbour in $\overline{X}$ and at most two neighbours in X. The result follows. $\square$

We shall see examples of this in Chapter 6.

Theorem 4.7 [Row5, Prop 4.2] and [Row6]. *Let X be a star set for μ. If $\mu \notin \{-1, 0\}$, then $\overline{X}$ is a location-dominating set.*

Proof. By the proof of Theorem 4.5, $\overline{X}$ is a dominating set since $\mu \neq 0$. Suppose by way of contradiction that $\overline{X}$ is not a location-dominating set, so that $\overline{\Gamma}(u) = \overline{\Gamma}(v)$ for some $u, v \in X$. From Remark 4.4 we know that $\mu P\mathbf{e}_u = \sum_{k \in \Delta(u)} P\mathbf{e}_k$ and so we have

$$\mu P\mathbf{e}_u = \sum_{k \in \overline{\Gamma}(u)} P\mathbf{e}_k + \sum_{k \in \Gamma(u)} P\mathbf{e}_k.$$

Similarly

$$\mu P\mathbf{e}_v = \sum_{k \in \overline{\Gamma}(v)} P\mathbf{e}_k + \sum_{k \in \Gamma(v)} P\mathbf{e}_k.$$

Subtracting one equation from the other we obtain

$$\mu P\mathbf{e}_u - \mu P\mathbf{e}_v = \sum_{k \in \Gamma(u)} P\mathbf{e}_k - \sum_{k \in \Gamma(v)} P\mathbf{e}_k$$

since $\overline{\Gamma}(u) = \overline{\Gamma}(v)$. Thus

$$\mu P\mathbf{e}_u - \mu P\mathbf{e}_v - \Big(\sum_{k \in \Gamma(u)} P\mathbf{e}_k - \sum_{k \in \Gamma(v)} P\mathbf{e}_k \Big) = 0. \tag{4.1}$$

There are two cases to consider: when $u \not\sim v$ and when $u \sim v$. As before, we exploit the linear independence of the vectors $P\mathbf{e}_j$ $(j \in X)$.

When u is not adjacent to v Equation (4.1) yields the solution $\mu = 0$ and $\Gamma(u) = \Gamma(v)$, a contradiction.

Now suppose that u is adjacent to v. Then $v \in \Gamma(u)$ and $u \in \Gamma(v)$ and we may write Equation (4.1) as follows:

$$\mu P\mathbf{e}_u - \mu P\mathbf{e}_v - ((P\mathbf{e}_v + \sum_{k \in \Gamma(u)\setminus\{v\}} P\mathbf{e}_k) - (P\mathbf{e}_u + \sum_{k \in \Gamma(v)\setminus\{u\}} P\mathbf{e}_k)) = 0.$$

A little re-arranging gives the equation

$$(\mu+1)P\mathbf{e}_u - (\mu+1)P\mathbf{e}_v - (\sum_{k \in \Gamma(u)\setminus\{v\}} P\mathbf{e}_k - \sum_{k \in \Gamma(v)\setminus\{u\}} P\mathbf{e}_k) = 0. \quad (4.2)$$

Again by virtue of linear independence this equation yields the solution $\mu = -1$ and $\Gamma(u) \setminus \{v\} = \Gamma(v) \setminus \{u\}$, a contradiction. The result follows. □

Remark 4.8 With an appropiate ordering we can write the adjacency matrix of a graph $A_G = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ where C is the adjacency matrix of the subgraph $G-X$. In the situation of Theorem 4.7 none of the columns of the matrix B would be the same. In [LiuRo] the idea of a location-dominating set is extended to a *k-location-dominating* set. Briefly, let $A_G^k = \begin{pmatrix} A_k & B_k^T \\ B_k & C_k \end{pmatrix}$. Then $\overline{X}$ is a k-location-dominating set if no two of the columns of B_k are the same.

The reader will recall that the overall objective is to characterize or identify a graph from the subgraph induced by $\overline{X}$ (together with its associated eigenvalue μ if necessary.) An application of Theorem 4.7 demonstrates that the problem of determining all graphs with a subgraph $G-X$ of prescribed size (for $\mu \notin \{-1, 0\}$) is a finite one.

Theorem 4.9 [Row1, Thm. 1.3]. *There are only finitely many graphs with an eigenspace $\mathcal{E}(\mu)$ ($\mu \notin \{-1, 0\}$) of prescribed codimension.*

Proof. Suppose that $\mu \notin \{-1, 0\}$ and suppose that $\mathcal{E}(\mu)$ has codimension t. Then $|\overline{X}| = t$. It follows that there are $2^t - 1$ non-empty subsets of $\overline{X}$. Let u be a vertex in X. There are fewer that 2^t possibilities for the $\overline{X}$-neighbourhood of u and so, since $\overline{X}$ is a location-dominating set, we have $|X| < 2^t$. Hence $|V(G)| < t + 2^t$ and the result follows. □

For $t \leq 4$ we have all possible μ, X and G, thanks to [Bell]. For $t \leq 5$ we refer the reader to [RoBe]. It should be noted that the bound given in Theorem 4.9 has since been improved upon. We state without proof the bound [Row2]:

$$|X| \leq \frac{1}{2}(t-1)(t+4)$$

when $t > 1$ and $\mu \notin \{-1, 0\}$.

The eigenvalues $-1, 0$ are essential exceptions. For example, in the case $\mu = 0$, if we consider the graph $\overline{K}_{n-t} \cup K_t$ which has distinct eigenvalues 0, $t-1$ and -1 with multiplicities $n-t$, 1 and $t-1$ respectively we see that $\mathcal{E}(0)$ has codimension t for all n. Thus there are an infinite number of graphs with the eigenspace $\mathcal{E}(0)$ of codimension t. Further examples appear in later chapters.

Theorem 4.7 does not exclude the possibility that $\overline{X}$ is a location-dominating set for $\mu \in \{-1, 0\}$. However these graphs are somewhat special and warrant a formal definition.

Definition 4.10 Suppose $\mu \in \{-1, 0\}$ and let X be a star set for μ in G. We say G is a *core graph* if it has the property $\overline{\Gamma}(u) \neq \overline{\Gamma}(v)$ for all $u,\ v \in X$.

These core graphs are the reduced graphs ($\mu = 0$) and coreduced graphs ($\mu = -1$) to be found in Ellingham's paper [Ell].

Remark 4.11 It should be noted that the proof of Theorem 4.9 depended on $\overline{X}$ being a location-dominating set. Since the core graphs have the property that $\overline{\Gamma}(u) \neq \overline{\Gamma}(v)$ for all $u, v \in X$, we can say that the bound $|V(G)| < t + 2^t$ can also be applied to core graphs.

Definition 4.12 Suppose $\mu \in \{-1, 0\}$ and let X be a star set for μ in G. Then the vertices $u, v \in X$ are called *duplicate vertices* if $\overline{\Gamma}(u) = \overline{\Gamma}(v)$.

To see the reason for this we need to re-examine the proof of Theorem 4.7. Let $u, v \in X$ be duplicate vertices. It follows from equation (4.1) that if $\mu = 0$ then u is not adjacent to v and $\Gamma(u) = \Gamma(v)$ whence $\Delta(u) = \Delta(v)$. In other words, the neighbourhood of u is a duplicate of the neighbourhood of v. Similarly if $\mu = -1$ then u is adjacent to v and $\Gamma(u) \setminus \{v\} = \Gamma(v) \setminus \{u\}$ which means (if we disregard the (u, v) edge) that u and v are adjacent to the same vertices in both X and $\overline{X}$. In this case the closed neighbourhoods are duplicated.

In the case of $\mu = -1$, a set of such vertices will induce a complete graph; when $\mu = 0$ such a set will be an independent set. We shall see that we can always add arbitrarily many vertices. Our main example of this appears in Chapter 9 where we give a detailed description of the core graphs which arise in the case where $\mu = -1$. We identify the vertices which can be duplicated and verify that any set of duplicate vertices induces a complete graph. In Chapter 6 where we deal exclusively with cubic graphs we obtain further examples of core graphs in the case where $\mu = -1$. Here adding duplicate vertices is not a possibility since this would compromise the condition that G is cubic. In Chapter Five we will see how these exceptional eigenvalues -1 and 0 appear in the context of the Reconstruction Theorem.

We shall now consider a particular type of star set and its relationship to the structure of regular graphs.

Definition 4.13 [Row1] A star set X is *uniform* if every vertex outwith the star set is adjacent to the same number of vertices within X.

If G is regular of degree k this amounts to saying that $G - X$ is regular of degree $k - b$ where $b = |\Delta(v) \cap X|$ for all $v \in \overline{X}$.

Example 4.14 For this example we shall denote a star set corresponding to μ_i by $S(\mu_i)$. Let G be the Petersen graph, with spectrum $\left\{ \begin{matrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{matrix} \right\}$.

Note that G is regular of degree three. A star partition for G is shown in Figure 4.1 where the vertices of $S(\mu_i)$ are labelled μ_i $(i = 1, 2, 3)$.

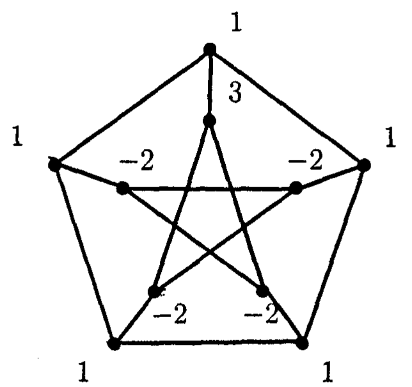


Figure 4.1: Petersen graph

If $X = S(-2)$, then X is not a uniform star set with $b = 1$ since $|\Delta(v) \cap X| = |\Gamma(v)|$ is not the same for every $v \in \overline{X}$. Indeed we could have $|\Gamma(v)| = 0$, $|\Gamma(v)| = 1$ or $|\Gamma(v)| = 2$, depending upon which vertex we consider.

However, if $X = S(1)$ then X is a uniform star set since each vertex outside X is adjacent to one vertex inside X. Note also that $G - X$ is a pentagon which is regular of degree two, as required.

These observations can be summarised as follows:

$S(1)$ is a uniform star set since, if $X = S(1)$ then for each $v \in \bar{X}$, $|\Delta(v) \cap X| = 1$. Equivalently $G - X$ is regular of degree $k - b = 2$.

Further note that $|\Delta(u) \cap \bar{X}| = 1$ for all $u \in X$ (Figure 4.2). In other words, the graph $G - \overline{X}$ is also regular. The following theorem shows that a similar result holds in general.

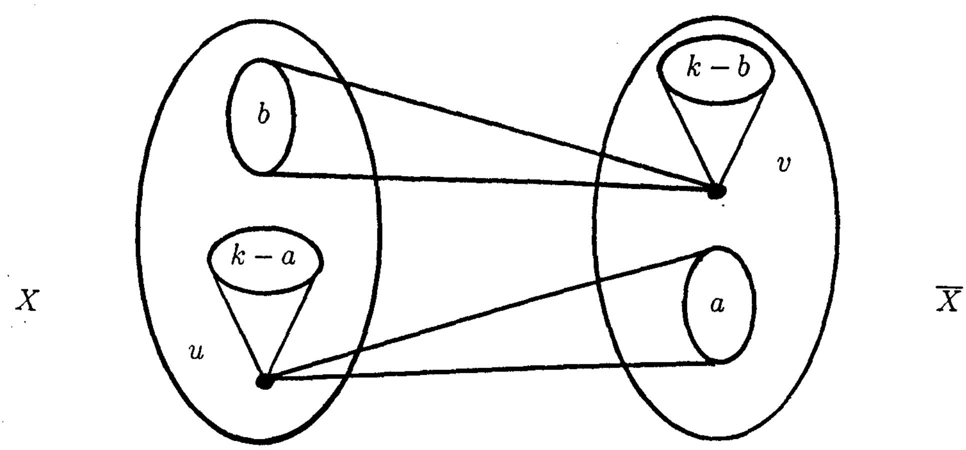


Figure 4.2: Uniform star set X

Theorem 4.15 [Rowl] *Let G be a regular graph of degree k and let X be a uniform star set for μ ($\mu \neq k$). Then there exists an integer a such that $|\overline{\Gamma}(u)| = a$ for all $u \in X$. Moreover, $a = k - b - \mu$ where $b = |\Gamma(v)|$ for all*

$v \in \overline{X}$. *In particular, μ is an integer and $-b \leq \mu$.*

Proof. Let u be vertex in X and let v be a vertex in $\overline{X}$. From Remark 4.4 we have

$$\mu P\mathbf{e}_u = \sum_{j \in \Delta(u)} P\mathbf{e}_j = \sum_{j \in \Gamma(u)} P\mathbf{e}_j + \sum_{j \in \overline{\Gamma}(u)} P\mathbf{e}_j$$

Summing over $u \in X$ we obtain

$$\mu \sum_{u \in X} P\mathbf{e}_u = \sum_{u \in X} \sum_{j \in \Gamma(u)} P\mathbf{e}_j + \sum_{u \in X} \sum_{j \in \overline{\Gamma}(u)} P\mathbf{e}_j.$$

Note that the degree of a vertex is equal to the number of neighbourhoods in which it appears and so this equation becomes

$$\mu \sum_{u \in X} P\mathbf{e}_u = \sum_{u \in X} d_u P\mathbf{e}_u + \sum_{v \in \overline{X}} bP\mathbf{e}_v$$

where d_u is the degree of u in $G - \overline{X}$ and $b = |X \cap \Delta(v)|$. Now $\mu \neq k$ and so μ is not a main eigenvalue. Consequently $P\mathbf{j} = \mathbf{0}$ where $\mathbf{j}$ is the all-one vector. In particular we have

$$\sum_{v \in \overline{X}} P\mathbf{e}_v = -\sum_{u \in X} P\mathbf{e}_u.$$

Hence

$$\mu \sum_{u \in X} P\mathbf{e}_u = \sum_{u \in X} d_u P\mathbf{e}_u + (-b \sum_{u \in X} P\mathbf{e}_u),$$

that is

$$\sum_{u \in X} (\mu - d_u + b) P\mathbf{e}_u = \mathbf{0}.$$

The vectors Pe_u ($u \in X$) are linearly independent and so we have $d_u = \mu + b$ for all $u \in X$. Hence $d_u = k - a$ where $a = k - b - \mu$. Clearly μ is an integer since a, k, and b are integers. Moreover $-b \leq \mu$ since $a - k \leq 0$. □

This information is summarised in Figure 4.3.

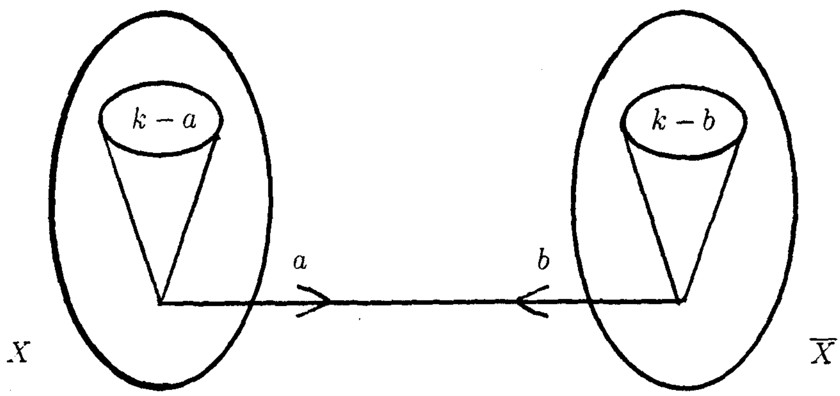


Figure 4.3

Further to Theorem 4.15 we note that, by counting $E(\overline{X}, X)$ in two ways we have $|X|a = |\overline{X}|b$. Moreover $G - \bar{X}$ is regular of degree $k - a$, and $G - X$ is regular of degree $k - b$.

We shall be using Theorem 4.15 in Chapter Six where we consider cubic graphs.

Chapter 5

Construction theory.

In Chapter 4 we saw how for a given graph G, having selected a star set X, we were able to describe some modest conditions on the structure of a part of the graph, namely the edge configuration between X and $\overline{X}$. These conditions were refined in the case where X was a uniform star set in G and G (hence also $G - X$) was regular: here we were able to state that the subgraph induced by X was also regular. Such results lead us to question what information is necessary in order to describe G fully. The following result is of major importance in this search. It shows that, for a given graph G, knowledge of three things, the subgraph $G - X$, the eigenvalue corrresponding to X and the edge set $E(\overline{X}, X)$ is equivalent to knowledge of the entire graph. This theorem has become known as the Reconstruction

Theorem and it appears in various forms in many places; [Ell, Thm 1.1], [CvRoS1, Thm 4.6.], [CvRoS, Thm 7.4.1.]. The converse appears in [Rowl, Thm 1.5.], [CvRoS, Thm 7.4.4.]. It is presented here in a combined form.

The Reconstruction Theorem.

Theorem 5.1 *Let X be a subset $V(G)$. Let the subgraphs induced by X and $\overline{X}$ have adjacency matrices A and C respectively; and let G have adjacency matrix $\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$, so that B represents $E(\overline{X}, X)$ with $b_{uv} = 1$ if and only if $u \sim v$. Let μ be an eigenvalue of G. Then X is a star set for μ if and only if μ is not an eigenvalue of C and $\mu I - A = B^T(\mu I - C)^{-1}B$.*

Proof. Let G be an n-vertex graph and let $m_\mu(G) = m$. Suppose that X is a star set for μ so that the matrix A has size $m \times m$. We have $A_G = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ and so $\mu I - A_G = \begin{pmatrix} \mu I - A & -B^T \\ -B & \mu I - C \end{pmatrix}$. Now dim $\mathcal{E}(\mu) = m_\mu(G) = m$ and so $\mathrm{r}(\mu I - A_G) = n - m$. By definition (cf. Theorem 2.3) μ is not an eigenvalue of $G - X$ and so is not an eigenvalue of C. Therefore $\ker(\mu I - C)$ is the zero subspace and $\mathrm{r}(\mu I - C) = n - m - 0 = n - m$. This means that the $n - m$ rows of $\mu I - C$ are linearly independent and so the last $n - m$ rows of $\mu I - A_G$ form a basis for the row-space of $\mu I - A_G$. Thus the first m rows of $\mu I - A_G$ are some linear combination of the last $n - m$ rows. Hence

$$(\mu I - A \,|\, -B^T) = L(-B \,|\, \mu I - C)$$

where L is some matrix.

Therefore $\mu I - A = L(-B)$ and $-B^T = L(\mu I - C)$. This yields the equation

$$\mu I - A = B^T(\mu I - C)^{-1}B$$

as required.

Conversely, suppose that μ is an eigenvalue of G but not of $G - X$ and that

$$\mu I - A = B^T(\mu I - C)^{-1}B, \tag{5.1}$$

where B has size $(n-m) \times m$. To show that X is a star set for μ it is sufficient to show that $m_\mu(G) = m$; equivalently that dim $\mathcal{E}(\mu) = m$.

Suppose that μ has eigenspace $\mathcal{E}(\mu) = \{\mathbf{v} \in I\!R^n : (\mu I - A_G)\mathbf{v} = \mathbf{0}\}$ and let $\mathbf{v} = \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{pmatrix}$ where $\mathbf{v_1} \in I\!R^m$, $\mathbf{v_2} \in I\!R^{n-m}$. Since $A_G = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ we have $\begin{pmatrix} \mu I - A & -B^T \\ -B & \mu I - C \end{pmatrix} \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$; that is

$$(\mu I - A)\mathbf{v_1} - B^T\mathbf{v_2} = \mathbf{0} \quad \text{and} \quad (-B)\mathbf{v_1} + (\mu I - C)\mathbf{v_2} = \mathbf{0}.$$

Clearly these equations are satisfied if and only if $\mathbf{v_2} = (\mu I - C)^{-1}B\mathbf{x}$ and $\mathbf{v_1} = \mathbf{x}$ where $\mathbf{x} \in I\!R^m$ since then we have $(\mu I - A)\mathbf{x} - B^T(\mu I - C)^{-1}B\mathbf{x} = \mathbf{0}$ by Equation (5.1), and $-B\mathbf{x} + (\mu I - C)(\mu I - C)^{-1}B\mathbf{x} = -B\mathbf{x} + B\mathbf{x} = \mathbf{0}$. Thus dim $\mathcal{E}(\mu) = m$ as required. $\square$

Remark 5.2 We note that in the situation of Theorem 5.1, the eigenspace of μ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in \mathbb{R}^m)$.

We also note that it is clear from Theorem 5.1 that deletion of k vertices from the star set will result in a graph with μ as an eigenvalue of multiplicity $m - k$ (see Remark 2.4).

If our aim were to catalogue graphs according to their adjacency matrices then applying Theorem 5.1 would result in a significant improvement in the case where the distinct eigenvalues of G are few and we can choose one of relatively large multiplicity. To see this we suppose that the rows and columns of A_G are ordered so that the first m rows of A_G correspond to the vertices in our chosen star set. Then it would be sufficient to record the last $n - m$ rows of A_G, together with the corresponding eigenvalue. Clearly it is to our benefit to select the eigenvalue of greatest multiplicity. With such information it is possible to reconstruct a graph uniquely.

If we lower our expectations and say that the reconstructed graph need not necessarily be unique, then we can start the reconstruction process with less information. It is at this point that the emphasis changes from *reconstructing* a specific graph to *constructing* (possibly) several graphs. In Chapter 6 we restrict ourselves by stipulating that the constructed graph is cubic and that the star set is uniform with $\mu \neq 3$. This gives us various possibilities to consider (see Table 6.1). In the case where $\mu = -2$ the constructed graph is unique, but when $\mu = -1$ and $G - X$ is regular of degree 2 (i.e. a union of cycles), the constructed graph was not unique; indeed it was found to belong to one of several infinite families of graphs.

In the chapters following Chapter 6 we explore the feasibility of reconstructing a graph from the subgraph induced by $\overline{X}$. This is done by constructing graphs from a given graph H which will be the subgraph induced by $\overline{X}$ in the constructed graph. We will find the following definitions and notation useful.

Definition 5.3 Let μ be an eigenvalue of G and let X be a star set for μ. Then the subgraph $G - X$ is a *star complement* for μ.

In particular μ is not an eigenvalue of the star complement. Note that a star set is a vertex set but the star complement is a graph. The definition of a star complement is given within the context of a graph. However it is useful to think of the graph H as standing independently of G since our method of construction is to add vertices to H in order to obtain G.

Notation 5.4 Let H be a graph. We write $H + u$ for any graph formed by adding a single vertex to H, and $H + u + v$ for any graph formed by adding two vertices to H in some way.

The graph H is a star complement for μ if and only if $\mu \notin Sp(H)$ and $\mu \in Sp(H+u)$ for some $H+u$ with $m_\mu(H+u) = 1$. Note that it is not sufficient to say simply that H is a star complement for μ when $\mu \notin Sp(H)$. However we can say that if $\mu \in Sp(G)$ and K is an induced subgraph of G such that $\mu \notin Sp(K)$ then K can always be extended to a star complement for μ. (The following proposition is essentially the same as [Row4, Proposition 1.1] and is given here without proof.)

Proposition 5.5 *Let μ be an eigenvalue of G and let K be an induced subgraph of G, say $K = G - Y$. If $\mu \notin Sp(K)$ then there exists a subset X of Y such that X is a star set for μ.*

In the case where $X = Y$, K is a star complement for μ and $|Y| = m_\mu(G)$. If X is a proper subset of Y then K can be extended to a star complement for μ by adding those vertices in Y which are not also in X.

Our aim is to construct relatively large graphs and so we consider the case where only one vertex can be added to H to be trivial. However we find that considering the case $G = H + u$ is essential when determining the possible values of μ.

Definition 5.6 We say the graph H is *μ-extendible* if $\mu \notin Sp(H)$ and $\mu \in Sp(H+u+v)$ with $m_\mu(H+u+v) = 2$ for some $H+u+v$.

In order to construct graphs with multiple eigenvalues it is necessary to restrict ourselves to those values of μ for which H is μ-extendible. The following theorem shows that if H is an induced subgraph of G, $\mu \notin Sp(H)$, then H is a star complement for μ if and only if H is μ-extendible to $H+u+v$ for every pair u, v of vertices not in $V(H)$.

The All-pairs Theorem.

Theorem 5.7 [Row2, Prop. 3.1] *Let G be given with $\mu \in Sp(G)$, and let X be a subset of $V(G)$ with $\mu \notin Sp(G-X)$ and $|X| > 1$. Let $H = G - X$ and let $H + u + v$ be the subgraph of G induced by $\overline{X} \cup \{u, v\}$ $(u, v \in X)$. Then X is a star set for μ if and only if μ is a double eigenvalue of $H + u + v$ $\forall$ $u, v \in X$.*

Proof. Let X be a star set for μ, with $m_\mu(G) = |X| = m$. Then μ is a double eigenvalue of $H + u + v$ for all $u, v \in X$ since deletion of any $m - 2$ vertices in X will result in a graph where the multiplicity of μ as an eigenvalue of G is 2 (see Remark 2.4).

Conversely, suppose that μ is a double eigenvalue of $H+u+v$ for all $u, v \in X$. To show that X is a star set for μ it is suffficient to show that $m_\mu(G) = m$. Let the graph $H + u + v$ have adjacency matrix $A_{H+u+v} = \begin{pmatrix} A_{uv} & B_{uv}^T \\ B_{uv} & C \end{pmatrix}$ where $A_{uv} = \begin{pmatrix} 0 & a_{uv} \\ a_{vu} & 0 \end{pmatrix}$ and $B_{uv} = (\mathbf{b}_u | \mathbf{b}_v)$. Since $\{u, v\}$ is a star set for μ in $H+u+v$ we can apply the Reconstruction Theorem to obtain $\mu I - A_{uv} = B_{uv}^T(\mu I - C)^{-1}B_{uv}$, that is

$$\mathbf{b}_u^T(\mu I - C)^{-1}\mathbf{b}_v = \begin{cases} \mu & \text{if} \quad u = v \\ -1 & \text{if} \quad u \sim v \\ 0 & \text{if} \quad u \not\sim v \end{cases}.$$

This equation holds for every pair u, v in X and so we have $B^T(\mu I - C)^{-1}B = \mu I - A$. Hence, by Theorem 5.1, X is a star set. □

In general, knowledge of the star complement is insufficient to construct a graph, uniquely or otherwise; knowledge of the eigenvalue corresponding to H is also necessary. The values for μ will be constrained by the condition that H is μ-extendible and so for certain star complements it is conceivable that there will be only one possible eigenvalue but we can offer no such example here. However in the case where $H \cong K_{2,5}$ we will see that μ has only two possible values, namely -1 and 1.

Propositions 5.8 and 5.10 show that the first step in determining the possible values for μ is to consider the eigenvalues of $H+u$ which are not eigenvalues of H.

Proposition 5.8 *Let H be μ-extendible, so that $m_\mu(H+u+v) = 2$ for some $H+u+v$ and $m_\mu(H) = 0$. Then $m_\mu(H+u) = 1$ and $m_\mu(H+v) = 1$.*

Proof. Since H is μ-extendible, the set $\{u, v\}$ is a star set for μ. The result follows. □

The importance of this proposition is that, given that H is a star complement corresponding to μ, if $\mu \notin Sp(H+u)$ then μ is not a double eigenvalue of $H + u + v$. Consequently any restrictions placed on the values of μ as an eigenvalue of $H + u$ will also apply to μ as an eigenvalue of $H + u + v$. The converse of Proposition 5.8 does not hold in general, as shown by the following example.

Example 5.9 Let $G = K_{r,s+2}$, with subgraph $H \cong K_{r,s}$. Then $G = H+u+v$ where $H + u \cong H + v \cong K_{r,s+1}$. Let $\mu = \pm\sqrt{r(s+1)}$. Then $m_\mu(H) = 0$, $m_\mu(H + u) = m_\mu(H + v) = 1$, but $m_\mu(H + u + v) = 0$.

Proposition 5.10 explains the outcome of this example and provides a partial converse to Proposition 5.8

Proposition 5.10 [Row4, Remark 3.3] *If* $m_\mu(H) = 0$ *and* $m_\mu(H + u) = m_\mu(H + v) = 1$, *then* $m_\mu(H + u + v) \in \{0, 2\}$.

Proof. Let $H+u+v = G$. It is sufficient to exclude the case where $m_\mu(G) = 1$. Suppose by way of contradiction that $m_\mu(G) = 1$. Let $H = G - Y$ so that $Y = \{u, v\}$. We have $m_\mu(H) = 0$. Since $|Y| \neq m_\mu(G)$, by Proposition 5.5 we can say without loss of generality that $\{u\}$ is a star set for μ and so $m_\mu(G - u) = 0$, that is $m_\mu(H + v) = 0$; a contradiction. □

From Theorem 5.7 we can see that, given a star complement H, together with a corresponding eigenvalue μ, one method of constructing G is to add vertices to H in such a way that $H + u + v$ satisfies the Reconstruction Theorem for every pair u, v of the additional vertices. By Theorem 4.9 we know that if $\mu \notin \{-1, 0\}$ then there are only finitely many possibilities for G. This stems from the observation that $|X|$ is bounded above, which in turn leads us to the following definition.

Definition 5.11 Let X be a star set for μ in G and let $H = G - X$. We say that X is *maximal* if it is not possible to add a vertex j to G in such a way that $m_\mu(G + j) = m_\mu(G) + 1$. We say then that G is a maximal graph with H as a star complement for μ.

In the case $\mu \in \{-1, 0\}$, a *maximal core graph* is a graph G to which it is not possible to add a non-duplicate vertex j such that $m_\mu(G + j) = m_\mu(G) + 1$.

Remark 5.12 Every core graph is contained in a maximal core graph.

This follows directly from Remark 4.11.

The Reconstruction Theorem says that a graph may be reconstructed from an eigenvalue, the edge set between the corresponding star set and its complement, and the subgraph induced by that complement. The All-Pairs

Theorem indicates how we can construct the graphs G with a given star complement H corresponding to μ. This is done by determining the graphs $H+u+v$ which have μ as a double eigenvalue and using this information to construct graphs with μ as a multiple eigenvalue. We start this process by selecting a μ-extendible graph H which will be the star complement for μ in the constructed graph G. In general, the tractability of the construction problem is contingent upon some measure of symmetry in H (so that only a limited number of graphs of the form $H+u$ need be considered), and also upon H having a small number of distinct eigenvalues (so that we can find a simple form for the inverse of the matrix $\mu I - C$.)

We apply the Reconstruction Theorem in the form $f(\mu)(\mu I - A) = B^T f(\mu)(\mu I - C)^{-1} B$ where $f(x)$ is the minimal polynomial of H. This matrix equation yields three polynomial equations of the form $g_1(\mu) = g_2(\mu) = g_3(\mu) = 0$. The first equation comes from the (u,u)-entry; the second comes from taking the difference between the (u,u)-entry and the (v,v)-entry; the third equation comes from the (u,v)-entry. From the first two equations we obtain information about the possible values for μ and from the third we obtain information about $E(X,\overline{X})$ in the form of the H-neighbourhoods of u and v. In the case where $H \cong K_{r,s} \cup tK_1$ we shall see that the intersection of the H-neighbourhoods of u and v determines whether or not u is adjacent to v. The H-neighbourhoods of u and v will govern how we add vertices to H to obtain G, each pair obeying the constraints laid down by the third equation. In general there will be more than one way of adding a selection of vertices, some being mutually exclusive and so we will not get a unique maximal graph. However, given H and μ it is sufficient to describe all maximal graphs since any other graph which arises will be an induced subgraph of one of these. This is because adjacency of vertices in X is determined by the H-neighbourhoods of these vertices. In the case where $H \cong K_{2,5}$ and $\mu \neq -1$ with $m_\mu(G) > 1$ we do obtain a unique maximal graph, Sch_{10}. In the case where $H \cong K_{1,16} \cup 6K_1$ and $\mu = 2$ the largest maximal graph is the MacLaughlin graph.

In the case of the exceptional eigenvalues -1 and 0 it is sufficient to describe the maximal core graphs. The reason for this is that the maximal core graph may be thought of as a homomorphic image of any graph in an infinite family of graphs, since each vertex may be duplicated *ad infinitum* (see Chapter 4 and Remark 5.14). We shall give explicit examples in the case where $H \cong K_{2,5}$ and $\mu = -1$ and the case where $H \cong K_{1,5}$ and $\mu = -1$.

We conclude this chapter by giving two examples of the application of the Reconstruction Theorem. The first concerns the characteristic polynomial of $H + u$.

Remark 5.13 Let $G = H + u$ and let H be a star complement corresponding to μ with adjacency matrix C. Using the usual notation we have $A_G = \begin{pmatrix} 0 & \mathbf{b}^T \\ \mathbf{b} & C \end{pmatrix}$. Thus by Theorem 5.1 we have

$$\mu = \mathbf{b}^T(\mu I - C)^{-1}\mathbf{b}. \tag{5.2}$$

The graph $H + u$ has characteristic polynomial

$$\begin{aligned} \Phi_{H+u}(x) &= \det \begin{pmatrix} x & -\mathbf{b}^T \\ -\mathbf{b} & xI - C \end{pmatrix} \\ &= \det(xI - C)(x - \mathbf{b}^T(xI - C)^{-1}\mathbf{b}) \end{aligned} \tag{5.3}$$

by Remark 1.5. Hence $\Phi_{H+u}(x) = \Phi_H(x)h(x)$ where $h(\mu) = 0$ is the equation obtained from Equation (5.2).

When applying the Reconstruction Theorem in the form

$$f(\mu)(\mu I - A) = B^T f(\mu)(\mu I - C)^{-1}B$$

where $f(x)$ is the minimal polynomial of H we obtain the equation $g_1(\mu) = 0$ where $g_1(x) = f(x)h(x)$. Hence

$$\Phi_{H+u}(x) = \frac{\Phi_H(x)g_1(x)}{f(x)}, \tag{5.4}$$

enabling the characteristic polynomial of $H + u$ to be found from $g_1(x)$.

It was noted in [Bell, Remark 2] that the characteristic polynomial of $H+u$ can be written in the form $\Phi_{H+u}(x) = x\det(xI - C) - \mathbf{b}^T\mathrm{adj}(xI - C)\mathbf{b}$ which we can obtain directly from Equation (5.3).

The second remark concerns the application of the Reconstruction Theorem in the case where $\mu \in \{-1, 0\}$.

Remark 5.14 Let X be a star set for μ in G. We can apply the Reconstruction Theorem for each pair u, v in X as we did for Theorem 5.7 to obtain the equation

$$\mathbf{b}_u^T(\mu I - C)^{-1}\mathbf{b}_v = \begin{cases} \mu & \text{if} \quad u = v \\ -1 & \text{if} \quad u \sim v \\ 0 & \text{if} \quad u \not\sim v \end{cases}.$$

When $\mu = -1$ this equation can be satisfied when $\mathbf{b}_u = \mathbf{b}_v$ and $u \sim v$. Similarly when $\mu = 0$ this equation is satisfied when $\mathbf{b}_u = \mathbf{b}_v$ and $u \not\sim v$.

In terms of the matrix B this means that B can consist of blocks of identical columns (see Remark 4.8); each block corresponding to a set of vertices which (in the case where $\mu = 0$) form an independent set, or (in the case where $\mu = -1$) induce a complete graph. These sets are sets of duplicate vertices (see Definition 4.12).

Chapter 6

Cubic graphs.

In this chapter we investigate the types of star complements which arise in the case where G is cubic and the star set corresponding to μ in G is uniform. We start by reviewing the results obtained in [Row1], commencing with a result which is applicable to all regular graphs. This is essentially [Row1, Thm 2.3(ii)] and it provides an example of core graphs and duplicate vertices.

Theorem 6.1 *Let G be a regular graph of degree k and let X be a uniform star set for μ ($\mu \neq k$). Let $\overline{X}$ be an independent set with $|\overline{X}| = t$. Then $\mu = -1$ and G is $t\,K_{k+1}$.*

Proof. By Theorem 4.15 we have $a + b = k - \mu$ where $a = |\overline{\Gamma}(u)|$ for all $u \in X$ and $b = |\Gamma(v)|$ for all $v \in \overline{X}$. Now $k - b = 0$ since $\overline{X}$ is an independent set and so $\mu = -a$. We apply the Reconstruction Theorem: $\mu I - A = B^T(\mu I - C)^{-1}B$. In this case the matrix C is the zero matrix and so we have

$$\mu^2 I - \mu A = B^T B \tag{6.1}$$

where $\mu \neq 0$. Recall that the rows of B correspond to vertices in $\overline{X}$, and the columns to vertices in X. Let columns u and v correspond to vertices u and v in X. Then the (u, v) entry of $B^T B$ is the number of vertices in $\overline{X}$ to which both u and v are adjacent. In particular the diagonal entries of $B^T B$ are a and so, by equating diagonal entries of Equation (6.1) we have $\mu^2 = a$. Thus $\mu^2 = -\mu$ whence $\mu = -1$ and $a = 1$. This means that each vertex in X is adjacent to precisely one vertex in $\overline{X}$. Substituting $\mu = -1$ into Equation (6.1) we obtain $A = B^T B - I$ and so $u, v \in X$ are adjacent if and only if they are adjacent to the same vertex in $\overline{X}$. Moreover, since $b = k$, each vertex in $\overline{X}$ is adjacent to k vertices in X. Thus each component of G is the complete graph K_{k+1} and the result follows. □

We give an example when $k = 3$ and $t = 2$ (Figure 6.1).

In the general case there is a unique maximal core subgraph, namely $t\,K_2$; the star set X comprises t sets of duplicate vertices, each inducing a complete graph K_k. Note that when G is a cubic graph, G is $t\,K_4$.

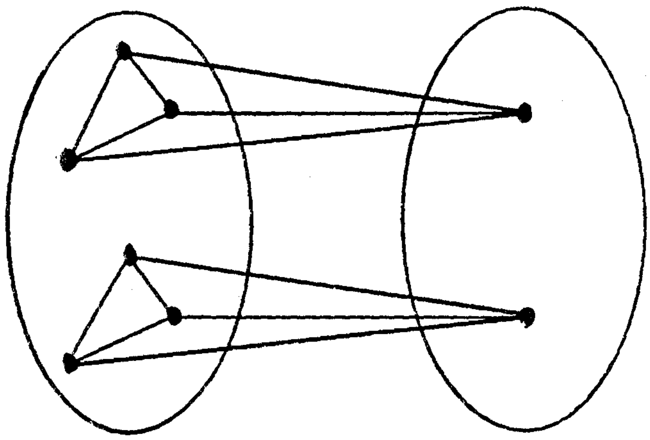


Figure 6.1

Having considered this general case we return to cubic graphs. Let G be a connected cubic graph so that $k = 3$. Let X be a uniform star set for μ ($\mu \neq 3$), and suppose that $\overline{X}$ is not an independent set. Let $r = k - b$ so that $\mu = r - a$. Since G is connected $b \neq 0$ and we have two possibilities for r, namely $r = 1, 2$. (The case when $r = 0$ has already been covered in Theorem 6.1.) We tabulate the possible values for a, b, r and μ in Table 6.1, and investigate the structure of G in two of these cases.

r	1			2		
b	2			1		
a	1	2	3	1	2	3
μ	0	-1	-2	1	0	-1

Table 6.1

First we provide details of the case $\mu = -2$ treated briefly in [Row1].Then we extend the technique to obtain further examples of cubic graphs in the case where the star set is uniform. In the following theorem we denote the block-diagonal matrix $\mathrm{diag}(D, D, \ldots, D)$ by $\mathrm{diag}D$.

Theorem 6.2 [Row1, Case $(r, \mu_i) = (1, -2)$] *Let G be a connected cubic graph. Let X be a uniform star set for $\mu = -2$. Then G is the Petersen graph.*

Proof. From Table 6.1 we see that $r = 1$ when $\mu = -2$.

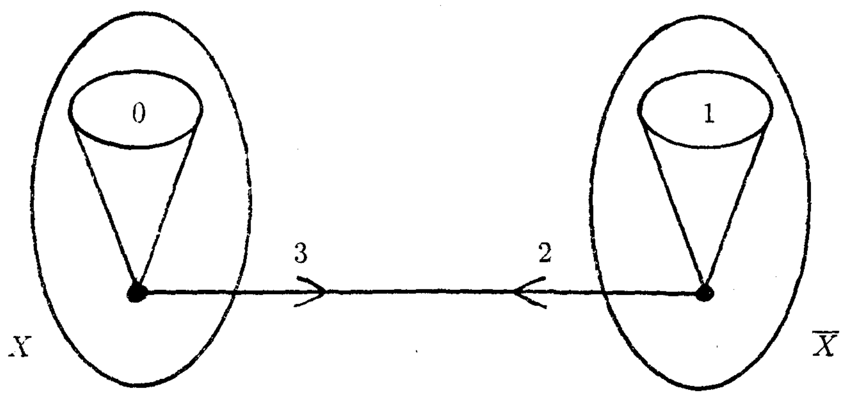


Figure 6.2: $(r, \mu) = (1, -2)$

Note that $k - a = 0$ and so X is an independent set. Moreover, since $k - b = 1$, the subgraph induced by $\overline{X}$ is regular of degree one, hence is of the form qK_2. We now have enough information to apply the Reconstruction Theorem: $A - \mu I = B^T(C - \mu I)^{-1}B$. Here A is the zero matrix and $\mu = -2$, and so we have $2I = B^T(C + 2I)^{-1}B$. Without loss of generality we can label the vertices of $G - X$ so that $C = \operatorname{diag}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. From Figure 6.2 it is clear that $|E(X, \bar{X})| = 3|X| = 2|\bar{X}| = 4q$, and so $q \equiv 0 \bmod 3$, say $q = 3h$ $(h = 1, 2, \ldots)$. The smallest graph which satisfies this edge condition arises when $h = 1$. Indeed we shall see that if $h > 1$ then G is not connected.

When $h = 1$ we have $|X| = 4$; hence C is a 6×6 matrix and B is a 6×4 matrix. Now $(C + 2I) = \operatorname{diag}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and so

$$(C + 2I)^{-1} = \operatorname{diag}\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1}\right) = \operatorname{diag}\left(\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\right).$$

Thus we have

$$6I = B^T \operatorname{diag}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} B. \tag{6.2}$$

Let $\mathbf{b}$ be the first column of B, then the first diagonal entry of the right-hand side of Equation (6.2) is:

$$(b_1, b_2, \ldots, b_6)\operatorname{diag}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{pmatrix}$$

$$= 2\left(b_1^2 - b_1b_2 + b_2^2 + b_3^2 - b_3b_4 + b_4^2 + b_5^2 - b_5b_6 + b_6^2\right)$$

and so we have a sum of like terms, say β_1, β_2, and β_3 where

$$\beta_1 + \beta_2 + \beta_3 = 3 \text{ and } \beta_1 = b_1^2 - b_1b_2 + b_2^2.$$

Note that $b_i = 0$ or 1 since B is an adjacency matrix. Further note that the components of $\mathbf{b}$ must sum to three since A is the zero matrix and G is regular of degree three.

If $(b_1, b_2) \neq (0,0)$ then $\beta_1 = 1$; hence there are three blocks $\neq (0,0)$; each of which is necessarily $(0,1)$ or $(1,0)$. Therefore, without loss of generality we can label the vertices of $\overline{X}$ so that $\mathbf{b} = (1\ 0\ 1\ 0\ 1\ 0)^T$ Next we consider the off-diagonal entries of Equation (6.2). Let $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{pmatrix}$ and $\mathbf{b}' = \begin{pmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_6 \end{pmatrix}$, where $\mathbf{b} \neq \mathbf{b}'$. Thus $0 = \mathbf{b}^T \mathrm{diag} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{b}'$. Multiplying out again gives the sum of three like terms, say β'_1,β'_2 and β'_3, where

$$\beta'_1 + \beta'_2 + \beta'_3 = 0 \text{ and } \beta'_1 = 2b_1b'_1 + 2b_2b'_2 - b_1b'_2 - b'_1b_2.$$

If $(b_1, b_2) = (b'_1, b'_2) \neq (0,0)$, then $\beta'_1 = 2$. If $(b_1, b_2) \neq (b'_1, b'_2)$ and both blocks are non-zero, then $\beta'_1 = -1$. Thus $\mathbf{b}'$ must consist of three blocks of $(1,0)$ or $(0,1)$, with two blocks different from those of $\mathbf{b}$. Thus, without loss of generality, we can write

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Hence $A_G = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ becomes

$$A_G = \begin{pmatrix} \mathbf{0} & & B^T & \\ 1100 & 01 & & \\ 0011 & 10 & & \\ 1010 & & 01 & \\ 0101 & & 10 & \\ 1001 & & & 01 \\ 0110 & & & 10 \end{pmatrix}$$

which is the adjacency matrix for the Petersen graph as shown in Figure 6.3.

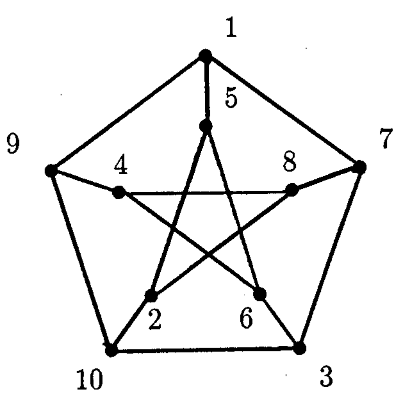


Figure 6.3: The Petersen graph

Note that $\{1, 2, 3, 4\}$ is a star set for the eigenvalue -2. $\square$

This procedure could be extended to non-connected graphs when $h > 1$ in which case the cubic graph G would be composed of h components, each one the Petersen graph.

The next example (Figure 6.5) can also be found in [Row1]. It is included here for comparison with further examples of the graph G obtained by the author and described below. Indeed it is conjectured that no other examples of the structure of G can be found in this case.

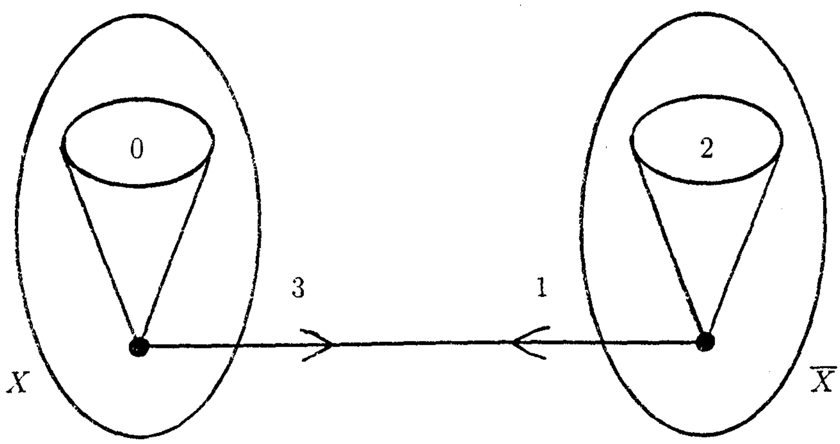


Figure 6.4: $(r, \mu) = (2, -1)$

Case $r = 2$, $\mu = -1$ Note that $k - a = 0$ and so X is an independent set, hence $A = O$. Also note that $G - X$ is regular of degree two and so is a union of cycles, see Corollary 4.6. Without loss of generality, we can write $C = \text{diag}(Z_1, Z_2, \ldots)$, where Z_j is the adjacency matrix of the j^{th} cycle in $G - X$. Let Q_j be a submatrix of B such that Q_j represents the edges between the vertices in X, and those in the j^{th} cycle in $\overline{X}$. Thus the adjacency matrix of the graph can be written:

$$A_G = \begin{pmatrix} O & Q_1^T & Q_2^T & \cdots \\ Q_1 & Z_1 & & \\ Q_2 & & Z_2 & \\ \vdots & & & \ddots \end{pmatrix}$$

We now apply the Reconstruction Theorem : $A - \mu I = B^T(C - \mu I)^{-1}B$.

Here $A = O$ and $\mu = -1$ and so we have $I = B^T(C + I)^{-1}B$. Now

$(C+I)^{-1} = (\text{diag}((Z_1+I), (Z_2+I), \ldots))^{-1} = \text{diag}((Z_1+I)^{-1}, (Z_2+I)^{-1}, \ldots)$

Note that the adjacency matrix of a cycle may be written $\begin{pmatrix} 0100..01 \\ 1010..00 \\ 0101..00 \\ \ddots \end{pmatrix}$, a circulant matrix which is denoted circul(0100..01); thus $(Z_j+I) = \text{circul}(110\ldots01)$. The invertibility of this matrix is dependent upon whether or not the cycle is of a length congruent to 0 mod 3, 1 mod 3, or 2 mod 3.

First we consider the case where the cycle is of a length congruent to 0 mod 3. The eigenvalues of an n-cycle are $2\cos\frac{2\pi}{n}k$, $(k = 1\ldots n)$ [Biggs, Prop 3.5]. If $n \equiv 0 \bmod 3$, say $n = 3h$, then the eigenvalues are $2\cos\frac{2\pi}{3h}k$. These include -1 when $h = k$. Since we cannot have -1 as an eigenvalue of $G - X$, we can say that no cycle in $G - X$ has length congruent to 0 mod 3. In this case $I + Z_j$ is not invertible. When the cycle has a length congruent to 1 mod 3,

$$(Z_j + I)^{-1} = \frac{1}{3}\text{circul}(1; 1, -2, 1; 1, -2, 1; \ldots; 1, -2, 1),$$

and when the cycle has a length congruent to 2 mod 3,

$$(Z_j + I)^{-1} = \frac{1}{3}\text{circul}(-1, 2; -1, -1, 2; \ldots; -1, -1, 2).$$

Let $Y_j = 3(Z_j + I)^{-1}$, so that the equation $I = B^T(C + I)^{-1}B$ becomes $3I = \sum_j Q_j^T Y_j Q_j$. It is observed in [Rowl] that this equation has many solutions and moreover that $G - X$ must always have at least one cycle of length congruent to 1 mod 3. One graph which satisfies these conditions is shown in Figure 6.5. The vertices of $G - X$ are shown in black.

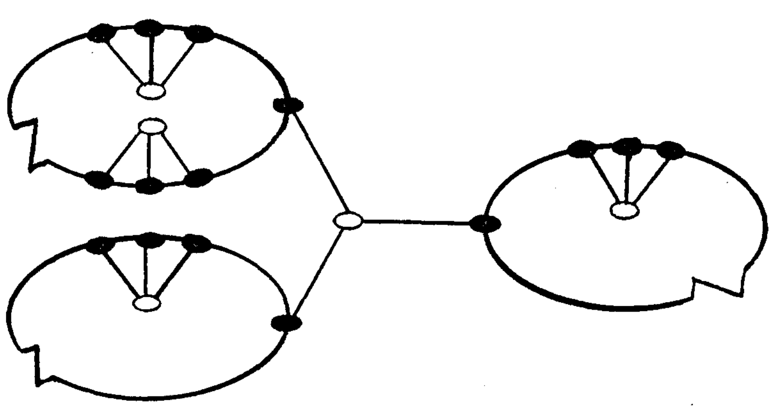


Figure 6.5

Note that $G - X$ consists of three cycles, each of a length congruent to 1 mod 3, say of lengths $3k_1 + 1$, $3k_2 + 1$ and $3k_3 + 1$. Furthermore $|X| =$

$k_1 + k_2 + k_3 + 1$. Deletion of the central vertex reduces the multiplicity of -1 as an eigenvalue of G by one. Apart from this, we can say that the occurrence of -1 as an eigenvalue arises from the configuration shown in Figure 6.6, since deletion of all but one of the other vertices in X results in a graph with -1 as a simple eigenvalue.

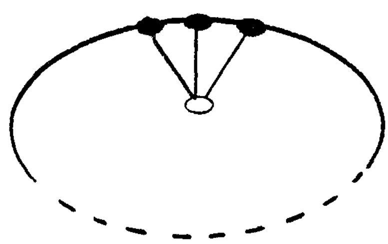


Figure 6.6

We extend the work done in [Row1] by investigating whether or not there are further examples, similar to Figure 6.5, but with the vertices in X adjoined to the cycles in different ways.

Let H be one of the cycles in $G-X$. In Figure 6.7 we give a diagrammatic representation of the graph H:

Here u, v and w are the lengths of the path between vertices x, y and z respectively. There are two possibilities to consider:

(i) that H is of some length congruent to 1 mod 3 in which case $u + v + w \equiv 1 \bmod 3$.

(ii) that H is of some length congruent to 2 mod 3 in which case $u + v +$

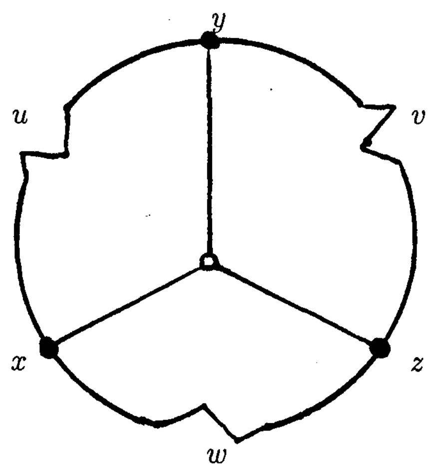


Figure 6.7

$w \equiv 2 \bmod 3$.

We wish to know for what values of u, v and w, -1 is an eigenvalue of H. Let $\mathbf{x} = (x_1, \ldots x_{u-1}, x_u, y_1, \ldots, y_v, z_1, \ldots, z_w, G)^T$ be such that $A_H\mathbf{x} = -\mathbf{x}$. Then -1 is an eigenvalue of H if and only if there exists a non-zero $\mathbf{x}$. It was suggested by P. Rowlinson that the graph H may be labelled with components of $\mathbf{x}$ as shown in Figure 6.8. The ensuing information could then be collated in matrix form.

Since $A_H\mathbf{x} = -\mathbf{x}$, by Remark 1.18 we have $-x_i = \sum_{j \sim i} x_j$. We consider the path between x_0 and x_u. The vertices $x_1, \ldots x_{u-1}$ are all of degree two, and so we have a recurrence relation:

$$x_j = -x_{j-1} - x_{j+1} \qquad (j = 1 \ldots, u-1).$$

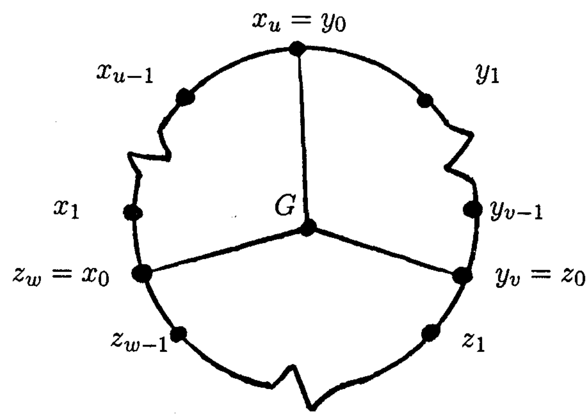


Figure 6.8

This has auxiliary equation $\omega^2 = -\omega - 1$ with solutions $\exp(2i\pi/3)$ and $\exp(-2i\pi/3)$. Thus the recurrence relation has solution

$$x_j = A\exp(2i\pi j/3) + B\exp(-2i\pi j/3) \qquad (j = 0, \ldots, u),$$

where A and B are constants. Similarly we have

$$y_j = C\exp(2i\pi j/3) + D\exp(-2i\pi j/3) \qquad (j = 0, \ldots, v),$$

and $$z_j = E\exp(2i\pi j/3) + F\exp(-2i\pi j/3) \qquad (j = 0, \ldots, w).$$

Note that $x_u = y_0$ and so

$$A\exp(2i\pi u/3) + B\exp(-2i\pi u/3) = C + D.$$

Two further equations are obtained similarly. Now note that

$$x_{u-1} + y_1 + G = -x_u = -y_0 \quad \text{since} \quad -x_i = \sum_{j\sim i} x_j$$

and so

$$\begin{aligned} A\exp(2i\pi(u-1)/3) \quad &+ \quad B\exp(-2i\pi(u-1)/3) \\ &+ \quad C\exp(2i\pi/3) + D\exp(-2i\pi/3) + G = -(C+D). \end{aligned}$$

Two further equations are obtained similarly. Finally note that

$$-G = A + B + C + D + E + F.$$

These equations together form a matrix equation $M(u,v,w)\mathbf{A} = \mathbf{0}$, where $\mathbf{A} = (A,B,C,D,E,F,G)^T$ and $M(u,v,w) =$

$$\begin{pmatrix} e^{2i\pi u/3} & e^{-2i\pi u/3} & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & e^{2i\pi v/3} & e^{-2i\pi v/3} & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & e^{2i\pi w/3} & e^{-2i\pi w/3} & 0 \\ e^{2i\pi(u-1)/3} & e^{-2i\pi(u-1)/3} & 1+e^{2i\pi/3} & 1+e^{-2i\pi/3} & 0 & 0 & 1 \\ 0 & 0 & e^{2i\pi(v-1)/3} & e^{-2i\pi(v-1)/3} & 1+e^{2i\pi/3} & 1+e^{-2i\pi/3} & 1 \\ 1+e^{2i\pi/3} & 1+e^{-2i\pi/3} & 0 & 0 & e^{2i\pi(w-1)/3} & e^{-2i\pi(w-1)/3} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Note that -1 is an eigenvalue of H if and only if this matrix equation has a non-zero solution; equivalently if and only if $\det M(u,v,w) = 0$.

Now observe that if $k \equiv k' \bmod 3$ then $\exp(2i\pi k/3) = \exp(2i\pi k'/3)$. Thus $\det M(u,v,w) = \det M(u',v',w')$ where $u \equiv u' \bmod 3$, $v \equiv v' \bmod 3$ and $w \equiv w' \bmod 3$. After rotating and reflecting as necessary we may assume that the remainders from u, v, w on division by three are in non-decreasing order. The possibilities are tabulated in Table 6.2.

First we consider the case where $u + v + w \equiv 1 \bmod 3$ and so we look at $\det M(0,0,1)$, $\det M(0,2,2)$ and $\det M(1,1,2)$. Of these, only the last is zero.

Thus we may conclude that -1 is an eigenvalue of H if and only if the paths between two of the vertices adjacent to the central vertex G are of length congruent to 1 mod 3, and the other path is of length congruent to 2 mod 3. This determines the configuration of vertices within one of the cycles of $G - X$.

u	v	w	sum mod 3
0	0	0	0
0	0	1	1
0	0	2	2
0	1	1	2
0	1	2	0
0	2	2	1
1	1	1	0
1	1	2	1
1	2	2	2
2	2	2	0

Table 6.2

Note that in a cycle of length congruent to 1 mod 3 there will always be one vertex remaining of degree 2 after we have joined all the other vertices to a vertex in X. This vertex must also be adjacent to a vertex in X, suggesting that $G - X$ is a union of three cycles, each of some length congruent to 1 mod 3. In Figure 6.9 we give three examples of possible configurations for a cycle of length seven, the smallest non-trivial example.

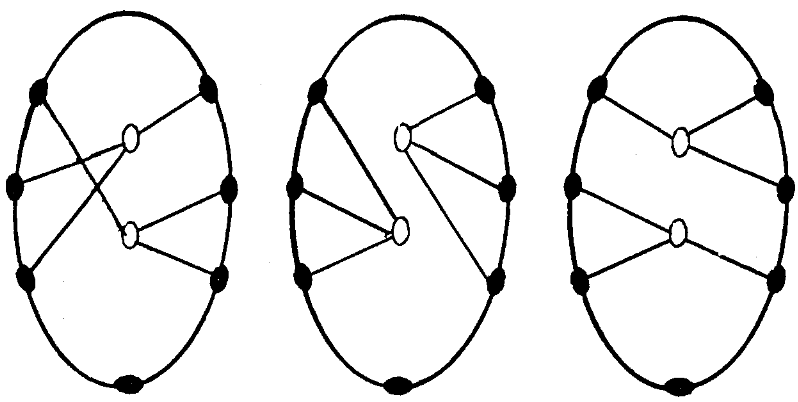


Figure 6.9

Thus in Figure 6.10, we are able to give another example of a graph where $G - X$ is a union of three cycles, similar to that shown in Figure 6.5.

Note that we can label the vertices of a graph with the components of an eigenvector following the rule $\mu x_i = \sum_{j \sim i} x_j$. Thus, in order to ascertain whether or not a graph has μ as an eigenvalue with multiplicity k, it is sufficient to show that we can label the graph in such a way as to illustrate that there are k linearly independent eigenvectors, in other words, that

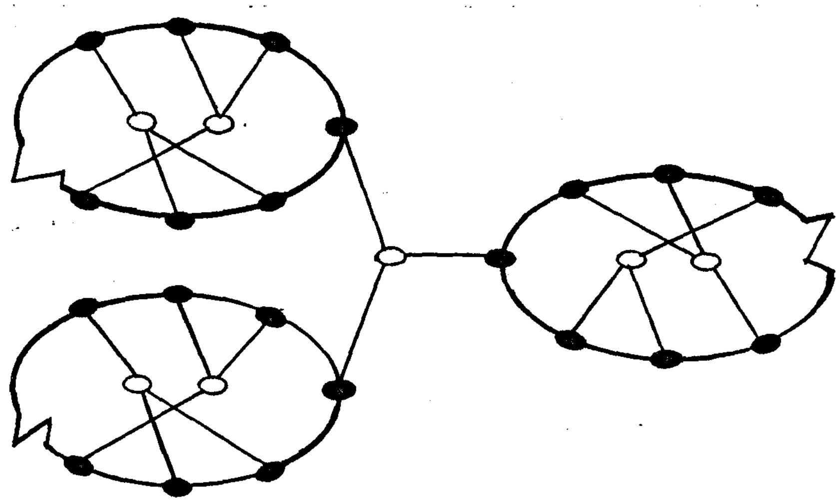


Figure 6.10

dim $\mathcal{E}(\mu) = k$. We illustrate this labelling in Figure 6.11.

Recall that here $\mu = -1$ with star set X such that $|X| = 7$, and so we want to find seven linearly independent vectors.

The first illustration shows six vectors; the second a further vector, and the seven vectors are linearly independent as required.

We now consider the case where $G - X$ has a cycle of a length congruent to 2 mod 3. Referring to Table 6.2 we see that we must consider three determinants. We observe that $\det M(0,0,2) \neq 0$, $\det M(1,2,2) \neq 0$ but $\det M(0,1,1) = 0$. Thus we may conclude that -1 is an eigenvalue of H if and only if the paths between two of the vertices adjacent to the central

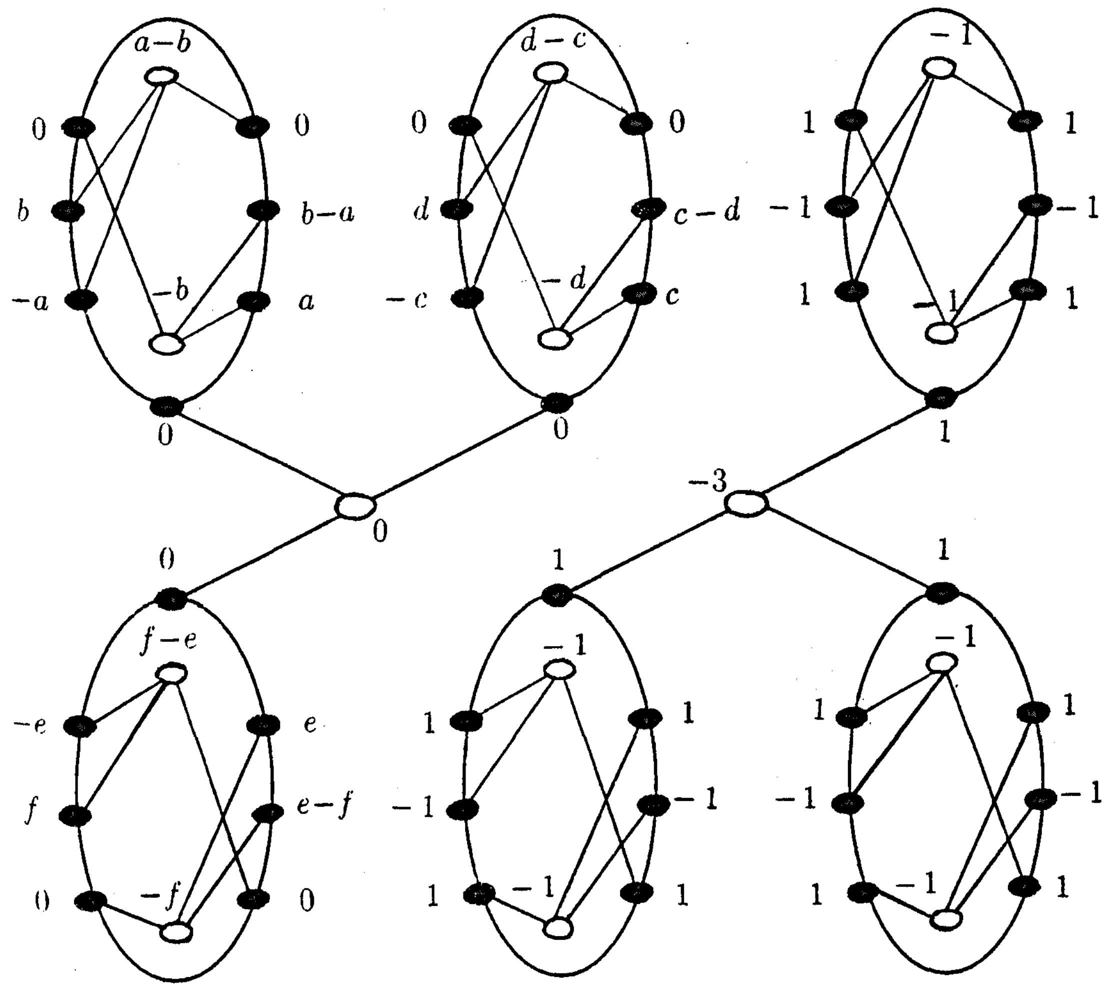


Figure 6.11

vertex G are of length congruent to 1 mod 3, and the other path is of length congruent to 0 mod 3. This determines the configuration of vertices within one of the cycles of $G - X$.

Note that in a cycle of length congruent to 2 mod 3 there will always be two vertices remaining of degree 2 after we have joined all the other vertices to a vertex in X. These vertices must also be adjacent to a vertex in X, suggesting that $G-X$ is a union of two cycles, one of some length congruent to 2 mod 3, the other of some length congruent to 1 mod 3.

We give three examples of possible configurations for a cycle of length eight in Figure 6.12.

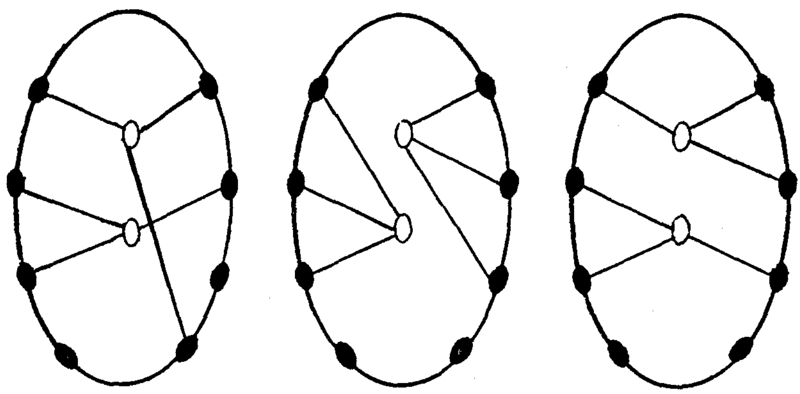


Figure 6.12

So far we have determined the possible configurations for the cycles in $G - X$ and given one example in Figure 6.10 of a graph where $G - X$ is a union of three cycles all of some length congruent to 1 mod 3. We have already stated that at least one of the cycles in $G - X$ must have length congruent to 1 mod 3. Further investigation suggests that $G - X$ always consists of three or fewer cycles. Of the possible combinations only the two already mentioned satisfy the condition $|\overline{X}| = 3|X|$. These are:

(i) $G-X$ is a union of three cycles, each of some length congruent to 1 mod 3.

(ii) $G-X$ is a union of two cycles, one of a length congruent to 1 mod 3, and the other of a length congruent to 2 mod 3.

This concludes the additional work done in the case where $r = 2$ and $\mu = -1$. In later chapters we will consider only graphs with eigenspaces of prescribed codimension; in other words we will specify the number of vertices in $G-X$. That is not the case here where the only condition is that G is cubic. We have found infinitely many examples for G. These examples are core graphs since no two vertices in X have the same $\overline{X}$-neighbourhood (see Definition 4.10). We cannot add duplicate vertices without relaxing the condition that G is cubic.

The investigation of other cases can be found in [Row1].

Chapter 7

Star complement $K_{r,s} \cup t\,K_1$.

In the next four chapters we shall be exploring the idea of constructing a graph G from a star complement H, and an associated eigenvalue μ. We let X be a star set for μ in the graph G constructed with $G - X = H$. The case where $G = H + u$ is trivial and so we shall assume that H is μ-extendible and that $|X| \geq 2$ except where otherwise stated. In Chapter 10 we shall restrict ourselves further and assume that $|X| \geq 3$ in the general case although we do give an example in the case $|X| = 2$.

Given a particular star complement our first objective is to find a range of possible values for μ. This task is expedited if we can assume that μ is an integer. In the case where $H \cong K_{r,s}$ in Chapter 8 we go to considerable lengths to exclude the case when μ is an not integer. We note that μ must be an integer if $|X|$ is large enough:

Remark 7.1 *Let X be a star set for μ in G and suppose that $H = G - X$. If $|X| \geq |V(H)|$ then μ is an integer.*

Let $|X| = m$ and $|V(H)| = t$. Suppose that μ is not an integer. Then it has an algebraic conjugate $\overline{\mu} \neq \mu$. The set X is a star set for both μ and $\overline{\mu}$, both of multiplicity m, and so G has at least $2m + 1$ eigenvalues. Thus $|V(G)| > 2m$, but $|V(G)| = m + t$ and so $m + t > 2m$ whence $m < t$ as required.

This means that if we wish the number of vertices added to H to exceed the number of vertices in H then we need only concern ourselves with integer eigenvalues.

Before we proceed any further it would be appropriate to explain why $K_{r,s} \cup t K_1$ was chosen as our star complement. Recall that the underlying purpose is to be able to identify, or at least classify in some way, a graph G by a star complement H and an associated eigenvalue. Ideally the ratio of $|V(G)|$ to $|V(H)|$ should be as high as possible and so we naturally turn to the strongly regular graphs as possible examples. In Chapter 1 we obtained a picture of McL_{112} by studying the construction of Higman's regular two-graph (see Example 1.26). This graph has 275 vertices; the eigenvalues are 112, 2 and -28 with multiplicities 1, 252 and 22 respectively. This indicates that we should be looking for a star complement with $275 - 252 = 23$ vertices corresponding to the eigenvalue 2. In Example 1.26 we observed that $K_{1,16} \cup 6\,K_1$ was an induced subgraph of McL_{112}. This subgraph has 23 vertices and does not have 2 as an eigenvalue; therefore it is a star complement for 2 in McL_{112}. Knowing this we knew that we should be able to construct the McL_{112} from $H \cong K_{1,16} \cup 6\,K_1$ and the associated eigenvalue 2, provided we could obtain enough information about how a vertex could be added to $K_{1,16} \cup 6\,K_1$.

First we consider the general case as far as practicable. Let H be a star complement $K_{r,s} \cup t\,K_1$ with $1 \leq r \leq s$ and let μ be the corresponding eigenvalue. We see immediately that $\mu \neq 0$ since 0 is an eigenvalue of $K_{r,s} \cup t\,K_1$. Thus, by Theorem 4.1, $V(H)$ is a dominating set. In particular, if H is connected, that is, if $t = 0$, then $H + u$ is connected.

We partition the vertex set of H into three independent sets, R, S and T with $|R| = r$, $|S| = s$ and $|T| = t$ so that each vertex in R is adjacent to every vertex in S. Let H be μ-extendible and write $H + u + v$ for some graph constructed by adding two vertices u and v. We suppose that u is of type (a, b, c), meaning that it is adjacent to a vertices in R, b vertices in S and c in T. Similarly v is adjacent to α vertices in R, β vertices in S and γ in T and so is of type (α, β, γ).

We find that it is more convenient to apply the Reconstruction Theorem in a modified form; we write:

$$f(\mu)(\mu I - A) = B^T f(\mu)(\mu I - C)^{-1} B \tag{7.1}$$

where $f(x) = x(x^2 - rs)$ is the minimal polynomial of H. We then apply this to $H+u+v$. The row of B^T corresponding to the vertex u has the form $\mathbf{b}_u^T = (\mathbf{b}_u^a|\mathbf{b}_u^b|\mathbf{b}_u^c)$, the weights of the sub-vectors being a, b and c. Similarly the row corresponding to the vertex v has the form $\mathbf{b}_v^T = (\mathbf{b}_v^\alpha|\mathbf{b}_u^\beta|\mathbf{b}_u^\gamma)$, the weights of the sub-vectors being α, β and γ. The matrix A is the adjacency matrix of the graph induced by $\{u, v\}$. The matrix C is the adjacency matrix of the star complement H so that C is the block matrix

$$C = \begin{pmatrix} O_{r\times r} & J_{r\times s} & O_{r\times t} \\ J_{s\times r} & O_{s\times s} & O_{s\times t} \\ O_{t\times r} & O_{t\times s} & O_{t\times t} \end{pmatrix}.$$

The first task is to find $(\mu I - C)^{-1}$. The reason for preferring star complements with few distinct eigenvalues now becomes apparent. There are three distinct eigenvalues of $K_{r,s} \cup t K_1$: $-\sqrt{rs}$, 0 and $\sqrt{rs}$ and so the minimal polynomial of C is a cubic. It follows that $\mu I - C$ satisfies a cubic and so $(\mu I - C)^{-1}$ is a quadratic in C. Clearly

$$C^2 = \begin{pmatrix} sJ & O & O \\ O & rJ & O \\ O & O & O \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} O & rsJ & O \\ rsJ & O & O \\ O & O & O \end{pmatrix},$$

with appropriate block sizes.

Let $(\mu I - C)^{-1} = xI + yC + zC^2$. Then

$$\begin{aligned} I &= (\mu I - C)(xI + yC + zC^2) \\ &= \mu x I + (\mu y - x)C + (\mu z - y)C^2 - zC^3 \\ &= \mu x I + (\mu y - x - zrs)C + (\mu z - y)C^2, \quad \text{since} \quad C^3 = rsC. \end{aligned}$$

Since the minimal polynomial of C has degree 3 the matrices I, C and C^2 are linearly independent and so we have the equations $\mu x - 1 = 0$, $\mu y - x - zrs = 0$ and $\mu z - y = 0$. Clearly $x = 1/\mu$ and $y = \mu z$ and so the second equation becomes $\mu^2 z - 1/\mu - zrs = 0$ whence

$$z = \frac{1}{\mu(\mu^2 - rs)} \quad \text{with} \quad y = \frac{1}{\mu^2 - rs}.$$

Thus, multiplying through by $f(\mu) = \mu(\mu^2 - rs)$, we have

$$\begin{aligned} f(\mu)(\mu I - C)^{-1} &= (\mu^2 - rs)I + \mu C + C^2 \\ &= \begin{pmatrix} \nu I + sJ_{r\times r} & \mu J_{r\times s} & O_{r\times t} \\ \mu J_{s\times r} & \nu I + rJ_{s\times s} & O_{s\times t} \\ O_{t\times r} & O_{t\times s} & \nu I_{t\times t} \end{pmatrix} \end{aligned}$$

where $\nu = \mu^2 - rs$.

Now $\mathbf{b}_u^T I \mathbf{b}_u = a + b + c$; $\mathbf{b}_u^T C \mathbf{b}_u = 2ab$ and $\mathbf{b}_u^T C^2 \mathbf{b}_u = a^2 s + b^2 r$. Putting these results together and equating the (u,u)-entries in Equation

(7.1) we obtain the equation

$$\mu^2(\mu^2 - rs) = (\mu^2 - rs)(a + b + c) + 2\mu ab + a^2 s + b^2 r. \tag{7.2}$$

Similarly, equating the (v, v)-entries yields the equation

$$\mu^2(\mu^2 - rs) = (\mu^2 - rs)(\alpha + \beta + \gamma) + 2\mu\alpha\beta + \alpha^2 s + \beta^2 r. \tag{7.3}$$

In order to equate the (u, v)-entries we note that $\mathbf{b}_u^T I \, \mathbf{b}_v = \rho$, the number of common neighbours in H of u and v; $\mathbf{b}_u^T C \, \mathbf{b}_v = a\beta + \alpha b$; and $\mathbf{b}_u^T C^2 \, \mathbf{b}_v = a\alpha s + b\beta r$. This yields the equation

$$\begin{aligned} -a_{uv}(\mu^2 - rs)\mu &= (\mu^2 - rs)\rho + a\alpha s + b\beta r + \mu(a\beta + \alpha b) \qquad (7.4) \\ & \text{where} \quad a_{uv} = \begin{cases} 1 & \text{if} \quad u \sim v \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that subtracting Equation (7.3) from Equation (7.2) yields a further equation. We summarise these results in the following remark.

Remark 7.2 *Let $H \cong K_{r,s} \cup t K_1$ be μ-extendible. Then the eigenvalue μ satisfies the three equations $g_1(\mu) = g_2(\mu) = g_3(\mu) = 0$ where*

$$\begin{aligned} g_1(x) &= (x^2 - rs)(a + b + c - x^2) + 2xab + a^2 s + b^2 r, & (7.5) \\ g_2(x) &= (x^2 - rs)(a + b + c - \alpha - \beta - \gamma) & (7.6) \\ & \quad +2x(ab - \alpha\beta) + s(a^2 - \alpha^2) + r(b^2 - \beta^2), & \\ g_3(x) &= (x^2 - rs)(\rho + a_{uv}x) + a\alpha s + b\beta r + x(a\beta + \alpha b). & (7.7) \end{aligned}$$

Here ρ *is the number of common neighbours in* H *of* u *and* v

and $a_{uv} = \begin{cases} 1 & if \quad u \sim v \\ 0 & otherwise. \end{cases}$

We now consider the particular case where $\mu = 2$, $r = 1$, $s = 16$ and $t = 6$.

Star complement $\mathbf{K_{1,16} \cup 6\,K_1}$.

Suppose that $H \cong K_{1,16} \cup 6\,K_1$ is μ-extendible, with $\mu = 2$.

The first step is to find the possible types for vertices u and v. We do this using the equation $g_1(2) = 0$ with $r = 1$, $s = 16$ and $t = 6$. Since $r = 1$, a can only take two values, 0 or 1; thus $a^2 = a$.

Let $a = 0$. Then $g_1(2) = -12b - 12c + 48 + b^2 = 0$. This yields a quadratic in b; $b^2 - 12b + 12(4 - c) = 0$. This equation has solution $b = 6 \pm 2\sqrt{3(c-1)}$ and so the possible values for c are $1, 2, 3, 4, 5$ and 6. We obtain integer solutions for b when $c \in \{1, 4\}$ and so the types arising when $a = 0$ are $(0, 0, 4)$, $(0, 6, 1)$ and $(0, 12, 4)$.

Let $a = 1$. Then $g_1(2) = -12b - 12c - 12 + 48 + 4b + 16 + b^2 = 0$. This yields a quadratic in b; $b^2 - 8b - 4(3c - 13) = 0$. This equation has solution $b = 4 \pm 2\sqrt{3(c-3)}$ and so the possible values for c are $3, 4, 5$ and 6. We obtain integer solutions for b when $c \in \{3, 6\}$ and so the types arising when $a = 1$ are $(1, 4, 3)$ and $(1, 10, 6)$.

This gives us 15 possible u, v pairs for $H + u + v$. We list the values for ρ obtained by solving $g_3(2) = 0$ for each pair in Table 7.1.

u, v pairs	ρ $u \sim v$	ρ $u \not\sim v$
$(0,0,4)$, $(0,0,4)$	-2	0
$(0,0,4)$, $(0,6,1)$	-2	0
$(0,0,4)$, $(0,12,4)$	-2	0
$(0,0,4)$, $(1,4,3)$	-2	0
$(0,0,4)$, $(1,10,6)$	-2	0
$(0,6,1)$, $(0,6,1)$	1	3
$(0,6,1)$, $(1,4,3)$	1	3
$(1,4,3)$, $(1,4,3)$	1	3
$(0,6,1)$, $(0,12,4)$	4	6
$(0,6,1)$, $(1,10,6)$	4	6
$(1,4,3)$, $(0,12,4)$	4	6
$(1,4,3)$, $(1,10,6)$	4	6
$(0,12,4)$, $(0,12,4)$	10	12
$(0,12,4)$, $(1,10,6)$	10	12
$(1,10,6)$, $(1,10,6)$	10	12

Table 7.1

The value for ρ must be a non-negative integer and so we see at once that we cannot have two adjacent vertices when one is of the type $(0,0,4)$. Indeed, inclusion of a vertex of type $(0,0,4)$ precludes the addition of any vertex other than a non-adjacent vertex of type $(0,6,1)$. Clearly this graph is not connected. Little more can be said of this graph but we give one example. In accordance with the Reconstruction Theorem it is sufficient to give an example of the matrix B.

Example.

$$B^T = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{pmatrix} 0 & 0000\,0000\,0000\,0000 & 00\,1111 \\ 0 & 1100\,0011\,1100\,0000 & 10\,0000 \\ 0 & 0011\,0011\,0011\,0000 & 10\,0000 \\ 0 & 0000\,1111\,0000\,1100 & 10\,0000 \\ 0 & 0101\,0111\,0000\,0010 & 01\,0000 \\ 0 & 1100\,1010\,0000\,1001 & 01\,0000 \end{pmatrix} \begin{array}{c} (0,0,4) \\ (0,6,1) \\ (0,6,1) \\ (0,6,1) \\ (0,6,1) \\ (0,6,1). \end{array}$$

This matrix is constructed using the rules that no row has any element in common with row a and that each subsequent row pair has only one or three elements in common, corresponding to $\rho = 1$ or 3. From this it is easy to see that the graph induced by the star set X is $K_2 \cup 4\,K_1$ since the row-pair (c, f) is the only one with just one vertex in common; all other row-pairs having three. It is almost certain that this graph is not maximal and it is certainly not unique. However it does serve to illustrate the general difficulty of constructing these graphs when there is no discernible pattern. (The reader is invited to see how many more rows he can add by hand to this matrix.) There is a computer program designed by M. Lepović (see [CvLRoS] for details) which in principle will find all maximal graphs for a given star complement. Unfortunately $K_{1,16} \cup 6\,K_1$ is too big, and the program is likely to require substantial modification if it is to deal successfully with even the restricted problem of vertices of type $(0,0,4)$ and $(0,6,1)$.

We shall restrict ourselves to connected graphs, and so vertices of type $(0,0,4)$ will not feature. We know that we should be able to construct the graph McL_{112} from $H \cong K_{1,16} \cup 6\,K_1$ and the associated eigenvalue 2. The question remains as to whether or not this is the only maximal graph which arises although it is highly unlikely. However we shall see that we can say that McL_{112} is the largest maximal graph which can be constructed from H. The process of construction was facilitated by an observation on the part of P. Rowlinson. In Example 1.26 we constructed a graph on the vertex set composed of the points and blocks of the unique Steiner system $S(4,7,23)$. We found that this graph was switching equivalent to McL_{162}, the complement of McL_{112}. It was observed that every step in the transformation from $S(4,7,23)$ to McL_{112} was reversible. This means that if we start with an arbitrary graph and perform the transformation in reverse, if the graph we obtain defines $S(4,7,23)$ then we may deduce that the initial graph was McL_{112}. We illustrate this reversibility process in the following theorem.

Theorem 7.3 *Let G be the largest connected maximal graph with star complement $H \cong K_{1,16} \cup 6\,K_1$ corresponding to the eigenvalue 2. Then $G \cong McL_{112}$.*

Proof. Let H_g be the graph arising from $S(4,7,23)$ which yields Higman's regular two-graph design. Fix a point-vertex in H_g and switch with respect to the neighbourhood of this vertex. Delete this vertex and take the complement to obtain McL_{112}. We now take the largest maximal graph with star complement $K_{1,16} \cup 6\,K_1$ and reverse the procedure, thus obtaining a graph which fully defines the unique Steiner system $S(4,7,23)$; the result follows.

Let X be a star set for 2 in G and let H be the star complement for 2 in G. As before we can partition the vertices of H into three independent sets, R, S and T with $|R| = 1$, $|S| = 16$ and $|T| = 6$ so that each vertex in S is adjacent to the vertex in R. It has already been shown that if a vertex is in X then it will be of type $(0,6,1)$, $(1,4,3)$, $(0,12,4)$ or $(1,10,6)$. We now partition the vertices of G into three sets, U, V and W where $W = S \cup T$, an independent set. Each vertex u in U is adjacent to 7 vertices in W, and each vertex v in V is adjacent to 16 vertices in W. Clearly the set U contains vertices of types $(0,6,1)$ and $(1,4,3)$, and V contains vertices of types $(0,12,4)$ and $(1,10,6)$ together with the single vertex in R. We now transform G in the following way: first we take the complement $\overline{G}$ of G;

then we add an isolated vertex x and switch with respect to the vertex set $W \cup V$. The vertex set of the resulting graph can be partitioned into two sets, say B and Y where $Y = W \cup \{x\}$ and $B = U \cup V$.

We now examine what happens to individual vertices under these transformations.

In G a vertex $u \in U$ is adjacent to 7 vertices in W; in $\overline{G}$ it is adjacent to $22 - 7 = 15$ in W. After switching it is again adjacent to 7 vertices in W. Moreover, since the set U and the vertex x are both outwith the switching set, any vertex in U remains non-adjacent to x after switching and so we can say that any vertex in U is adjacent to 7 vertices in Y. Similarly a vertex $v \in V$ is first adjacent to 16 vertices in W in G. In $\overline{G}$ it is adjacent to $22 - 16 = 6$ vertices in W. This vertex is within the switching set and so remains adjacent to 6 vertices in W after switching. However the vertex x is outwith the switching set and so v becomes adjacent to x after switching. Thus v is adjacent to 7 vertices in Y in the final graph. We note in passing that the graph induced by Y is K_{23}.

We now examine what happens to pairs of vertices under these transformations.

Consider a pair u_1, u_2 of vertices in U which are adjacent in G. In G the number of common neighbours in W is 1. In $\overline{G}$ this number becomes

$22-(7+7-1)=9$ and we have $u_1 \not\sim u_2$. After switching we still have $u_1 \not\sim u_2$ but now the number of common neighbours in $W \cup \{x\} = Y$ is 1.

We consider each possible pair in the same way and write $\rho(W)$ for the number of common neighbours in W. The results are presented in Table 7.2.

vertex pair	$\rho(W)$ in G	$\rho(W)$ in $\overline{G}$	$\rho(W \cup \{x\})$ after switching
$u_1 \sim u_2$	1 $u_1 \sim u_2$	$22-(7+7-1)=9$ $u_1 \not\sim u_2$	1 $u_1 \not\sim u_2$
$u_1 \not\sim u_2$	3 $u_1 \not\sim u_2$	$22-(7+7-3)=11$ $u_1 \sim u_2$	3 $u_1 \sim u_2$
$v_1 \sim v_2$	10 $v_1 \sim v_2$	$22-(16+16-10)=0$ $v_1 \not\sim v_2$	1 $v_1 \not\sim v_2$
$v_1 \not\sim v_2$	12 $v_1 \not\sim v_2$	$22-(16+16-12)=2$ $v_1 \sim v_2$	3 $v_1 \sim v_2$
$u \sim v$	4 $u \sim v$	$22-(7+16-4)=3$ $u \not\sim v$	$7-4=3$ $u \sim v$
$u \not\sim v$	6 $u \not\sim v$	$22-(7+16-6)=5$ $u \sim v$	$7-6=1$ $u \not\sim v$

Table 7.2

From this we can see that each vertex in B can be identified with a 7-set taken from the 23 points or vertices in Y, each set incident in either 1 or 3 points. Thus for a given 4-set of Y, there exists at most one 7-set containing it. There are ${}^{23}C_4$ possible 4-sets. However each 7-set contains ${}^{7}C_4$ 4-sets and so the maximum number of 7-sets is $\frac{{}^{23}C_4}{{}^{7}C_4} = 253$. Thus, when we have the maximum star set in G, we have the maximum set B and so we have precisely one 7-set containing a given 4-set taken from 23 points. This fully describes the unique Steiner triple system $S(4, 7, 23)$ and the graph constructed is H_g. Thus we may conclude that if $H \cong K_{1,16} \cup 6\, K_1$ and $\mu = 2$ then the largest maximal graph is McL_{112}. $\square$

Doubtless this is not the only maximal graph but it is certainly the largest. What is not known is how big G has to be in order to ensure that it is a subgraph of McL_{112}. Let $\mathcal{F}$ be any family of 6-subsets of a 23-set such that any two members of $\mathcal{F}$ intersect in 1 or 3 elements. We require the least value M such that if $|\mathcal{F}| \geq M$ then $\mathcal{F}$ can be embedded in $S(4, 7, 23)$. This is an outstanding problem in design theory. Note that if $|X| \geq M$ then G is an induced subgraph of McL_{112}.

Chapter 8

Star complement $K_{r,s}$.

In the previous chapter we examined the case where $K_{r,s} \cup t\,K_1$ was the star complement. Only one general result was given in Remark 7.2. We now let $t = 0$ and explore how much can be said about μ in the general case where the star complement H is a graph isomorphic to $K_{r,s}$ $(r \leq s)$.

First we examine the possible eigenvalues for $H + u$. In order to do this we must first find the generic minimal polynomial for $H + u$. Initially this was done by considering a divisor of $H + u$. It is this method which we now illustrate.

Let the vertices of H be divided into two independent sets, R and S such that $|R| = r$, $|S| = s$ and each vertex in R is adjacent to every vertex in S. Let $H + u$ be a graph formed by adding a vertex u adjacent to a vertices in

R and b vertices in S, so that there are $(r+1)(s+1)$ possibilites for $H+u$, dependent on the values of a and b.

Thus $V(H+u)$ may be partitioned into five sets: $\{u\}$, $\{w \in R : w \sim u\}$, $\{w \in R : w \not\sim u\}$, $\{w \in S : w \not\sim u\}$, and $\{w \in S : w \sim u\}$ with cardinalities 1, a, $r-a$, $s-b$ and b respectively. Thus the corresponding divisor of $H+u$ has adjacency matrix

$$D = \begin{pmatrix} 0 & a & 0 & 0 & b \\ 1 & 0 & 0 & s-b & b \\ 0 & 0 & 0 & s-b & b \\ 0 & a & r-a & 0 & 0 \\ 1 & a & r-a & 0 & 0 \end{pmatrix}$$

and characteristic polynomial

$$\det(xI - D) = x(b^2r + a^2s + (a+b)(x^2 - rs) + 2abx - x^2(x^2 - rs)).$$

By [CvDS, Thm 4.5], the characteristic polynomial of D divides the characteristic polynomial of $H+u$. By the Interlacing Theorem, the eigenvalues of $H+u$ interlace with those of H, giving:

$$\mu_1 \geq \sqrt{rs} \geq \mu_2 \geq 0 \geq \ldots \geq 0 \geq \mu_{r+s} \geq -\sqrt{rs} \geq \mu_{r+s+1}.$$

Thus we can see that there are at most four non-zero eigenvalues of $H+u$ and we may deduce that the minimal polynomial of $H+u$ is $\det(xI - D)$.

It follows that if μ is a non-zero eigenvalue of $H+u$ then μ satisfies

$$\mu^2(\mu^2 - rs) = (\mu^2 - rs)(a+b) + a^2 s + b^2 r + 2ab\mu. \tag{8.1}$$

We note in passing that this method can be extended to find the minimal polynomial of $H+u+v$.

Another way in which we can find the minimal polynomial of $H+u$ is by applying the Reconstruction Theorem to the graph $H+u+v$. From Remark 5.13 we have Equation (5.4):

$$\Phi_{H+u}(x) = \frac{\Phi_H(x) g_1(x)}{f(x)},$$

where $f(x)$ is the minimal polynomial of H and g_1 is as given in Remark 7.2 with $t = 0 = c$. Here $H \cong K_{r,s}$ and so $\Phi_H(x) = (x^2 - rs)x^{n-2}$. It follows that

$$\Phi_{H+u}(x) = \frac{(x^2 - rs)x^{n-2} g_1(x)}{(x^2 - rs)x} = x^{n-3} g_1(x)$$

and so the minimal polynomial of $H+u$ is simply $x\, g_1(x)$. Now $\mu \neq 0$ since $0 \in Sp(H)$ and so we can say that in the case where $H \cong K_{r,s}$ is a star complement for μ, μ satisfies $g_1(x) = 0$.

Lemma 8.1 *Let H be a star complement $K_{r,s}$ with corresponding eigenvalue μ, so that $\mu \in Sp(H+u)$ and $\mu \notin Sp(H)$. Then $H+u$ is connected and either $|\mu| > \sqrt{rs}$ or $|\mu| < \sqrt{rs}$, with*

$$-\frac{1}{4}\left(\sqrt{1+4(r+s)}+1\right) \leq \mu \leq \frac{1}{4}\left(\sqrt{1+4(r+s)}-1\right)$$

when $|\mu| < \sqrt{rs}$.

Proof. The eigenvalues of $K_{r,s}$ are $-\sqrt{rs}, 0, \ldots, 0, \sqrt{rs}$. By definition μ is not an eigenvalue of $K_{r,s}$, in particular μ is non-zero. Therefore $H + u$ must be connected since the addition of an isolated vertex would produce a graph with the same distinct eigenvalues, the multiplicity of zero merely increasing by one. Put more simply, since $\mu \neq 0$, $V(H)$ is a dominating set. Consequently, since H is connected, $H + u$ is also connected. Furthermore it is clear that either $|\mu| < \sqrt{rs}$ or $|\mu| > \sqrt{rs}$.

Now suppose that $|\mu| < \sqrt{rs}$. We consider the expression

$$(a - a')^2 s + (b - b')^2 r + 2(a - a')(b - b')\mu$$

and compare the coefficients of a and b with those in Equation (8.1). This gives us the following equations which we use to define a' and b'.

$$2a's + 2\mu b' = rs - \mu^2$$

and

$$2b'r + 2\mu a' = rs - \mu^2.$$

This yields

$$b' = \frac{1}{2}(s - \mu) \quad \text{and} \quad a' = \frac{1}{2}(r - \mu).$$

Hence we may rewrite Equation (8.1) as follows:

$$\frac{s}{4}(2a - r + \mu)^2 + \frac{r}{4}(2b - s + \mu)^2 + \frac{2\mu}{4}(2a - r + \mu)(2b - s + \mu) =$$

$$\mu^2(\mu^2 - rs) + \frac{s}{4}(r-\mu)^2 + \frac{r}{4}(s-\mu)^2 + \frac{2\mu}{4}(r-\mu)(s-\mu).$$

Further simplification gives us:

$$(rs - \mu^2)X^2 + (rY + \mu X)^2 = r(r + s - 2\mu - 4\mu^2)(rs - \mu^2) \qquad (8.2)$$

where

$$X = 2a - r + \mu \quad \text{and} \quad Y = 2b - s + \mu.$$

From this we can see that when $\mu^2 < rs$, we must have $(r+s-2\mu-4\mu^2) \geq 0$. Hence, when $|\mu| < \sqrt{rs}$, we have

$$-\frac{1}{4}\left(\sqrt{1+4(r+s)}+1\right) \leq \mu \leq \frac{1}{4}\left(\sqrt{1+4(r+s)}-1\right).$$

□

Lemma 8.2 *Let $H \cong K_{r,s}$ be a star complement with corresponding eigenvalue μ. Then $|\mu| \leq \lambda$ where λ is the maximal index of the generic graph $H + u$.*

Proof. By Remark 1.11 the absolute value of each eigenvalue of a graph does not exceed its index.

The maximal index of the generic graph $H + u$ is the index of $H + u$ when u is adjacent to every vertex in H since the deletion of an edge will never increase the value of the index, [CvDS, Thm 0.7]. The result follows.

□

Lemma 8.3 *Let H be μ-extendible and let μ be positive. Then*

$$\mu \leq \frac{1}{4}\left(\sqrt{1+4(r+s)}-1\right).$$

Proof. By the Interlacing Theorem a graph $H+u+v$ has at most six non-zero eigenvalues, of which three are positive. If μ is a double eigenvalue of $H+u+v$ it must also be an eigenvalue of $H+u$. Suppose that $\mu > \sqrt{rs}$: then μ is the index of $H+u+v$ with multiplicity 2. By Remark 1.13 this implies that the graph is not connected. However we have already established that $H+u+v$ must be connected since both $H+u$ and $H+v$ are connected, and so $\mu < \sqrt{rs}$. The result follows from Lemma 8.1. □

From these lemmas we can see that the eigenvalues we are looking for fall into three intervals:

(*i*) $-\lambda \leq \mu < -\sqrt{rs}$ where λ is the maximal index;

(*ii*) $-\frac{1}{4}(\sqrt{1+4(r+s)}+1) \leq \mu < 0$ and

(*iii*) $0 < \mu \leq \frac{1}{4}(\sqrt{1+4(r+s)}-1)$.

If we can now show that μ is an integer, or that μ^2 is an integer, then for a particular r and s we will know precisely which eigenvalues to consider. The following propositions attempt to do just that. For the remainder of this chapter we consider the graphs $H+u+v$ where $H \cong K_{r,s}$. We partition the vertex set of H into two independent sets R and S as before. We suppose that u is of type (a,b), meaning that it is adjacent to a vertices in

R and b vertices in S. Similarly we say that v is of type (α, β). We apply the Reconstruction Theorem and in accordance with Remark 7.2 find that $g_1(\mu) = g_2(\mu) = g_3(\mu) = 0$ where

$$\begin{aligned}
g_1(x) &= (x^2 - rs)(a + b - x^2) + 2xab + a^2 s + b^2 r \\
g_2(x) &= (x^2 - rs)(a + b - \alpha - \beta) \\
&\qquad +2x(ab - \alpha\beta) + s(a^2 - \alpha^2) + r(b^2 - \beta^2) \\
g_3(x) &= (x^2 - rs)(\rho + a_{uv}x) + a\alpha s + b\beta r + x(a\beta + \alpha b)
\end{aligned}$$

where ρ is the number of common neighbours in H of u and v and $a_{uv} = \begin{cases} 1 & \text{if} \quad u \sim v \\ 0 & \text{otherwise.} \end{cases}$

When u and v are vertices of a different type then the eigenvalue μ will be an eigenvalue of both $H + u$ and $H + v$. Thus, if we know the spectra of all the graphs $H + u$ we can identify possible eigenvalues as they will appear in more than one spectrum. However, if u and v are vertices of the same type then, on this basis, every eigenvalue in the spectrum of each $H + u$ is a candidate. Fortunately we are able to impose quite stringent conditions in this case using the relevant propositions.

In the following proofs we shall make use of the fact that if μ is not an integer and we have an equation of the form $P\mu + Q = 0$ where P and Q are both rational, then $P = 0$ and $Q = 0$.

Proposition 8.4 *Let $H \cong K_{r,s}$ be a star complement for μ in $H+u+v$. Let u and v be non-adjacent vertices of the same type. Then $\mu^2 = a+b-\rho$. Furthermore, if μ is not an integer then $ab = 0$ and $a^2s + b^2r + \rho(a+b-\rho - rs) = 0$ with*

$$|\mu| \leq \frac{1}{4}\left(\sqrt{1+4(r+s)} - 1\right).$$

Proof. We have

$$g_3(\mu) = (\mu^2 - rs)\rho + 2ab\mu + a^2s + b^2r.$$

Taking the difference $g_1(\mu) - g_3(\mu)$ and dividing by $(\mu^2 - rs)$ we get $\mu^2 = a+b-\rho$, as required. Now suppose that μ is not an integer. Using the substitution $\mu^2 = a+b-\rho$ in $g_3(\mu) = 0$ we obtain

$$0 = (a+b-\rho-rs)\rho + 2ab\mu + a^2s + b^2r,$$

which is an equation of the form $P\mu + Q = 0$ where $P = 2ab$ and $Q = a^2s + b^2r + \rho(a+b-\rho-rs)$. Since μ is not an integer we have $ab = 0$ and $a^2s + b^2r + \rho(a+b-\rho-rs) = 0$. Moreover, since μ^2 is an integer, $-\mu$ is an algebraic conjugate of μ, and both are double eigenvalues of $H+u+v$. Since one of these must be positive, the result follows from Lemma 8.3. □

Proposition 8.5 *Let $H \cong K_{r,s}$ be a star complement for μ in $H+u+v$.*

If $H+u+v$ is bipartite then either μ is an integer, or μ^2 is an integer with $|\mu| \leq \frac{1}{4}\left(\sqrt{1+4(r+s)}-1\right)$.

Proof. Suppose that μ is a non-integral double eigenvalue of the bipartite graph $H+u+v$ and that μ^2 is not an integer. Then μ has an algebraic conjugate $\bar{\mu}$ such that $\pm\mu$ and $\pm\bar{\mu}$ are four distinct eigenvalues of $H+u+v$. Each is non-zero with multiplicity two. However, by the Interlacing Theorem, $H+u+v$ has at most six non-zero eigenvalues. Consequently either μ is an integer, or μ^2 is an integer and then, as in Proposition 8.4, $|\mu| \leq \frac{1}{4}\left(\sqrt{1+4(r+s)}-1\right)$. □

For the following proposition it is necessary to restrict the value of r so that $r \geq 2$. We shall deal with the case $r=1$ separately.

Here we find that $\mu^2+\mu=h$ where h is bounded in terms of r and s. In Chapter 9, when $H \cong K_{2,5}$, by considering the values of h we are able to evaluate μ when μ is an integer, and to eliminate the case when μ is not an integer.

Proposition 8.6 *Let $H \cong K_{r,s}$ be a star complement for μ in $H+u+v$. Let u and v be adjacent vertices of the same type (a,b) and let $a+b-\rho=h$. Suppose that $r \geq 2$. Then $|\mu| < \sqrt{rs}$ and $\mu^2+\mu=h$ where h is a non-*

negative integer such that

$$\sqrt{1+4h} \leq \frac{1}{2}\left(\sqrt{1+4(r+s)}+1\right).$$

Furthermore, if μ is not an integer then

$$(a+b-h-ab)^2 = a^2b^2 + a + b - a^2s - b^2r.$$

Proof. We have

$$g_3(\mu) = (\mu^2 - rs)(\rho+\mu) + 2ab\mu + a^2s + b^2r.$$

Taking the difference $g_1(\mu) - g_3(\mu)$ and dividing through by $(\mu^2 - rs)$ we get $\mu^2 + \mu = a + b - \rho = h$. It follows that $\mu = -\frac{1}{2} \pm \frac{\sqrt{1+4h}}{2}$ and so $|\mu| \leq \frac{1}{2}(1+\sqrt{1+4h})$. Now $(r+s) \geq 2(a+b) - \rho = 2h + \rho$ and so

$$0 \leq h \leq \frac{r+s}{2}. \tag{8.3}$$

We improve upon this bound by showing first that $|\mu| \leq \sqrt{rs}$. (Note that $|\mu| \leq \sqrt{rs}$ implies that $|\mu| < \sqrt{rs}$ since $\sqrt{rs}$ is an eigenvalue of H.) Clearly $|\mu| \leq \frac{1}{2}(1+\sqrt{1+2(r+s)})$ and so in order to prove that $|\mu| \leq \sqrt{rs}$, it is sufficient to show that $1+\sqrt{1+2(r+s)} \leq 2\sqrt{rs}$. Squaring both sides we have

$$1 + 2\sqrt{1+2(r+s)} + 1 + 2(r+s) \leq 4rs,$$

equivalently,

$$\sqrt{1+2(r+s)} \leq 2rs - (r+s+1).$$

Again we square both sides to obtain

$$1+2(r+s) \leq 4r^2s^2 - 4r^2s - 4rs^2 + r^2 - 2rs + s^2 + 1 + 2(r+s),$$

equivalently $4r^2s^2 - 4r^2s - 4rs^2 + (r-s)^2 \geq 0$, that is

$$2r^2s(s-2) + 2rs^2(r-2) + (r-s)^2 \geq 0. \tag{8.4}$$

This inequality holds for $2 \leq r \leq s$, and so we have $|\mu| < \sqrt{rs}$. We now apply Lemma 8.1 in the case where $|\mu| < \sqrt{rs}$.

When $\mu < 0$ we have $\frac{1}{2}(1+\sqrt{1+4h}) \leq \frac{1}{4}(\sqrt{1+4(r+s)}+1)$ and so $\sqrt{1+4h} \leq \frac{1}{2}(\sqrt{1+4(r+s)}-1)$.

Similarly when $\mu > 0$ we have $\frac{1}{2}(-1+\sqrt{1+4h}) \leq \frac{1}{4}(\sqrt{1+4(r+s)}-1)$ and so $\sqrt{1+4h} \leq \frac{1}{2}(\sqrt{1+4(r+s)}+1)$, as required.

Now suppose that μ is not an integer. Substituting $a+b-\rho-\mu$ for μ^2 in $g_3(\mu)=0$ we have an equation of the form $P\mu+Q=0$ where

$$P = 2ab+a+b+1-2\rho-rs$$

and

$$Q = \rho(a+b+1-\rho-rs) + a^2s + b^2r - a - b.$$

Since μ is not an integer, $P=0$ and $Q=0$ and so $Q = \rho(\rho-2ab) + a^2s + b^2r - a - b$. Completing the square we have $(\rho-ab)^2 = a^2b^2+a+b-a^2s-b^2r$ and the result follows. $\square$

Proposition 8.7 *Let $H \cong K_{r,s}$ be a star complement for μ in $H+u+v$. Let u and v be adjacent vertices of the same type (a,b) and let $a+b-\rho=h$. Let $r=1$. Then $\mu^2+\mu=h$ where h is a non-negative integer, and we have one of the following.*

(a) When $s \geq 6$, $|\mu| < \sqrt{s}$ and h is such that

$$\sqrt{1+4h} \leq \frac{1}{2}(\sqrt{5+4s}+1).$$

(b) When $s < 6$, $h \in \{0,1,2,3\}$.

Furthermore when μ is not an integer, $H+u+v$ is a 5-cycle.

Proof. That $\mu^2+\mu=h$ follows directly from Proposition 8.6.

The final bound for h found in Proposition 8.6 depended on $|\mu| < \sqrt{rs}$ which in turn depended upon the validity of the inequality (8.4). When $r=1$ we need to show that $|\mu| < \sqrt{s}$. The inequality (8.4) becomes $s(s-6) \geq -1$. Clearly this holds when $s \geq 6$ and thus we obtain the bound

$$\sqrt{1+4h} \leq \frac{1}{2}(\sqrt{5+4s}+1)$$

in this case.

When $s < 6$ we must be content with the weaker bound given by (8.3) in Proposition 8.6, namely $0 \leq h \leq \frac{r+s}{2}$, and so $h \in \{0,1,2,3\}$ as required.

Now suppose that μ is not an integer. We apply the condition given in

Proposition 8.6 with $r = 1$:

$$(a + b - h - ab)^2 = a^2b^2 + a + b - a^2s - b^2.$$

Since $r = 1$ we have only two cases to consider, $a = 0$ and $a = 1$.

When $a = 1$ we have $(1 - h)^2 = 1 + b - s$, that is $h(h-2) = b - s$. Since $b - s \leq 0$, $h \in \{0, 1, 2\}$. If $h = 0, 2$ then μ is an integer and so $h = 1$ with $b = s - 1$ and $\mu^2 + \mu = 1$. However if we apply $g_1(\mu) = 0$ in this case where $r = 1$, $a = 1$ and $b = s - 1$ we obtain the equation

$$g_1(\mu) = (\mu^2 - s)(s - \mu^2) + 2\mu(s - 1) + s + (s - 1)^2.$$

Now $\mu^2 = 1 - \mu$ and so we have

$$\begin{aligned} g_1(\mu) &= (1 - \mu - s)(s - 1 + \mu) + 2\mu(s - 1) + s(s - 1) + 1 \\ &= (s - 1) + 1 - \mu^2 \\ &= s - 1 + \mu = 0. \end{aligned}$$

Thus $\mu = 1 - s$, an integer, and we have a contradiction when $a = 1$.

When $a = 0$, $b \neq 0$ and we have $(b - h)^2 = b(1 - b) \geq 0$ whence $b = 1 = h$ and $\rho = 0$. Again we apply $g_1(\mu) = 0$ to obtain the equation

$$g_1(\mu) = (\mu^2 - s)(1 - \mu^2) + 1.$$

Since $\mu^2 = 1 - \mu$ we have

$$g_1(\mu) \quad = \quad (1 - \mu - s)\mu + 1$$

$$= \mu(1-s)-(1-\mu)+1$$

$$= \mu(2-s)=0,$$

which offers no contradiction. Indeed, since $\mu \neq 0$, $s = 2$. The maximum number of vertices of type $(0,1)$ that we can add with $\rho = 0$ is 2 and so we have a 5-cycle. $\square$

Remark 8.8 *Let X be a star set in G for the non-integer eigenvalue μ. If $G - X \cong K_{1,s}$ and X contains 2 adjacent vertices of the same type then $s = 2$, $\mu = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ and $G \cong C_5$.*

This follows by inspection of one-vertex extensions of $K_{1,2}$. This graph will be rediscovered in Proposition 10.4 in Chapter 10.

For the proof of the following proposition it is necessary that $r < s$.

Proposition 8.9 *Let $H \cong K_{r,s}$ $(r < s)$ be a star complement for μ in $H+u+v$. Let u and v be of different types (a,b) and (α,β) and let $a+b = \alpha+\beta$. Then $ab \neq \alpha\beta$ and μ is an integer.*

Proof. Suppose by way of contradiction that $ab = \alpha\beta$. Then

$$a^2+b^2 = (a+b)^2 - 2ab = (\alpha+\beta)^2 - 2\alpha\beta = \alpha^2+\beta^2$$

and so $a^2 - \alpha^2 = \beta^2 - b^2$. However, from $g_2(\mu) = 0$ we have $s(a^2-\alpha^2) =$

$r(\beta^2 - b^2)$ and so either $r = s$ or both $a^2 - \alpha^2$ and $\beta^2 - b^2$ are zero. Since $r \neq s$ and a, b, α, β are non-negative we get $a = \alpha$ and $b = \beta$, a contradiction.

Now suppose that $ab \neq \alpha\beta$. Then $g_2(\mu) = 0$ where

$$g_2(x) = 2x(ab - \alpha\beta) + s(a^2 - \alpha^2) + r(b^2 - \beta^2).$$

This is an equation of the form $P\mu + Q = 0$, where $P = 2(ab - \alpha\beta) \neq 0$ and $Q = s(a^2 - \alpha^2) + r(b^2 - \beta^2)$. In this instance both P and Q are non-zero integers and so μ is rational. Therefore μ is an integer. □

Proposition 8.10 *Let $H \cong K_{r,s}$ be a star complement for μ in $H + u + v$. Let u and v be of different types (a, b) and (α, β) and let $a + b \neq \alpha + \beta$. Then μ is an integer unless*

$$G(2H - G^2 + rs + a + b) - 2ab = 0$$

and

$$H(H - G^2 + rs + a + b) + rs(a + b) - a^2 s - b^2 r = 0,$$

where

$$G = \frac{2(ab - \alpha\beta)}{a + b - \alpha - \beta}$$

and

$$H = \frac{s(a^2 - \alpha^2) + r(b^2 - \beta^2)}{a + b - \alpha - \beta} - rs.$$

Proof. We re-write $g_2(\mu) = 0$ as follows:

$$\mu^2 + G\mu + H = 0$$

where

$$G = \frac{2(ab - \alpha\beta)}{a + b - \alpha - \beta}$$

and

$$H = \frac{s(a^2 - \alpha^2) + r(b^2 - \beta^2)}{a + b - \alpha - \beta} - rs.$$

From this equation we obtain the expressions $\mu^2 = -(G\mu + H)$ and $\mu^4 = \mu(2GH - G^3) + H(H - G^2)$. Substituting these into $g_1(\mu) = 0$ we obtain an equation of the form $P\mu + Q = 0$ where

$$P = G(2H - G^2 + rs + a + b) - 2ab$$

and

$$Q = H(H - G^2 + rs + a + b) + rs(a + b) - a^2s - b^2r.$$

Now μ is an integer unless $P = Q = 0$. □

The following proposition is an improvement upon Proposition 8.10 in the case where u and v are non-adjacent. We utilise the fact that $g_3(\mu) = 0$ yields a quadratic in this case.

Proposition 8.11 *Let $H \cong K_{r,s}$ be a star complement for μ in $H + u + v$. Let u and v be non-adjacent vertices of different types (a, b) and (α, β), with*

$a+b \neq \alpha+\beta$, and suppose that μ is not an integer. Then

$$(\alpha b - a\beta)(s(a^2+\alpha^2) - r(\beta^2+b^2)) = 0.$$

Proof. Firstly let $\rho = 0$. Then $g_3(\mu) = a\alpha s + b\beta r + \mu(a\beta + \alpha b) = 0$ and so μ is an integer unless $a\beta + \alpha b = 0$ and $a\alpha s + b\beta r = 0$. If $a\alpha s + b\beta r = 0$ then $a\alpha = 0$ and $b\beta = 0$. Since $a\alpha = 0$ either $a = 0$ or $\alpha = 0$. If $a = 0$ then $\alpha b = 0$ and $b\beta = 0$. Now $b \neq 0$ because we cannot add an isolated vertex and so $\alpha = \beta = 0$, a contradiction. We also get a contradiction when $\alpha = 0$ by symmetry. It follows that $\rho \neq 0$.

The minimal polynomial of the algebraic number μ is the unique monic polynomial f in $\mathbb{Q}[x]$ of least degree such that $f(\mu) = 0$. Now $g_2(\mu) = 0$ is a quadratic and so f divides g_2. Thus either f has degree 1, in which case μ is an integer, or f has degree 2 in which case $f = \gamma g_2$ for some $\gamma \in \mathbb{Q}$.

We re-write $g_2(x)$ as $x^2(a+b-\alpha-\beta) + 2x(ab-\alpha\beta) + d$, where

$$d = s(a^2-\alpha^2) + r(b^2-\beta^2) - rs(a+b-\alpha-\beta).$$

Thus

$$f(x) = x^2 + 2x\frac{(ab-\alpha\beta)}{(a+b-\alpha-\beta)} + \frac{d}{(a+b-\alpha-\beta)}.$$

From $g_3(\mu) = 0$ we can see that, since $\rho \neq 0$, μ also satisfies the quadratic

$$h(x) = x^2 + x\frac{a\beta+\alpha b}{\rho} + \frac{a\alpha s + b\beta r}{\rho} - rs,$$

and so, when μ is not an integer we have $h = f$. Equating coefficients, we obtain two equations:

$$\frac{2(ab - \alpha\beta)}{(a + b - \alpha - \beta)} = \frac{(a\beta + \alpha b)}{\rho}$$

and

$$\frac{s(a^2 - \alpha^2) + r(b^2 - \beta^2)}{(a + b - \alpha - \beta)} - rs = \frac{(a\alpha s + b\beta r)}{\rho} - rs.$$

Rearranging these we get

$$2\rho(ab - \alpha\beta) = (a\beta + \alpha b)(a + b - \alpha - \beta) \tag{8.5}$$

and

$$\rho(s(a^2 - \alpha^2) + r(b^2 - \beta^2)) = (a + b - \alpha - \beta)(a\alpha s + b\beta r).$$

Combining these two equations we obtain:

$$2(ab-\alpha\beta)(a+b-\alpha-\beta)(a\alpha s+b\beta r) = (a\beta+\alpha b)(a+b-\alpha-\beta)(s(a^2-\alpha^2)+r(b^2-\beta^2))$$

which simplifies to

$$(\alpha b - a\beta)(s(a^2 + \alpha^2) - r(\beta^2 + b^2)) = 0$$

as required. □

In the case where u and v are adjacent, $g_3(\mu) = 0$ yields a cubic in μ. Proposition 8.11 is already elaborate and in view of the more elaborate nature of analogous results involving the cubic g_3, we take the above to represent the limit of what is worthwhile in the general case. In specific cases such as $K_{2,5}$ it is easier in practice to examine the eigenvalues of $H+u$ and $H+v$, for if μ is a double eigenvalue of $H+u+v$ it must be an eigenvalue of both $H+u$ and $H+v$. Nevertheless the results of this Chapter will prove useful in Chapter 9 ($H \cong K_{2,5}$) and Chapter 10 where we consider the case $r=1$ (ie. $H \cong K_{1,s}$ $(s \geq 1)$).

Chapter 9

Star complement $K_{2,5}$.

Throughout this chapter we let H be a graph isomorphic to $K_{2,5}$. We apply the Reconstruction Theorem to $H+u+v$ and in accordance with Remark 7.2 we find that $g_1(\mu) = g_2(\mu) = g_3(\mu) = 0$ where

$$\begin{aligned}
g_1(x) &= (x^2-10)(a+b-x^2) + 2xab + 5a^2 + 2b^2 \\
g_2(x) &= (x^2-10)(a+b-\alpha-\beta) \\
&\quad +2x(ab-\alpha\beta) + 5(a^2-\alpha^2) + 2(b^2-\beta^2) \\
g_3(x) &= (x^2-10)(\rho + a_{uv}x) + 5a\alpha + 2b\beta + x(a\beta + \alpha b) \\
&\quad \text{where} \quad a_{uv} = \begin{cases} 1 & \text{if} \quad u \sim v \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and ρ is the number of common neighbours of u and v in H.

Proposition 9.1 can also be proved by inspecting the spectra of 17 graphs in Table 9.1, as was done in [JacRo]. We proceed as far as possible without the use of a computer.

Proposition 9.1 *Let μ be an eigenvalue of $H+u$ but not of H and let μ be an integer. If $H \cong K_{2,5}$, then either $\mu = 1$ and u is of type $(1,1)$ or $(0,3)$ or $\mu = -1$ and u is of type $(0,3)$, $(2,1)$ or $(1,5)$.*

Proof. First we suppose that $|\mu| > \sqrt{10}$. Let λ be the maximal possible index of $H+u$. Then λ is the index of $H+u$ when u is of type $(2,5)$. The eigenvalues of this graph satisfy the equation $g_1(x) = 0$ where

$$g_1(x) = (x^2 - 10)(7 - x^2) + 20x + 20 + 50 = -x(x^3 - 17x - 20).$$

We let $f(x) = x^3 - 17x - 20$. Now $f(3) < f(\sqrt{10}) < f(4) < 0$ and $f(5) > 0$ and so $4 < \lambda < 5$. It follows that μ could be ± 4 by Lemma 8.2. Suppose $\mu = -4$; then $g_1(-4) = 6(a+b) - 8ab + 5a^2 + 2b^2 - 96 = 0$, a quadratic in b. This equation has no integer solution for b when $a \in \{0,1,2\}$. Therefore $\mu = -4$ is not an eigenvalue of $H+u$. Similarly when $\mu = 4$ we obtain no integer solutions for b and so $\mu = 4$ is not an eigenvalue of $H+u$.

Secondly we suppose that $|\mu| < \sqrt{10}$. By Lemma 8.1 we have

$$-\frac{1}{4}(\sqrt{29}+1) \leq \mu \leq \frac{1}{4}(\sqrt{29}-1)$$

and so μ could be ± 1. In this case where $r = 2$ and $s = 5$, Equation (8.2) becomes

$$(10-\mu^2)X^2+(2Y+\mu X)^2=2(7-2\mu-4\mu^2)(10-\mu^2)$$

where

$$X=2a-2+\mu \quad \text{and} \quad Y=2b-5+\mu.$$

When $\mu=1$ we get

$$9X^2+(2Y+X)^2=18=9+9$$

and so

$$(X,Y)\in\{(1,1),(1,-2),(-1,2),(-1,-1)\}.$$

Thus $(a,b)\in\{(1,1),(0,3)\}$ and we may conclude that when $\mu=1$ is an eigenvalue of $H+u$ then u is of type $(1,1)$ or $(0,3)$.

When $\mu=-1$, we get

$$9X^2+(2Y-X)^2=90$$

and so

$$(X,Y)\in\{(1,5),(1,-4),(-1,4),(-1,-5),(3,0),(3,3),(-3,0),(-3,-3)\}.$$

Thus $(a,b)\in\{(0,3),(2,1),(1,5)\}$ and we may conclude that when $\mu=-1$ is an eigenvalue of $H+u$ then u is of type $(0,3)$, $(2,1)$ or $(1,5)$. □

Proposition 9.2 *Let H be a star complement for μ in $H+u+v$. If $H \cong K_{2,5}$, then μ is ± 1.*

Proof. By Proposition 9.1 it is sufficient to show that μ is an integer. We suppose by way of contradiction that μ is not an integer and consider five cases. (We say u is of type (a,b) and v is of type (α,β).)

(i) Suppose that u and v are of the same type and that u is not adjacent to v. By Proposition 8.4, μ^2 is an integer and $|\mu| \leq \frac{1}{4}\left(\sqrt{29}-1\right) < \sqrt{2}$ and so $\mu = \pm 1$, an integer.

(ii) Suppose that u and v are of the same type and that u is adjacent to v. By Proposition 8.6, $\mu^2+\mu = h$ where $0 \leq h \leq \frac{r+s}{2}$ whence $h = 1,2,3$. When $h = 2$ we have $\mu^2+\mu-2 = 0$ and so $\mu = -2$ or 1; both integer values. Thus $h \neq 2$. When $h = 1$, we have $\mu^2+\mu-1 = 0$ and so μ is not an integer. Similarly μ is not an integer when $h = 3$ because then we have $\mu^2+\mu-3 = 0$. When μ is not an integer we can apply the condition

$$(a+b-h-ab)^2 = a^2b^2+a+b-5a^2-2b^2$$

from Proposition 8.6. If $a = 0$ we have $(b-h)^2 = b(1-2b)$; a contradiction since $b(1-2b) < 0$ and so $a \neq 0$. If $a = 1$ we have $(1-h)^2 = -b^2+b-4$ and we obtain a quadratic in b, namely $b^2-b+(4+(1-h)^2) = 0$, which has no real solutions and so $a \neq 1$. However, if $a = 2$ we have $(2-b-h)^2 = 2b^2+b-18$

which yields the quadratic

$$b^2 + b(5 - 2h) - (h^2 - 4h + 22) = 0.$$

Thus

$$b = \frac{1}{2}\left(2h - 5 \pm \sqrt{8h^2 - 36h + 113}\right).$$

Now suppose that $h = 1$. When $a = 2$, $b = \frac{1}{2}(-3 \pm \sqrt{85})$ and we have no integer solutions for b. Now suppose that $h = 3$. When $a = 2$, $b = \frac{1}{2}(1 \pm \sqrt{77})$ and we have no integer solutions for b.

From this we may conclude that in this case μ is an integer.

(iii) Suppose that u and v are of different types (a, b) and (α, β) and that $a + b = \alpha + \beta$; then by Proposition 8.9 μ is an integer since in this case we have $r < s$.

(iv) Suppose that u and v are non-adjacent vertices of different types (a, b) and (α, β) and that $a + b \neq \alpha + \beta$. Since μ is not an integer, we must fulfil the condition given in Proposition 8.11 with $r = 2$ and $s = 5$: $(\alpha b - a\beta)(5(a^2 + \alpha^2) - 2(\beta^2 + b^2)) = 0$. We address three possibilities.

The first is that $\alpha b = a\beta = 0$. In this case either $a = \alpha = 0$ or $b = \beta = 0$ and so $H + u + v$ is bipartite. From Proposition 8.5 we see that μ^2 is an integer with $|\mu| \leq \frac{1}{4}(\sqrt{29} - 1)$ whence $\mu^2 = 1$ and so μ is an integer.

The second possibility is that $\alpha b = a\beta \neq 0$. Without loss of generality we let $\alpha \leq a$ and let

$$\frac{a}{\alpha} = \frac{b}{\beta} = t \quad \text{so that} \quad t = 1 \text{ or } 2.$$

If $t = 1$ we have $a = \alpha$ and $b = \beta$; but u and v are of different types and so $t \neq 1$. We are left with $t = 2$. This means that $a = 2$ and $\alpha = 1$ and that $b = 2\beta$. Equation (8.5) becomes $2\rho(3\beta) = 4\beta(1+\beta)$ so that

$$\rho = \frac{2(1+\beta)}{3}.$$

Now ρ is a non-negative integer and so $\beta = 2$ with $b = 4$. (We cannot have $\beta = 5$ since then $b = 10$.) Thus we have a possible double eigenvalue of the graph $H + u + v$ when u is of type $(2, 4)$ and v is of type $(1, 2)$. However, in this case $g_2(\mu) = 3\mu^2 + 12\mu + 9 = 0$ and so $\mu = -1$ or -3; an integer.

The third possibility is that $5(a^2 + \alpha^2) - 2(\beta^2 + b^2) = 0$. Again without loss of generality we assume that $\alpha \leq a$. Here $a^2 + \alpha^2$ must be even, and so either $a = \alpha$, or $a = 2$ with $\alpha = 0$.

If $a = 2$ and $\alpha = 0$ then $\beta^2 + b^2 = 10 = 9 + 1$. Of the two possible u, v pairs we discount the pair $(2, 1), (0, 3)$ since $a + b \neq \alpha + \beta$. The other possibility is that u is of type $(2, 3)$ and v is of type $(0, 1)$. From Equation (8.5) we have

$$\rho = \frac{(a\beta + \alpha b)(a + b - \alpha - \beta)}{2(ab - \alpha\beta)}$$

and so in this case $\rho = \frac{2}{3}$. But ρ must be an integer and so the graph

$H + u + v$ does not have a double eigenvalue when u is of type $(2, 3)$ and v is of type $(0, 1)$.

If $a = \alpha = 0$ then $b = \beta = 0$ and u and v are of the same type.

If $a = \alpha = 1$ then $\beta^2 + b^2 = 5$. Thus u is of type $(1, 2)$ and v is of type $(1, 1)$. However in this case $\rho = \frac{3}{2}$ and so ρ is not an integer.

If $a = \alpha = 2$ then $\beta^2 + b^2 = 20$. Thus u is of type $(2, 4)$ and v is of type $(2, 2)$ in which case $g_2(\mu) = \mu^2 + 4\mu + 2 = 0$ and so $\mu^2 = -(4\mu + 2)$. Using this substitution we re-write $g_1(\mu) = 0$ as follows.

$$(4\mu + 12)(4\mu + 2 + a + b) - 2\mu ab - 5a^2 - 2b^2 = 0.$$

Further simplification yields the equation

$$2\mu(2a + 2b - ab - 4) - (16 + 5a^2 + 2b^2) = 0,$$

an equation of the form $P\mu + Q = 0$ where $P = 2(2a + 2b - ab - 4)$ and $Q = -(16 + 5a^2 + 2b^2)$. Now $(a, b) \in \{(2, 2), (2, 4)\}$ and in both cases we have $P = 0$ but $Q \neq 0$. It follows that the graph $H + u + v$ does not have a double eigenvalue when u is of type $(2, 2)$ and v is of type $(2, 4)$.

(v) Suppose that u and v are adjacent vertices of different types (a, b) and (α, β) and that $a + b \neq \alpha + \beta$. We examine the spectra of the connected graphs $H + u$ given in Table 9.1 and note that $\mu = \pm 2\sqrt{3}$ is the only non-integral eigenvalue that appears in more than one graph. In one case u is of type $(2,0)$; in the other u is of type $(0,3)$. We consider the graph $H+u+v$ where u is of type $(2,0)$ and v is of type $(0,3)$. If $\pm 2\sqrt{3}$ is indeed a double eigenvalue of this graph then $g_1(\mu) = g_2(\mu) = g_3(\mu) = 0$ when $\mu = \pm 2\sqrt{3}$. However, if $g_3(\mu) = 0$ when $\mu = \pm 2\sqrt{3}$ then we have non-integral values for ρ, namely $\pm 8\sqrt{3}$, and so we cannot satisfy $g_3(\mu) = 0$ in this case. We conclude that $\mu = \pm 2\sqrt{3}$ is not a double eigenvalue of $H + u + v$.

Finally we conclude that when μ is a double eigenvalue of $H+u+v$ but not of H then μ is an integer. Therefore $\mu = \pm 1$ by Proposition 9.1. □

In the case where u and v are different types, the conclusion of Proposition 9.2 is also obtainable by considering the spectra in Table 9.1. However in the case where u and v are vertices of the same type it would be necessary to examine the spectra of every graph $H + u + v$ since every non-zero eigenvalue of every $H + u$ is then a possibility (Proposition 5.10).

Remark 7.1 provided us with one reason for assuming that μ is an integer value in the special case where the constructed graph G must have $|V(G)| > 2|V(H)|$. The following remark gives an additional reason for considering only integer values for μ.

Remark 9.3 Let H be a star complement for μ. Suppose that G_1 is obtained from H by adding two adjacent vertices of type (a, b) and (α, β); and G_2 is obtained from H by adding two non-adjacent vertices again of type

(a, b) and (α, β). Let ρ_1 denote the number of common H-neighbours of the two additional vertices in G_1; similarly let ρ_2 denote the number of common H-neighbours in G_2.

Now suppose that μ is an eigenvalue of both G_1 and G_2 with $m_\mu(G_1) = m_\mu(G_2) = 2$. We can apply the Reconstructon Theorem to both G_1 and G_2; in particular $g_3(\mu) = 0$ in both cases (see Remark 7.2). From this we obtain an equation for μ in terms of ρ_1 and ρ_2; namely $\mu = \rho_1 - \rho_2$, and as both ρ_1 and ρ_2 are clearly integers, μ must also be an integer.

From this we may deduce that if μ is a *non-integral* eigenvalue of G_1 with $m_\mu(G_1) = 2$ then $m_\mu(G_2) = 0$ (Proposition 5.10), and *vice versa.*

We summarize the results so far.

Let G be a graph with star set X corresponding to μ such that $|X| \geq 2$. If $H \cong K_{2,5}$ is a star complement for μ in G then $\mu = \pm 1$.

When $\mu = 1$, a vertex in X is of type $(1, 1)$ or $(0, 3)$.

When $\mu = -1$, a vertex in X is of type $(0, 3)$, $(2, 1)$ or $(1, 5)$.

We have not yet determined if the star set can include vertices of every type. In order to do this we need to ascertain which pairs of vertices u, v can be added simultaneously, see Theorem 5.7. This is done by considering the values for ρ obtained by solving the equation $g_3(\mu) = 0$ in each case.

We give the values found when $\mu = 1$ in Table 9.2. We can now show

(a, b)	Non-zero eigenvalues of $H + u$				Repeated eigenvalues
(1,0)	± 3.24422,	± 0.689248			
(0,1)	± 3.1964,	± 0.884878			
(2,0)	$\pm 2\sqrt{3}$				†
(1,1)	3.38469,	1,	-1.20972,	-3.17497	†
(0,2)	± 3.30136,	± 1.0493			
(2,1)	3.6737,	0.654334,	-1,	-3.32803	†
(1,2)	3.57621,	1.03808,	-1.44479,	-3.1695	
(0,3)	$\pm 2\sqrt{3}$,	± 1			†
(2,2)	3.9066,	0.691586,	-1.38026,	-3.21793	
(1,3)	3.79481,	0.932061,	-1.48055,	-3.24633	
(0,4)	± 3.66103,	± 0.772577			
(2,3)	4.1468,	0.583738,	-1.56706,	-3.16348	
(1,4)	4.0231,	0.71045,	-1.34045,	-3.3931	
(0,5)	$\pm\sqrt{15}$				
(2,4)	4.38548,	0.366677,	-1.55705,	-3.1951	
(1,5)	4.25153,	0.328502,	-1,	-3.58003	†
(2,5)	4.61849,	-1.21834,	-3.31033		

Table 9.1: Non-zero eigenvalues of $H + u$.

vertices of type $(a,b),(\alpha,\beta)$	ρ $u \sim v$	ρ $u \not\sim v$
$(1,1),(1,1)$	0	1
$(0,3),(0,3)$	1	2
$(0,3),(1,1)$	0	1

Table 9.2: $\mu = 1$.

that when G is the maximal graph constructed from $H \cong K_{2,5}$, G is the Schläfli graph, Sch_{10}.

Theorem 9.4 *Let G be a maximal graph with star set X corresponding to $\mu \neq -1$, with $|X| \geq 2$. Let H be the star complement corresponding to μ with $H \cong K_{2,5}$. Then G is unique and G is the Schläfli graph, Sch_{10}.*

Proof. Clearly $\mu = 1$. Let G be a graph with star complement $H \cong K_{2,5}$ corresponding to the eigenvalue 1, and let the vertices of H be divided into two independent sets R and S where $|R| = 2$ and $|S| = 5$. By Proposition 9.1 the remaining vertices of G are of two types and so can be divided into two further sets, Q and P so that the vertices in Q are of type $(1,1)$, and those in P are of type $(0,3)$. Now $\overline{X}$ is a location dominating set since $\mu \notin \{0,-1\}$ and so the map $u \mapsto \bar{\Gamma}(u)$ $(u \in X)$ is injective. It follows that the number of possible $\overline{X}$-neighbourhoods for a vertex of type $(0,3)$ is ${}^5C_3$. Similarly for a vertex of type $(1,1)$ the number is ${}^2C_1 \times {}^5 C_1$. Thus the two sets P and Q could both contain as many as 10 vertices. Indeed it is possible to add all vertices of both types at the same time without contradicting the values for ρ given in Table 9.1 and it is this which makes this graph unique. We now describe G.

Let the vertices in Q be labelled a and a' for $a = 2, 3, 4, 5, 6$; the vertices in P be labelled ij $(2 \leq i < j \leq 6)$; the vertices in R be labelled 1 and $1'$, and those in S be labelled ij $(i = 1, j = 2, 3, 4, 5, 6)$. Then $a \not\sim b$ and $a' \not\sim b'$ so that Q is composed of two independent sets, $a \sim b'$ if and only if $a \neq b$, $a \sim (ij)$ and $a' \sim (ij)$ if and only if $a \in \{i, j\}$ and $ij \sim ab$ if and only if $\{i, j\} \cap \{a, b\}$ is empty. Thus G is Sch_{10}, precisely as described in Example 1.25.

□

That there is a unique maximal graph in the case when $\mu = 1$ and $H \cong K_{2,5}$ is perhaps unusual. In the case where $\mu = 2$ and $H \cong K_{1,16} \cup 6\, K_1$, we were able to determine only that the *largest* maximal graph was unique. We give other examples of this uniqueness property:

(i) The Clebsch graph is the unique maximal graph with $\mu \neq -1$ and $H \cong K_{1,5}$, $|X| > 1$, [Row4, Thm3.6].

(ii) Among s-regular graphs the Higman-Sims graph [CvL, p.107] is the unique maximal graph with $\mu = 2$ and $H \cong K_{1,s}$ [Row3, Cor.2.3]. A detailed description of this is given in Example 10.15.

For a more recent example we are indebted to [Bell1].

(*iii*) For odd $t > 3$, $L(K_t)$ is the unique maximal graph with $\mu = -2$ and $H \cong C_t$. For a proof of this when $t \geq 37$ see [CvRoS3, Thm. 4.1].

(*iv*) If $t \notin \{8, 12, 13\}$ then $\overline{L(K_t)}$ is the unique maximal graph with $\mu = 1$ and $H \cong K_{1,t-3} \cup 2K_1$ [CvRoS4, Thm. 2.1].

One consequence of this uniqueness property is that all graphs constructed from $H \cong K_{2,5}$ corresponding to $\mu = 1$ are subgraphs of Sch_{10}. We shall exploit this fact when determining the regular graphs with $H \cong K_{2,5}$.

Before that we shall briefly touch upon the automorphism group of the Schläfli graph, $\mathrm{Aut}(\mathrm{Sch}_{10})$. We should note that the subgraph $\langle P \rangle$, induced by P, is the Petersen graph. Now Aut(Pe) is isomorphic to the symmetric group $S(5)$. In describing the following regular graphs we shall use implicitly the fact that $\mathrm{Sym}\{2, 3, 4, 5, 6\}$ induces a subgroup of $\mathrm{Aut}(\mathrm{Sch}_{10})$ which fixes R and S setwise. In Figure 9.1 we provide what we hope will be a useful diagram of Sch_{10}.

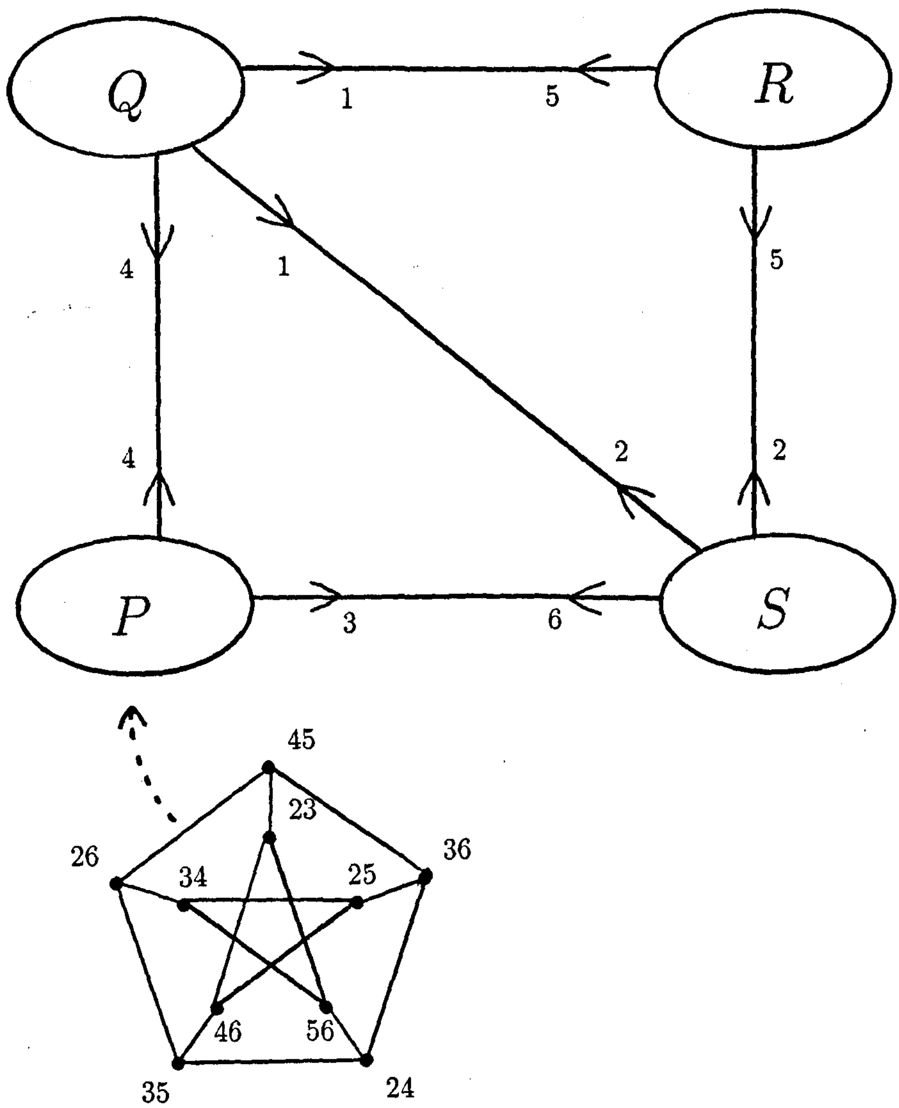


Figure 9.1: Diagram of Sch_{10}

We shall now determine the regular graphs which have $K_{2,5}$ as a star complement corresponding to the eigenvalue $\mu = 1$. Let G be an n-vertex, k-regular subgraph of Sch_{10} with star set X for the eigenvalue $\mu = 1$. Let H be the corresponding star complement. Let $X = P \cup Q$ where P is the set of vertices of type $(0, 3)$ and Q is the set of vertices of type $(1, 1)$. Let the vertex set of H be partitioned into two independent sets R and S in the usual way.

Counting edges in two ways, we have $|E(R, X)| = 2(k - 5) = |E(Q, R)|$ and so $|Q| = 2(k - 5)$. Now $|E(S, X)| = 5(k - 2)$, but $|E(S, Q)| = |Q| = 2(k - 5)$ so that $|E(S, P)| = 5(k - 2) - 2(k - 5) = 3k$. Thus $|P| = k$. Also note that $n = k + 2(k - 5) + 7 = 3(k - 1)$.

The following notation will be useful. Let $D(P)$ be the ordered degree sequence of the vertices in the subgraph induced by P. Let $N(P, Q)$ be the sequence of numbers corresponding to the number of edges from a vertex in P to the subgraph induced by Q. We take $N(P, Q)$ to have the same order as $D(P)$. Since these graphs are regular we have $d(P) + n(P, Q) = k - 3$ where $d(P)$ and $n(P, Q)$ are corresponding numbers in these sequences. Similarly $d(Q) + n(Q, P) = k - 2$.

Since $5 \leq k \leq 9$ we have 5 cases to consider.

(i) Degree 5.

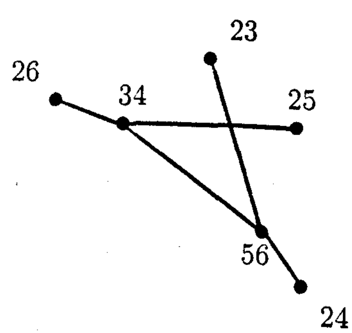
Here Q is the empty set and $|P| = 5$. The subgraph induced by P is regular of degree 2, hence is any 5-cycle within the Petersen graph. Without loss of generality $P = \{23, 45, 36, 24, 56\}$ and G is determined.

(ii) Degree 6.

Here $|Q| = 2$ and $|P| = 6$. Each of the vertices in R has exactly one neighbour in Q so without loss of generality, either (a) $Q = \{2, 2'\}$ and the vertices in Q are not adjacent, or (b) $Q = \{2, 3'\}$ where the vertices are adjacent.

(a) $Q = \{2, 2'\}$.

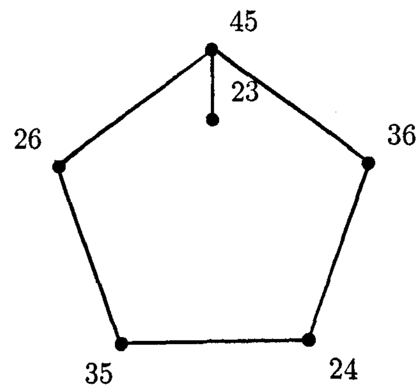
Here $N(Q, P) = (4, 4)$ and so $\{23, 24, 25, 26\} \subset P$. Then, without loss of generality, either (i) $P = \{23, 24, 25, 26, 34, 35\}$ or (ii) $P = \{23, 24, 25, 26, 34, 56\}$. In both cases $N(P, Q) = (2, 2, 2, 2, 0, 0)$ and so we require $D(P) = (1, 1, 1, 1, 3, 3)$. This condition is only satisfied in case (ii), and so G is determined. The subgraph induced by P is:



(b) $Q = \{2, 3'\}$.

Here $N(Q,P) = (3,3)$ and we have two possibilities; either (i) 2 and $3'$ have a common neighbour in P in which case $23 \in P$ or (ii) they do not, in which case $23 \notin P$.

(i) If $23 \in P$ then $N(P,Q) = (2,1,1,1,1,0)$ and so $D(P) = (1,2,2,2,2,3)$. The only subgraph of the Petersen graph with this degree sequence which contains 23 is of the form



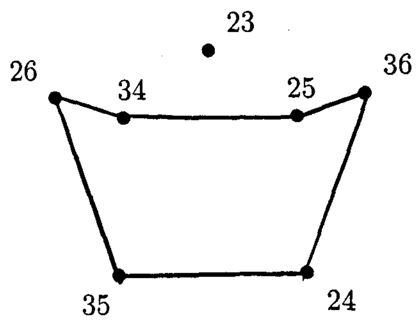
and so G is uniquely determined.

(ii) If $23 \notin P$ then $P = \{24, 25, 26, 34, 35, 36\}$ whence $N(P,Q) = (1,1,1,1,1,1)$. Thus G is uniquely determined since $\langle P \rangle$ is the only 6-cycle within the Petersen graph which does not go through 23, and in which each vertex is adjacent to either 2 or $3'$.

(iii) Degree 7. Here $|Q| = 4$ and $|P| = 7$. Each vertex in R has exactly two neighbours in Q and so either (a) $D(Q) = (1,1,1,1)$, (b) $D(Q) =$ $2,1,1,2)$ or (c) $D(Q) = (2,2,2,2)$.

(a) $D(Q) = (1,1,1,1)$.

Without loss of generality let $Q = \{2, 3, 2', 3'\}$ and note that $N(Q, P) = (4, 4, 4, 4)$. Hence $P = \{23, 24, 25, 26, 34, 35, 36\}$ with $N(P, Q) = (4, 2, 2, 2, 2, 2, 2)$ and $D(P) = (0, 2, 2, 2, 2, 2, 2)$ as required. Thus the subgraph induced by P is

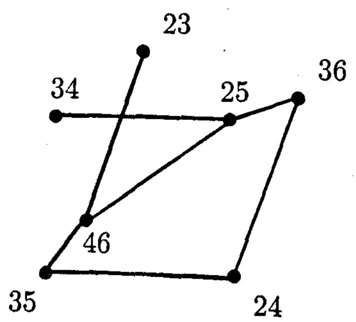


and G is determined.

(*b*) $D(Q) = (2, 1, 1, 2)$.

Here $N(Q, P) = (3, 4, 4, 3)$. Without loss of generality let $Q = \{2, 3, 3', 4'\}$ and so $\{23, 34, 35, 36, \} \subseteq P$ since vertices 3 and $3'$ both have four neighbours in P. This provides one neighbour each for 2 and $4'$ and so we must have $24 \in P$ also. Without loss of generality the remaining two vertices in P are either 25 and 45, or 25 and 46, with $N(P, Q) = (3, 3, 2, 2, 2, 1, 1)$ in both cases. However we require $D(P) = (1, 1, 2, 2, 2, 3, 3)$ which is satisfied only when $P = \{23, 34, 35, 36, 24, 25, 46\}$. Thus the subgraph induced by P

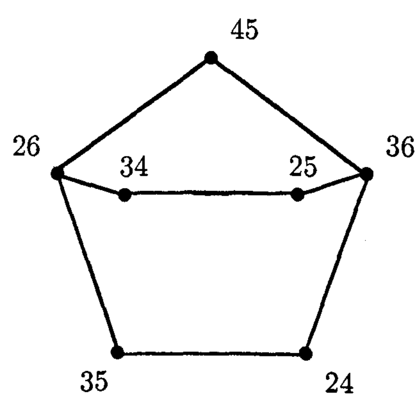
is



and G is determined.

(c) $D(Q) = (2, 2, 2, 2)$.

Here $N(Q, P) = (3, 3, 3, 3)$ and $N(P, Q) = (1, 1, 2, 2, 2, 2, 2)$ with $D(P) = (3, 3, 2, 2, 2, 2, 2)$. Without loss of generality let the subgraph induced by P be



and so Q is either $\{2, 3, 4', 5'\}$ or $\{2, 3', 4, 5'\}$. These graphs are non-isomorphic since when $Q = \{2, 3, 4', 5'\}$ we have four triangles, namely those induced by $\{34, 3, 4'\}$, $\{25, 2, 5'\}$, $\{35, 3, 5'\}$ and $\{24, 2, 4'\}$ whilst when $Q =$

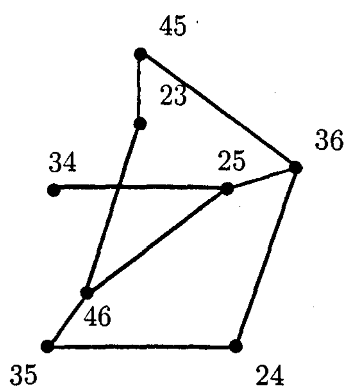
$\{2, 3', 4, 5'\}$ we have only three; $\{34, 3', 4\}$, $\{25, 2, 5'\}$ and $\{45, 4, 5'\}$.

(iv) Degree 8.

Here $|Q| = 6$ and $|P| = 8$. Each vertex in R has exactly three neighbours in Q and so without loss of generality, Q is either (a) $\{2, 3, 4, 3', 4', 5'\}$, (b) $\{2, 3, 4, 2', 3', 4'\}$ or (c) $\{2, 3, 4, 4', 5', 6'\}$.

(a) $Q = \{2, 3, 4, 3', 4', 5'\}$.

Here $D(Q) = (3, 2, 2, 2, 2, 3)$ and so $N(Q, P) = (3, 4, 4, 4, 4, 3)$. The vertices 3,4,3′ and 4′ each require four neighbours in P. Consequently we must have $\{23, 34, 35, 36, 24, 45, 46\} \subseteq P$. Thus the vertices 2 and 5′ require one more neighbour in P and so we have $25 \in P$. Thence $P = \{25, 23, 34, 35, 36, 24, 45, 46\}$ with $N(P, Q) = (2, 3, 4, 3, 2, 3, 3, 2)$ and $D(P) = (3, 2, 1, 2, 3, 2, 2, 3)$ as required. The subgraph induced by P is



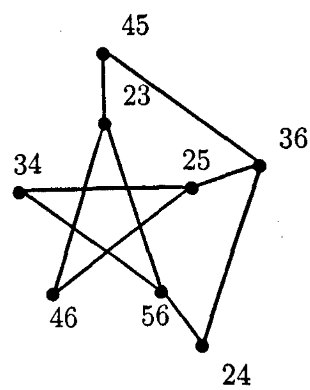
and so G is determined.

(b) $Q = \{2, 3, 4, 2', 3', 4'\}$.

Here the subgraph induced by Q is regular of degree 2 and so each vertex in Q must have four neighbours in P. In view of this we must have $\{23, 24, 25, 26, 34, 35, 36, 45, 46\} \subseteq P$, but this would mean that $|P| \geq 9$ which is a contradiction and so this case does not arise.

(c) $Q = \{2, 3, 4, 4', 5', 6'\}$.

Here $D(Q) = (3, 3, 2, 2, 3, 3)$ and so $N(Q, P) = (3, 3, 4, 4, 3, 3)$. Consequently $\{24, 34, 45, 46\} \subseteq P$, each vertex having three neighbours in Q. The total number of edges between Q and P is twenty and so we have $N(P, Q) = (3, 3, 3, 3, 2, 2, 2, 2)$ with $D(P) = (2, 2, 2, 2, 3, 3, 3, 3)$. Without loss of generality we say $P = \{24, 34, 45, 46, 23, 25, 36, 56\}$ so that the subgraph induced by P is



and G is determined.

(v) Degree 9.

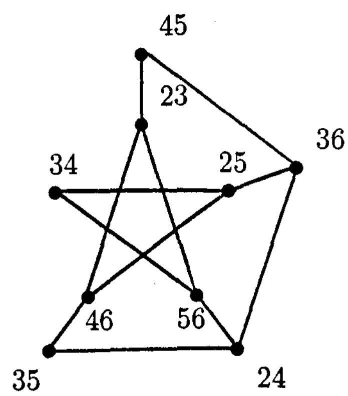
Here $|Q| = 8$ and $|P| = 9$. Each vertex in R has exactly four neighbours in Q and so without loss of generality either (a) $Q = \{2, 3, 4, 5, 2', 3', 4', 5'\}$ or (b) $Q = \{2, 3, 4, 5, 3', 4', 5', 6'\}$.

(a) $Q = \{2, 3, 4, 5, 2', 3', 4', 5'\}$.

Here the subgraph induced by Q is regular of degree 3 and so each vertex in Q must be adjacent to four vertices in P. Consequently we must have $\{23, 24, 25, 26, 34, 35, 36, 45, 46, 56\} \subseteq P$ which would mean that $|P| \geq 10$. This is a contradiction, and so this case does not arise.

(b) $Q = \{2, 3, 4, 5, 3', 4', 5', 6'\}$.

Here $D(Q) = (4, 3, 3, 3, 3, 3, 3, 4)$ and so $N(Q, P) = (3, 4, 4, 4, 4, 4, 4, 3)$, with vertices 2 and $6'$ having three neighbours in P and the remaining vertices in Q have four neighbours in P. Consequently $P = \{23, 34, 35, 36, 24, 45, 46, 25, 56\}$ with $N(P, Q) = (3, 4, 4, 3, 3, 4, 3, 3, 3)$ and $D(P) = (3, 2, 2, 3, 3, 2, 3, 3, 3)$ as required. Thus the subgraph induced by P is



and G is determined.

Thus we may conclude that there is a total of eleven regular subgraphs of Sch_{10} which have $H \cong K_{2,5}$ as a star complement corresponding to the eigenvalue 1.

We conclude this chapter by examining the graphs which arise when the eigenvalue is -1. As a preliminary we describe three graphs: G_{14}, G_{17} and G_{18}.

The graph $\mathbf{G_{14}}$.

The vertex set of G_{14} consists of four independent sets, R, S, U and V where $|R| = |U| = 2$ and $|S| = |V| = 5$. The subgraphs induced by $R \cup S$, $R \cup V$, $U \cup S$ and $U \cup V$ are all $K_{2,5}$ and the subgraphs induced by $S \cup V$ and $R \cup U$ are $5K_2$ and $2K_2$ respectively. This fully describes the edge set of G_{14}. (This is the graph known as $5K_2 \nabla 2K_2$; every vertex in $5K_2$ is adjacent to every vertex in $2K_2$.) The spectrum of this graph contains the eigenvalue -1 with multiplicity 7. Any one of the $K_{2,5}$ subgraphs may be taken as a star complement corresponding to -1.

It may help to visualize this graph as being drawn on the walls of a house. We draw one $K_{2,5}$ on each of the front and the back walls with the five vertices uppermost; the edges in between providing some rather inconvenient cross-bracing. For the graphs G_{17} and G_{18} we simply put the roof on.

For these next two graphs it is convenient to label the vertices in S and V so that $s_i \sim v_j$ if and only if $i = j$.

The graph G_{17}.

The vertex set of G_{17} is that of G_{14} together with one other independent set T_1, where $T_1 = \{t_1, t_3, t_5\}$. The complete edge set of G_{17} is the edge set of G_{14} together with the edge sets $E(T_1, S)$ and $E(T_1, V)$ where $E(T_1, S) = \{(t_i, s_i), (t_i, s_2), (t_i, s_4) : i = 1, 3, 5\}$. Similarly $E(T_1, V) = \{(t_i, v_i), (t_i, v_2), (t_i, v_4) : i = 1, 3, 5\}$.

The spectrum of this graph includes the eigenvalue -1 with multiplicity ten. Any one of the $K_{2,5}$ subgraphs may be taken as a star complement corresponding to -1.

The graph G_{18}.

The vertex set of G_{18} is that of G_{14} together with one other independent set T_2, where $T_2 = \{t_1, t_2, t_3, t_4\}$. The complete edge set of G_{18} is the edge set of G_{14} together with the edge sets $E(T_2, S)$ and $E(T_2, V)$ where

$$\begin{aligned} E(T_2, S) &= \{(t_1, s_i) : i = 1, 2, 3\} \cup \{(t_2, s_i) : i = 2, 3, 4\} \cup \\ &\quad \{(t_3, s_i) : i = 3, 4, 1\} \cup \{(t_4, s_i) : i = 4, 1, 2\}. \end{aligned}$$

As before with G_{17}, $E(T_2, V)$ mimics $E(T_2, S)$.

The spectrum of this graph includes the eigenvalue -1 with multiplicity eleven. Any one of the $K_{2,5}$ subgraphs may be taken as a star complement corresponding to -1.

Remark 9.5 The graph G_{14} can be obtained from either G_{18} or G_{17} on deletion of the T-sets.

Remark 9.6 Let $G_{17} - t$ be the graph obtained from G_{17} by deleting one of the vertices from T_1. Then $G_{17} - t$ is an induced subgraph of G_{18}.

Here the constructed graph is denoted by G' and the maximal core graph by G. We will show that G_{14} is the unique maximal core graph when G' is regular although in itself it is not regular. The graphs G_{17} and G_{18} are the two maximal core graphs which arise when G' is not regular. We shall deal with this case first.

We should point out that since $\mu = -1$ we should expect duplicate vertices inducing a complete subgraph.

Theorem 9.7 *Let G be a maximal core graph with $H \cong K_{2,5}$ as a star complement for the eigenvalue $\mu = -1$ with $|X| \geq 2$. Then either $G = G_{17}$ or $G = G_{18}$.*

Proof. We divide the vertex set of $K_{2,5}$ into two independent sets R and S as before and say that a vertex u is of type (a, b) if it is adjacent to a vertices in R and b vertices in S. From Proposition 9.1 we know that when

vertices of type (a,b), (α,β)	ρ $u \sim v$	ρ $u \not\sim v$
(0,3),(0,3)	3	2
(2,1),(2,1)	3	2
(1,5),(1,5)	6	5
(0,3),(2,1)	1	0
(0,3),(1,5)	(4)	3
(2,1),(1,5)	2	(1)

Table 9.3: $\mu = -1$.

the eigenvalue μ is -1, the vertices which may be added to form a star set are of the type $(0,3)$, $(2,1)$, and $(1,5)$. Again we obtain values for ρ by solving the equation $g_3(-1) = 0$ for each possible pair of vertices and present the results in Table 9.3. We can make the following observations from Table 9.3.

When u is of type $(2,1)$ and v is of type $(1,5)$ then u must be adjacent to v since their common H-neighbourhood contains precisely two vertices. Furthermore, when u is of type $(0,3)$ and v is of type $(1,5)$ their common H-neighbourhood contains precisely three vertices and so u and v cannot be adjacent.

When u and v are adjacent vertices of the same type they have exactly the same $\overline{X}$ neighbourhood and so they are duplicate vertices as expected. It follows that if u and v are vertices of the same type in G then $u \not\sim v$. When u and v are both of type $(2, 1)$, both are adjacent to the two vertices in R. Therefore they must be adjacent to different vertices in S and so the maximum number of vertices of type $(2, 1)$ which can be added is five. Similarly the maximum number of vertices of type $(1, 5)$ which can be added is two. The only constraint on adding two vertices of the type $(0, 3)$ is that $\rho = 2$; that is, they have exactly two common neighbours in S. This can be considered in terms of the families of 3-sets taken from a 5-set and intersecting in 2-sets. There are just two possibilities for such a family: without loss of generality either (i) $\mathcal{F} = \{\ \{1,2,4\}, \{2,3,4\}, \{2,4,5\}\ \}$ or (ii) $\mathcal{F} = \{\ \{1,2,3\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}\ \}$. Clearly, in case (i) the maximum number of vertices of type $(0, 3)$ which can be added is three; in case (ii) the maximum number is four. In both cases it is possible to have the maximum number of each type of vertex present at the same time. In case (i) G is the graph G_{17} and in case (ii) G is the graph G_{18}.

□

Note that the subgraph induced by the vertices of type $(2, 1)$ and those of type $(1, 5)$ is $K_{2,5}$ in both G_{17} and G_{18}. Furthermore, if we let u be a vertex of type $(2, 1)$ so that u is adjacent to a vertex, say w, in S, then the vertex u is adjacent to precisely those vertices of type $(0, 3)$ which are also adjacent to the vertex w. Thus the vertices of type $(2, 1)$ and those in S are interchangeable sets. Similarly the vertices in R and those of type $(1, 5)$ are not adjacent to the vertices of type $(0, 3)$ and so the two $K_{2,5}$ subgraphs of the maximum core graphs are interchangeable.

Theorem 9.8 *Let G be a maximal core graph of a k-regular graph G' with $H \cong K_{2,5}$ as a star complement for the eigenvalue $\mu = -1$. Then G is the graph G_{14} and $k \equiv -1 \bmod 9$.*

Proof. By Theorem 9.7 we know that G is either G_{17}, G_{18} or a subgraph of one of these graphs. The proof of this theorem hinges on showing that G' cannot contain any vertices of the type $(0, 3)$. We suppose by way of contradiction that G' contains vertices of the type $(0, 3)$.

Let the vertex set of $K_{2,5}$ be divided into two independent sets $R = \{r_1, r_2\}$ and $S = \{s_1, s_2, s_3, s_4, s_5\}$ as before.

First we suppose that G is either G_{18} or a subgraph of G_{18} which includes at least one vertex of type $(0, 3)$. Since G' is regular we must allow for the addition of duplicate vertices and so we can describe G' as follows.

Let the vertices of X be divided into eleven sets, each of which induces a complete graph. Say the vertices of type $(1, 5)$ are in one of two U-sets, a vertex in U_i, $i = 1, 2$ being adjacent to r_i in R, all vertices in S and all vertices of type $(2, 1)$. Let the vertices of type $(2, 1)$ be in one of five V-sets, a vertex in V_i, $i = 1, \ldots, 5$ being adjacent to s_i in S, all vertices in R and to all vertices in the U-sets. The vertices of type $(0, 3)$ are in the T-sets, a vertex in T_1 being adjacent to s_2, s_3, and s_4; in T_2 to s_1, s_3, and s_4, and similarly for T_3 and T_4, the vertex s_5 being non-adjacent to any vertex of type $(0, 3)$. Any of the U-, V-, or T-sets may be empty.

Considering the possible common H-neighbourhoods, it follows that a vertex in V_1 is adjacent to all vertices in T_2, T_3 and T_4, a vertex in V_2 to all vertices in T_1, T_3 and T_4, and so forth. No vertex in V_5 is adjacent to any vertex of type $(0, 3)$.

Let $|U_i| = u_i$ $(i = 1, 2)$, $|V_i| = v_i$ $(i = 1, 2, 3, 4, 5)$ and $|T_i| = t_i$ $(i = 1, 2, 3, 4)$.

Let $d(r_1)$ be the degree of r_1; note that since $d(r_1) = d(r_2)$ we must have $|U_1| = |U_2|$ and so we write $|U_i| = u$.

Counting the degrees of the vertices in R and S we obtain the following:

$$
\begin{aligned}
d(r_i) &= u + v + 5 \quad (i = 1, 2) \\
&\qquad \text{where} \quad v = \sum_{i=1}^{5} v_i, \\
d(s_i) &= 2u + t + v_i - t_i + 2 \quad (i = 1, 2, 3, 4)
\end{aligned}
$$

$$\text{where} \quad t = \sum_{i=1}^{4} t_i, \quad \text{and}$$

$$d(s_5) = 2u + v_5 + 2.$$

From this we obtain a set of linear equations:

$$\begin{aligned} u + v &= k - 5, \\ 2u + t + v_i - t_i &= k - 2, \quad i = 1, \ldots, 4 \quad \text{and} \\ 2u + v_5 &= k - 2. \end{aligned}$$

Summing over vertices in S we get

$$10u + 4t + \sum_{i-1}^{5} v_i - \sum_{i=1}^{4} t_i = 5(k-2)$$

whence

$$10u + 3t + v = 5(k-2).$$

Now $v = k - 5 - u$ and so we have

$$u = \frac{1}{9}(4k-5) - \frac{1}{3}t$$

From this we get solutions for v_i,

$$v_i = \frac{1}{9}(k-8) - \frac{1}{3}t + t_i \quad (i = 1, 2, 3, 4)$$

and

$$v_5 = \frac{1}{9}(k-8) + \frac{2}{3}t.$$

Now suppose that at least one of the U- or V-sets is non-empty then by counting degrees we have either

$k = u_i - 1 + 6 + v,$ for a vertex in U_i $(i = 1, 2)$, or

$k = v_i - 1 + 2u + t - t_i + 3,$ for a vertex in V_i $(i = 1, 2, 3, 4)$, or

$k = v_5 - 1 + 2u + 3,$ for a vertex in V_5, whence $u + v = k - 5$, $v_i + 2u + t - t_i = k - 2$ and $v_5 + 2u = k - 2$ as before. Thus the presence of both U- and V-sets is consistent with our solutions.

However, suppose that at least one T-set is present, say T_i, then we have

$$k = t_i - 1 + 3 + v - v_5 - v_i$$

whence

$$v - v_i - v_5 = k - 2 - t_i.$$

From our previous solutions, summing over v_i we have

$$v - v_i - v_5 = \frac{3}{9}(k - 8) - \frac{3}{3}t + t - t_i$$

and so

$$k - 2 = \frac{1}{3}(k - 8)$$

whence $\frac{2}{3}(k + 1) = 0$. This is so only when $k = -1$. However, we must have $k \geq 1$ and so we have a contradiction. Thus we may conclude that G is neither G_{18} nor a subgraph of G_{18} which includes at least one vertex of type $(0, 3)$.

Secondly we suppose that G is either G_{17} or a subgraph of G_{17} which includes at least one vertex of type $(0,3)$. We consider G' and divide the vertices of type $(2,1)$ and $(1,5)$ as before. However the vertices of type $(0,3)$ now fall into three sets T_1, T_3 and T_5, so that a vertex in T_i is adjacent to s_i, s_2 and s_4, and also to those vertices in V_i, V_2 and V_4, for $i = 1, 2, 3$. As before, $|U_1| = |U_2| = u$, and $v = \sum_{i=1}^{5} v_i$, but now we let $t = t_1 + t_3 + t_5$.

From the degrees of vertices in R and S we obtain

$$\begin{aligned} d(r_i) &= u + v + 5, \\ d(s_i) &= v_i + 2u + t_i + 2 \quad (i = 1, 3, 5) \quad \text{and} \\ d(s_i) &= v_i + 2u + t + 2. \quad (i = 2, 4). \end{aligned}$$

Summing over the vertices in S we get $v + 10u + 3t + 10 = 5k$ and so

$$u = \frac{1}{9}(4k - 5) - \frac{1}{3}t \quad \text{since} \quad v = k - 5 - u.$$

From this we obtain solutions for v_i:

$$\begin{aligned} v_i &= \frac{1}{9}(k - 8) + \frac{2}{3}t - t_i \quad (i = 1, 3, 5) \quad \text{and} \\ v_i &= \frac{1}{9}(k - 8) - \frac{1}{3}t \quad (i = 2, 4). \end{aligned}$$

As before the presence of the U- and V-sets is consistent with these solutions.

By Remark 9.6 the case where only one or two T-sets are present has already been covered. Therefore we can assume that T_1, T_3 and T_5 are all non-empty. Then, considering any vertex of type $(0,3)$ we have

$$k = t_i - 1 + v_i + v_2 + v_4 + 3 \quad (i = 1, 3, 5)$$

whence $v_i + v_2 + v_4 = k - 2 - t_i$. However, suppose that at least one T-set, say T_1, is also present, then $d(t_1) = t_1 - 1 + v_1 + v_2 + v_4 + 3$ whence $v_1 + v_2 + v_4 = k - 2 - t_1$. From our previous solutions we have $v_i + v_2 + v_4 = \frac{1}{3}(k-8) - t_i$ and so $k - 2 = \frac{1}{3}(k-8)$ as before. Thus we may conclude that G is neither G_{17} nor a subgraph of G_{17} which includes at least one vertex of type $(0,3)$.

Hence, by Remark 9.5, G is G_{14} as required.

We have shown that, if G is k-regular, then $\overline{X}$ is divided into seven sets, U_1, U_2, $V_1, \ldots, V_5$ with

$$u = \frac{1}{9}(4k-5) \quad \text{and} \quad v = \frac{1}{9}(k-8),$$

and so $k \equiv -1 \bmod 9$, since both u and v are integers. Note that given G, the star complement $K_{2,5}$ may be induced by any seven vertices, taken one from each set.

$\square$

We tabulate the first few parameter sets for G.

$k \equiv$	$\lvert U^*\rvert$	$\lvert V^*\rvert$	$\lvert E(G)\rvert$	$\lvert V(G)\rvert$
$-1 \bmod 9$	$\frac{4}{9}(k+1)$	$\frac{1}{9}(k+1)$	$\frac{13k(k+1)}{18}$	$\frac{13(k+1)}{9}$
8	4	1	52	13
17	8	2	221	26
26	12	3	507	39
35	15	4	910	52
44	18	5	1430	65

We note in passing that no subgraph of such a regular graph is isomorphic to a subgraph of Sch_{10}. However a subgraph of the graphs arising from G_{17} and G_{18} may be since deletion of all vertices of types $(2,1)$ and $(1,5)$ will result in a graph which has a star complement $H \cong K_{2,5}$ corresponding to both $\mu = 1$ and $\mu = -1$.

Theorem 9.8 provides us with an example of a unique maximal core graph for the eigenvalue $\mu = -1$, subject to the condition that the constructed graph is a k-regular graph. We can also give an example of a unique maximal core graph for the eigenvalue $\mu = 0$, subject to the condition that the core graph is connected.

Example 9.9 [P. Rowlinson] For any integer $t > 1$, the cocktail-party graph $\overline{tK_2}$ is the unique maximal reduced connected graph having K_t as a star complement for 0.

Chapter 10

Star complement $K_{1,s}$.

The case where G is an s-regular graph with $K_{1,s}$ as a star complement is analysed in [Row3] where the regularity of the graph is exploited in much the same way as was illustrated in Chapter 6. It was found that when $\mu = 1$ G is the Clebsch graph, and when $\mu = 2$ G is the Higman-Sims graph. The case where the star complement is $K_{1,5}$ can be found in [Row4]. In this chapter we will deal with the general case when $|X| \geq 3$. As we have pointed out previously, it is sufficient to determine the graphs where X is maximal as any other graph will be an induced subgraph of a maximal graph.

One way of tackling this case would have been to apply the results found in Chapter 8; however, because the parameters are so much simpler it is clearer to start with the three equations arising from the application of the

Reconstruction Theorem as they appear in Chapter 7. Here $r = 1$ and $t = 0 = c = \gamma$.

Let G be a graph with star set X corresponding to μ such that $|X| \geq 3$. Let $H \cong K_{1,s}$ be the corresponding star complement in G. As before, the vertex set of H is divided into two, a singleton set and the set S with $|S| = s$. We say a vertex is of type (a, b) if it is joined to a vertices in the singleton set and b vertices in set S.

We will show that $\mu^2 < s$ and that μ^2 is an integer. Furthermore we will show that it is possible to determine the specific values that s can take for a given eigenvalue, along with the possible types of the vertices in X. We give examples of graphs which can arise for particular eigenvalues and provide tables of additional parameter sets $(\mu, s, (a, b)(\alpha, \beta), \rho)$ for $|\mu| \leq 3$. (As before, ρ is the number of common neighbours in H of u and v.)

An example where $|X| = 2$ does arise: here $\mu = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ and G is a 5-cycle (see proof of Proposition 10.4.)

We observe that a can take only two values, 0 or 1 and so if there are at least three vertices joined to H there will be one pair of vertices of types (a, b) and (a, β) for some a. It follows that in the case where $|X| \geq 3$ we can let $\alpha = a$ in $g_2(\mu) = g_3(\mu) = 0$; we can also let $a^2 = a$. After simplifying we obtain

$$\begin{aligned}
g_1(x) &= (x^2-s)(a+b-x^2)+2xab+as+b^2 \\
g_2(x) &= (b-\beta)(x^2-s+2xa+b+\beta) \\
g_3(x) &= (x^2-s)(\rho+a_{uv}x)+as+b\beta+xa(\beta+b) \\
&\qquad \text{where} \quad a_{uv}=\begin{cases} 1 & \text{if} \quad u\sim v \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and ρ is the number of common neighbours in H of u and v.

The first step is to establish bounds for the eigenvalue μ. We do this by considering the possible types (a,b) of the vertex u in $H+u$.

Lemma 10.1 *Let H be a star complement for μ in $H+u$ with $H\cong K_{1,s}$ and let u be a vertex of type (a,b).*

(i) If $a=0$, then $b=\frac{1}{2}\left(s-\mu^2\pm\sqrt{(s-\mu^2)(s-5\mu^2)}\right)$.

(ii) If $a=1$, then $b=\frac{1}{2}\left(s-\mu^2-2\mu\pm\sqrt{(s-\mu^2)(s-5\mu^2-4\mu)}\right)$.

Proof. The eigenvalues of $H+u$ satisfy the equation $g_1(x)=0$.

(*i*) When $a=0$ we have $g_1(\mu)=(\mu^2-s)(b-\mu^2)+b^2=0$. Arranging this as a quadratic in b we get $b^2-b(s-\mu^2)-\mu^2(\mu^2-s)=0$ and so

$$\begin{aligned}
b &= \frac{1}{2}\left(s-\mu^2\pm\sqrt{(s-\mu^2)^2+4\mu^2(\mu^2-s)}\right) \\
&= \frac{1}{2}\left(s-\mu^2\pm\sqrt{(s-\mu^2)(s-5\mu^2)}\right).
\end{aligned}$$

(*ii*) When $a = 1$ we have $g_1(\mu) = (\mu^2 - s)(1 + b - \mu^2) + 2\mu b + s + b^2 = 0$ which can be written as

$$\begin{aligned} b^2 - b(s - \mu^2 - 2\mu) + s + (1 - \mu^2)(\mu^2 - s) &= 0; \\ b^2 - b(s - \mu^2 - 2\mu) + s + \mu^2 - s - \mu^4 + \mu^2 s &= 0; \\ b^2 - b(s - \mu^2 - 2\mu) - \mu^2(\mu^2 - 1 - s) &= 0. \end{aligned} \quad (10.1)$$

Thus

$$\begin{aligned} b &= \frac{1}{2}\left((s - 2\mu - \mu^2) \pm \sqrt{(s - 2\mu - \mu^2)^2 + 4\mu^2(\mu^2 - 1 - s)}\right) \\ &= \frac{1}{2}\left((s - \mu^2 - 2\mu) \pm \sqrt{(s - \mu^2)(s - 5\mu^2 - 4\mu)}\right). \end{aligned}$$

The result follows. □

Proposition 10.2 *Let H be a star complement for μ in $H+u$. If $H \cong K_{1,s}$ then $H + u$ is connected and one of the following holds:*

(*a*) *When $\mu^2 < s$, $-\frac{1}{5}\left(2 + \sqrt{4 + 5s}\right) \le \mu \le \sqrt{\frac{s}{5}}$.*

(*b*) *When $\mu^2 > s$, $-\frac{1}{2}\left(1 + \sqrt{1 + 8s}\right) \le \mu < -\sqrt{s}$.*

Proof. That $H + u$ is connected follows directly from Lemma 8.1.

(*a*) Let $\mu^2 < s$. We consider the expressions obtained for b in Lemma 10.1. When $a = 0$, in order to ensure that we get real solutions for b we must have $(s - 5\mu^2) \ge 0$, since $(s - \mu^2) > 0$. Hence $-\sqrt{\frac{s}{5}} \le \mu \le \sqrt{\frac{s}{5}}$. Similarly, when $a = 1$ we must have $(s - 5\mu^2 - 4\mu) \ge 0$. The equation

$5\mu^2+4\mu-s=0$ has solutions $\mu=\frac{1}{5}\left(-2\pm\sqrt{4+5s}\right)$ and so we have $-\frac{1}{5}\left(2+\sqrt{4+5s}\right)\leq\mu\leq\frac{1}{5}\left(\sqrt{4+5s}-2\right)$. Now $-\frac{1}{5}\left(2+\sqrt{4+5s}\right)\leq-\sqrt{\frac{s}{5}}$ and $\sqrt{\frac{s}{5}}\geq\frac{1}{5}\left(\sqrt{4+5s}-2\right)$ and the result follows.

(*b*) Let $\mu^2>s$. If μ is positive then by Lemma 8.3 $\mu\leq\frac{1}{4}\left(\sqrt{5+4s}-1\right)$. In particular $\mu<\sqrt{s}$, a contradiction, and so μ is negative with $\mu<-\sqrt{s}$. By Lemma 8.2 we know that $|\mu|\leq\lambda$ where λ is the index of the graph $H+u$ when u is of type $(1,s)$. This graph has minimal polynomial $x(x+1)(x^2-x-2s)$, which is obtained from $g_1(x)=0$ when $a=1$ and $b=s$ and so $\lambda=\frac{1}{2}(1+\sqrt{1+8s})$. In particular, since μ is negative, we have $-\frac{1}{2}\left(1+\sqrt{1+8s}\right)\leq\mu$ as required. □

Thus far we have established bounds for μ as an eigenvalue for $H+u$. For the next two propositions we shall consider the graphs $H+u+v$ where u is of type (a,b) and v is of type (a,β). We suppose that H is μ-extendible so that μ is not an eigenvalue of H, but is a double eigenvalue of $H+u+v$.

In the following proofs we again make use of the fact that if μ is not an integer and we have an equation of the form $P\mu+Q=0$ where P, Q are rationals, then $P=0$ and $Q=0$.

Proposition 10.3 *Let $H\cong K_{1,s}$ be a star complement for μ in $H+u+v$. Let u and v be non-adjacent vertices of types (a,b) and (a,β) respectively and suppose that μ is not an integer. Then μ^2 is an integer with $|\mu|\leq\sqrt{\frac{s}{5}}$*

and $a = 0$.

Proof. First we suppose that $b = \beta$. Then

$$g_3(\mu) - g_1(\mu) = (\mu^2 - s)(\rho + \mu^2 - a - b) = 0.$$

Now $(\mu^2 - s) \neq 0$ and so $\mu^2 = a + b - \rho$, an integer. Using this substitution we obtain

$$g_3(\mu) = (a + b - \rho - s)\rho + as + b^2 + 2ab\mu = 0.$$

Since μ is not an integer, we have $ab = 0$ and $(a+b-\rho-s)\rho + as + b^2 = 0$. Suppose that $a = 1$. Then $b = 0$, and both u and v are of type $(1, 0)$, with $\rho = 1$. It follows that $\mu^2 = 1 + 0 - 1 = 0$, and $\mu = 0$. But μ is not an integer and so we have $a = 0$. Moreover $|\mu| \leq \sqrt{\frac{s}{5}}$ because $\mu^2 = b - \rho \leq b \leq s$ and we can apply Proposition 10.2(a), in this case applied to $|\mu|$, an algebraic conjugate of μ. Observe that this does not exclude the possibility that $a = 1$ when μ is an integer.

Secondly we suppose that $b \neq \beta$. Then

$$g_2(\mu) = (b - \beta)(\mu^2 - s + 2\mu a + b + \beta) = 0,$$

whence $\mu^2 - s = -(2\mu a + b + \beta)$. From this we have

$$\begin{aligned} g_3(\mu) &= -\rho(2\mu a + b + \beta) + as + b\beta + \mu a(b + \beta) \\ &= a(b + \beta - 2\rho)\mu + as + b\beta - \rho(b + \beta) = 0. \end{aligned}$$

Since μ is not an integer we have $a(b+\beta-2\rho) = 0$ and $as+b\beta-\rho(b+\beta) = 0$. Now suppose that $a = 1$. Then $b - \rho + \beta - \rho = 0$ and so $b = \beta = \rho$. But $b \neq \beta$ and so we have $a = 0$. Moreover, when $a = 0$, we have $\mu^2 = s - b - \beta$ and so μ^2 is an integer. Again $\mu^2 < s$. Thus, by again applying Proposition 10.2(a) to $|\mu|$, we have $|\mu| \leq \sqrt{\frac{s}{5}}$. □

Proposition 10.4 *Let $H \cong K_{1,s}$ be a star complement for μ in $H+u+v$. Let u and v be adjacent vertices of types (a,b) and (a,β) respectively. Then μ is an integer.*

Proof. Suppose that μ is not an integer.

Suppose first that $b = \beta$. Then

$$g_3(\mu) - g_1(\mu) = (\mu^2 - s)(\mu^2 + \mu + \rho - a - b) = 0$$

and so

$$\mu^2 + \mu = a + b - \rho. \tag{10.2}$$

From this we have

$$g_3(\mu) = (a + b - \rho - \mu - s)(\rho + \mu) + as + b^2 + 2\mu ab = 0$$

and so we have

$$\mu(a + b - 2\rho - s + 2ab + 1) + \rho(a + b - \rho - s + 1) + as + b^2 - a - b = 0.$$

Since μ is not an integer we have

$$\begin{aligned} a+b-2\rho-s+2ab+1 &= 0 && (10.3) \\ \text{and} \quad \rho(a+b-\rho-s+1)+as+b^2-a-b &= 0. && (10.4) \end{aligned}$$

From Equation (10.3) we have $a+b-\rho-s+1=\rho-2ab$. Substituting this into Equation (10.4) we obtain $\rho(\rho-2ab)=a+b-as-b^2$; on completing the square we get $(\rho-ab)^2=ab^2+a+b-as-b^2$.

Now suppose that $a=0$; then $\rho^2=b(1-b)\geq 0$. But $b\neq 0$ since adding an isolated vertex does not affect the distinct eigenvalues, and so we have $b=1$ with $\rho=0$. It follows that $\mu^2+\mu=1$ and so $\mu=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}$. Moreover $s=a+b-2\rho+2ab+1=2$ and so the graph is a 5-cycle. However, since we are only considering the case when $|X|\geq 3$, this graph is excluded.

Now suppose that $a=1$. Then

$$(\rho-b)^2=b^2+1+b-s-b^2=1+b-s\geq 0. \qquad (10.5)$$

This gives us $s-1\leq b$ and so either $b=s-1$ or $b=s$. Since $\mu\notin\{-1,0\}$, if $b=s-1$ then $\rho=b$. For then both u and v are of type $(1,s-1)$ and necessarily $\rho=s-2+1=s-1=b$. However, if we use the substitution $\rho=b$ in Equation (10.3) we have $1+b-2b-s+2b+1=2+b-s=0$ whence $b=s-2$, a contradiction. It follows that $b\neq s-1$. If $b=s$ then both u and v are of type $(1,s)$ and $\rho=1+s$. This means that u and v have the same H-neighbourhood indicating that μ is -1, a contradiction.

Secondly we suppose that $b \neq \beta$. The equation $g_2(\mu) = 0$ yields

$$\mu^2 - s = -(2a\mu + b + \beta)$$

as before, but now, since u and v are adjacent,

$$g_3(\mu) = -(2a\mu + b + \beta)(\rho + \mu) + as + b\beta + a\mu(b + \beta) = 0.$$

Multiplying out and substituting for μ^2 we obtain

$$2a(2a\mu + b + \beta - s) - \mu(b + \beta) - 2a\mu\rho - \rho(b + \beta) + as + b\beta + a\mu(b + \beta) = 0.$$

Collecting terms we obtain an equation of the form $P\mu + Q = 0$; specifically,

$$\mu(4a^2 - (b + \beta) - 2a\rho + a(b + \beta)) + 2a(b + \beta - s) - \rho(b + \beta) + as + b\beta = 0.$$

Since μ is not an integer we have $2a(2a - \rho) + (b + \beta)(a - 1) = 0$ and $(b + \beta)(2a - \rho) - as + b\beta = 0$.

Now suppose that $a = 0$; then $b + \beta = 0$ whence $b = 0 = \beta$; a contradiction.

Now suppose that $a = 1$; then $\rho = 2$ and $s = b\beta$. Again using $\mu^2 - s = -(2a\mu + b + \beta)$ we have

$$g_1(\mu) = -(2\mu + b + \beta)(1 + b + 2\mu + b + \beta - s) + 2\mu b + s + b^2 = 0.$$

This becomes

$$\begin{aligned}
-4\mu^2 - 2\mu(1+2b+\beta-s+b+\beta-b) - (b+\beta)(1+2b+\beta-s)+s+b^2 &= 0; \\
2(3-2(b+\beta)+s)\mu + (b+\beta)(3-2b-\beta+s)+b^2-3s &= 0.
\end{aligned}$$

Since μ is not an integer we have $3-2(b+\beta)+s=0$, whence $3-2b-\beta+s=\beta$. Using this substitution we obtain $\beta(b+\beta)+b^2-3s=0$. Since $s=b\beta$ this becomes $b^2-2b\beta+\beta^2=0$ whence $b=\beta$; a contradiction. Thus μ is an integer. □

Corollary 10.5 *Let G be a graph with star complement $H \cong K_{1,s}$ corresponding to μ and let the star set X consist of adjacent vertices of the same type $(1,b)$. Then $\mu=-1$ and G has a maximal core graph which is one of the following:*

(i) K_3 with a pendant edge.

(ii) $K_2 \nabla s K_1$.

Proof. By the proof of Proposition 10.4 we have two cases to consider: *(i)* $b=s-1$ and *(ii)* $b=s$. In both cases $\mu=-1$ by Equation (10.2).

(i) We consider $g_1(-1)=0$ when $a=1$ and $b=s-1$.

$$g_1(-1)=(1-s)^2-2(s-1)+s+(s-1)^2=2-s=0$$

whence $s=2$ and so u and v are both of type $(1,1)$. Now $g_3(-1)=(1-2)(\rho-1)+1=2-\rho=0$ whence $\rho=2$. Thus the maximal core graph of G is K_3 with a pendant edge, the duplicated vertex in G being one of the vertices with degree 2.

(ii) In the case where $b = s$ we do not obtain a specific value for s from $g_1(-1) = 0$. However we can obtain a value for ρ in terms of s since $g_3(-1) = (1-s)(\rho-1-s) = 0$ whence $\rho = 1+s$. It follows that the maximal core graph of G is $K_2 \nabla s K_1$, the duplicated vertex in G being one of the vertices in K_2. □

Given s, Proposition 10.2 provides us with search intervals for μ. Numerical investigation failed to produce a situation when μ was such that $\mu^2 > s$. The following result shows that we have $\mu^2 < s$ in all cases. Note that Proposition 10.3 does not cover the case where μ is an integer.

Proposition 10.6 *Let $H \cong K_{1,s}$ be a star complement for μ in $H+u+v$. Let u and v be vertices of types (a, b) and (a, β) respectively. Then $\mu^2 < s$.*

Proof. We suppose by way of contradiction that $\mu^2 > s$. It follows from Proposition 10.2 that μ is negative. Also note that since $s \geq 1$ we have $\mu < -1$.

First we show that u and v are the same type. Lemma 10.1 gives us general solutions for b. Suppose that $a = 0$; then there is at most one non-negative integer solution for b since $\mu^2 > s$. Now suppose that $a = 1$ and that u is of type $(1, b)$ and v is of type $(1, \beta)$ with $b \neq \beta$. Let $k(x) =$

$x^2 - x(s - \mu^2 + 2|\mu|) - \mu^2(\mu^2 - 1 - s)$. Then, from Equation (10.1), we have $k(b) = k(\beta) = 0$. It follows that $b+\beta = s-\mu^2+2|\mu|$ and $b\beta = -\mu^2(\mu^2-1-s)$. Now $b\beta \geq 0$ and so $\mu^2 - 1 - s \leq 0$. Hence $\mu^2 = s + 1$. Substituting this value into $g_1(x) = 0$ (together with $a = 1$) we obtain the equation $b - s - 2|\mu|b + s + b^2 = 0$; that is $b(b + 1 - 2|\mu|) = 0$ and so $b \in \{0, 2|\mu| - 1\}$. Now suppose that u is of type $(1, 0)$ and that v is of type $(1, 2|\mu| - 1)$; then $\rho = 1$ and so from $g_3(x) = 0$ we have $1 - a_{uv}|\mu| + s - |\mu|(2|\mu| - 1) = 0$. Hence $|\mu|(1 - a_{uv}) - \mu^2 = 0$. If $u \not\sim v$ then we have $|\mu| - \mu^2 = 0$, that is $\mu = -1$ whence $s = 0$; a contradiction. If $u \sim v$ then $\mu = 0$; a contradiction. Thus we may conclude that u and v are of the same type (a, b).

We have $g_3(\mu) - g_1(\mu) = (\mu^2 - s)(\rho - a_{uv}|\mu| - a - b + \mu^2) = 0$; that is

$$\rho - a_{uv}|\mu| - a - b + \mu^2 = 0. \tag{10.6}$$

Now suppose that $a = 1$ and that $u \sim v$; then by Corollary 10.5, $\mu = -1$, thus contradicting $\mu^2 > s$. Suppose that $a = 1$ and that $u \not\sim v$ then from Equation (10.6) we have $\rho - 1 = b - \mu^2$. Now $b - \mu^2 < 0$ and so $\rho < 1$, but $\rho \geq 1$ since u, v are both of type $(1, b)$ and so we have a contradiction. Suppose that $a = 0$ and $u \not\sim v$. From Equation (10.6) we have $\rho = b - \mu^2$, but $\rho \geq 0$ and so we have a contradiction.

This leaves us with the case where u and v are of type $(0, b)$ and $u \sim v$. By Proposition 10.4, μ is an integer and so $\mu \leq -2$ since $\mu < -1$. The number of vertices in S which are adjacent to either u or v or both is $b-\rho+b-\rho+\rho=2b-\rho$ and so $2b-\rho\leq s$. From Equation (10.6) we have $b=\rho+\mu^2-|\mu|$, hence

$$\rho+2\mu^2-2|\mu|\leq s<\mu^2$$

and so

$$\rho<\mu^2-2|\mu|\leq 0, \quad \text{since} \quad \mu\leq -2$$

and we have a contradiction. Thus we may conclude that $\mu^2<s$.

□

Before we go on to give specific examples it may be useful to summarize our results so far.

Let G be a graph with star set X corresponding to μ such that $|X|\geq 3$. If $H\cong K_{1,s}$ is a star complement for μ in G then $\mu^2<s$ and μ^2 is an integer. Let u be a vertex in X of type (a,b) and consider the induced subgraph $H+u$. If μ^2 is an integer but μ is not, then u is of type $(0,b)$. If μ is an integer then u can be of type $(0,b)$ or $(1,b)$. Lemma 10.1 gives us expressions for b in the two cases $a=0$ and $a=1$. We use the fact that b must be an integer to obtain an equation of the form $n^2=m^2+l^2$ where n,m and l are integers. The solutions for such an equation were given in Remark 1.24 and are used in this instance to give values for s and b for a given μ.

Remark 10.7 Let u be of type $(0, b)$. Then from Lemma 10.1 we have $b = \frac{1}{2}\left(s - \mu^2 \pm \sqrt{(s-\mu^2)(s-5\mu^2)}\right)$. Since b is an integer we have

$$(s-\mu^2)(s-5\mu^2) = m^2 \tag{10.7}$$

where m is an integer. Without loss of generality we take $m \geq 0$. We rewrite Equation (10.7) as follows:

$$\left(n - \frac{(5\mu^2 - \mu^2)}{2}\right)\left(n + \frac{(5\mu^2 - \mu^2)}{2}\right) = m^2, \quad \text{where} \quad n = s - \frac{(5\mu^2 + \mu^2)}{2}.$$

That is

$$(n - 2\mu^2)(n + 2\mu^2) = m^2, \quad \text{where} \quad n = s - 3\mu^2.$$

Hence we have the equation $n^2 = m^2 + (2\mu^2)^2$ with integer solutions given by

$$(m, 2\mu^2, n) \quad = \quad (t(x^2 - y^2), 2txy, t(x^2 + y^2)) \quad x \geq y \geq 0,$$

where x, y and t are integral parameters.

Let $2\mu^2 = 2txy$. Then

$$\begin{aligned} s &= n + 3\mu^2 = t(x^2 + y^2) + 3txy \\ \text{and} \quad b &= \frac{1}{2}(n + 2\mu^2 \pm m) \\ &= \frac{1}{2}(t(x^2 + y^2) + 2txy \pm t(x^2 - y^2)) \\ &= tx(x+y) \quad \text{and} \quad ty(x+y). \end{aligned}$$

Note that in this case where $a = 0$, the values we obtain for s and b are dependent solely on the value of μ^2. Thus we obtain the same values regardless of the sign of the eigenvalue μ.

Remark 10.8 Let u be of type $(1, b)$. Then from Lemma 10.1 we have $b = \frac{1}{2}\left(s - \mu^2 - 2\mu \pm \sqrt{(s-\mu^2)(s-5\mu^2-4\mu)}\right)$. Since b is an integer we have

$$(s - \mu^2)(s - 5\mu^2 - 4\mu) = m^2 \tag{10.8}$$

where m is an non-negative integer. As before we can rewrite Equation (10.8) to obtain

$$(n - 2(\mu^2 + \mu))(n + 2(\mu^2 + \mu)) = m^2, \quad \text{where} \quad n = s - (3\mu^2 + 2\mu).$$

Hence we have the equation $n^2 = m^2 + (2(\mu^2 + \mu))^2$ with integer solutions given by

$$(m, 2(\mu^2 + \mu), n) \;=\; (t(x^2 - y^2), 2txy, t(x^2 + y^2)) \quad x \geq y \geq 0,$$

where x, y and t are integral parameters.

Let $2(\mu^2 + \mu) = 2txy$. Then

$$\begin{aligned} s &= n + 3\mu^2 + 2\mu = t(x^2 + y^2) + 3txy - \mu \\ \text{and } b &= \frac{1}{2}(s - \mu^2 - 2\mu \pm m) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(n + 3\mu^2 + 2\mu - \mu^2 - 2\mu \pm m) \\
&= \frac{1}{2}(t(x^2 + y^2) + 2\mu^2 \pm t(x^2 - y^2)) \\
&= tx^2 + \mu^2 \quad \text{and} \quad ty^2 + \mu^2.
\end{aligned}
$$

Having found the possible types for the vertex u we then consider the whole graph G and obtain values for ρ for every possible u, v pair of vertices in X. In order to do this we must also consider the possibility that u is of type (a, b) and v is of type (α, β) where $a \neq \alpha$ and so we use the equation $g_3(\mu) = 0$ as given in Remark 7.2 with $r = 1$, namely

$$
\rho = \frac{a\alpha s + b\beta + \mu(\alpha b + a\beta)}{s - \mu^2} - a_{uv}\mu \tag{10.9}
$$

where $a_{uv} = \begin{cases} 1 & \text{if} \quad u \sim v \\ 0 & \text{otherwise.} \end{cases}$

For our first set of examples we examine the graphs arising when the eigenvalue is -1. As a preliminary we describe graphs $G(s)$, G_{15} and G_{16}.

The graph G(s).

The vertex set of $G(s)$ consists of two singleton sets R and U, and two independent sets, S and V where $|S| = |V| = s$. The subgraphs induced by $R \cup S$, $R \cup V$, $U \cup S$ and $U \cup V$ are all $K_{1,s}$ and the subgraphs induced by $S \cup V$ and $R \cup U$ are sK_2 and K_2 respectively. This fully describes the edge set of $G(s)$. (This graph is known as $sK_2 \nabla K_2$.) The spectrum of this graph contains the eigenvalue -1 with multiplicity $s+1$. Any one of the $K_{1,s}$ subgraphs may be taken as a star complement corresponding to -1.

In particular $G(5)$ is the graph $5K_2 \nabla K_2$.

For these next two graphs it is convenient to label the vertices in S and V so that $s_i \sim v_j$ if and only if $i = j$.

The graph $\mathbf{G_{15}}$.

The vertex set of G_{15} is that of $G(5)$ together with one other independent set T_1, where $T_1 = \{t_1, t_2, t_3\}$. The complete edge set of G_{15} is the edge set of $G(5)$ together with the edge sets $E(T_1, S)$ and $E(T_1, V)$ where $E(T_1, V) = \{(t_1, v_1), (t_1, v_2), (t_2, v_1), (t_2, v_3), (t_3, v_2), (t_3, v_3)\}$. Similarly $E(T_1, S) = \{(t_1, s_1), (t_1, s_2), (t_2, s_1), (t_2, s_3), (t_3, s_2), (t_3, s_3)\}$.

The spectrum of this graph includes the eigenvalue -1 with multiplicity nine. Any one of the $K_{1,5}$ subgraphs may be taken as a star complement corresponding to -1.

The graph $\mathbf{G_{16}}$.

The vertex set of G_{16} is that of $G(5)$ together with one other independent set T_2, where $T_2 = \{t_2, t_3, t_4, t_5\}$. The complete edge set of G_{16} is the edge set of $G(5)$ together with the edge sets $E(T_2, S)$ and $E(T_2, V)$ where $E(T_2, S) = \{(t_i, s_i) : i = 2, 3, 4, 5\} \cup \{(t_i, s_1) : i = 2, 3, 4, 5\}$. As before with G_{15}, $E(T_2, V)$ mimics $E(T_2, S)$.

The spectrum of this graph includes the eigenvalue -1 with multiplicity ten. Any one of the $K_{1,5}$ subgraphs may be taken as a star complement corresponding to -1.

We now come to the reason for considering the particular case $s = 5$. We let $H \cong K_{1,s}$ and determine the possible types of the vertices in X using the expressions given in Lemma 10.1 and the results given in Remarks 10.7 and 10.8.

Let $a = 1$. Then $b = \frac{1}{2}(s + 1 \pm \sqrt{(s-1)^2})$ and so the vertices which may be added are of the type $(1, 1)$ and $(1, s)$. Let $a = 0$. From Remark 10.7 we have $2\mu^2 = 2txy$ and so $(t, x, y) = (1, 1, 1)$. Thus $s = 5$ and $b = 2$. Hence when $H \cong K_{1,5}$ the vertices which may be added are of the type $(1, 1)$, $(1, 5)$ and $(0, 2)$. Using Equation (10.9) we obtain values for ρ and present the results in Table 10.1.

We now consider the graphs arising in the case where $H \cong K_{1,5}$ is a star complement for -1. We will show that the graphs G_{15} and G_{16} are the two maximal core graphs which arise. The following result can also be found in [Row4].

u, v of type:		ρ	
$(1,1),(1,1)$		$1+a_{uv}$	
$(1,1),(1,5)$	$(1,1),(1,s)$	$1+a_{uv}$	$1+a_{uv}$
$(1,1),(0,2)$		a_{uv}	
$(1,5),(1,5)$	$(1,s),(1,s)$	$5+a_{uv}$	$s+a_{uv}$
$(1,5),(0,2)$		$2+a_{uv}$	
$(0,2),(0,2)$		$1+a_{uv}$	

Table 10.1: $\mu = -1$.

Theorem 10.9 *Let G be a maximal core graph with $H \cong K_{1,5}$ as a star complement for the eigenvalue $\mu = -1$ with $|X| \geq 3$. Then either $G = G_{15}$ or $G = G_{16}$.*

Proof. We divide the vertex set of H into two; a singleton set and the set S with $|S| = 5$. When u is of type $(1,1)$ and v is of type $(1,5)$ then u must be adjacent to v since their common H-neighbourhood contains precisely two vertices. Furthermore, when u is of type $(0,2)$ and v is of type $(1,5)$ their common H-neighbourhood contains precisely two vertices and so u and v cannot be adjacent.

When u and v are adjacent vertices of the same type they have exactly the same H-neighbourhood and so they are duplicate vertices as expected. It follows that if u and v are vertices of the same type in G then $u \not\sim v$. When u and v are both of type $(1,1)$, both are adjacent to the vertex of degree 5 in H. Therefore they must be adjacent to different vertices in S

and so the maximum number of vertices of type $(1, 1)$ which can be added is five. Similarly the maximum number of vertices of type $(1, 5)$ which can be added is one. The only constraint on adding two vertices of the type $(0, 2)$ is that $\rho = 1$; that is, they have exactly one common neighbour in S. This can be considered in terms of the families of 2-sets taken from a 5-set and intersecting in a 1-set. There are just two possibilities for such a family: without loss of generality either (i) $\mathcal{F} = \{ \{1, 2\}, \{1, 3\}, \{2, 3\} \}$ or (ii) $\mathcal{F} = \{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\} \}$. Clearly, in case (i) the maximum number of vertices of type $(0, 2)$ which can be added is three; in case (ii) the maximum number is four. In both cases it is possible to have the maximum number of each type of vertex present at the same time. In case (i) G is the graph G_{15} and in case (ii) G is the graph G_{16}.

□

Example 10.10 Let G be the maximal core graph with $H \cong K_{1,s}$ as a star complement for -1 with $|X| \geq 3$. If $s \neq 5$ then G is $G(s)$ as shown in Figure 10.1.

For our second set of examples we examine the graphs arising when the eigenvalue is 1. We have already seen that when $a = 0$, $s = 5$ and $b = 2$. Now let $a = 1$. From Remark 10.8 we have $2(\mu^2 + \mu) = 2txy$ and so $(t, x, y) = (2, 1, 1)$ and $(1, 2, 1)$. When $(t, x, y) = (2, 1, 1)$ we have $s = 9$ and

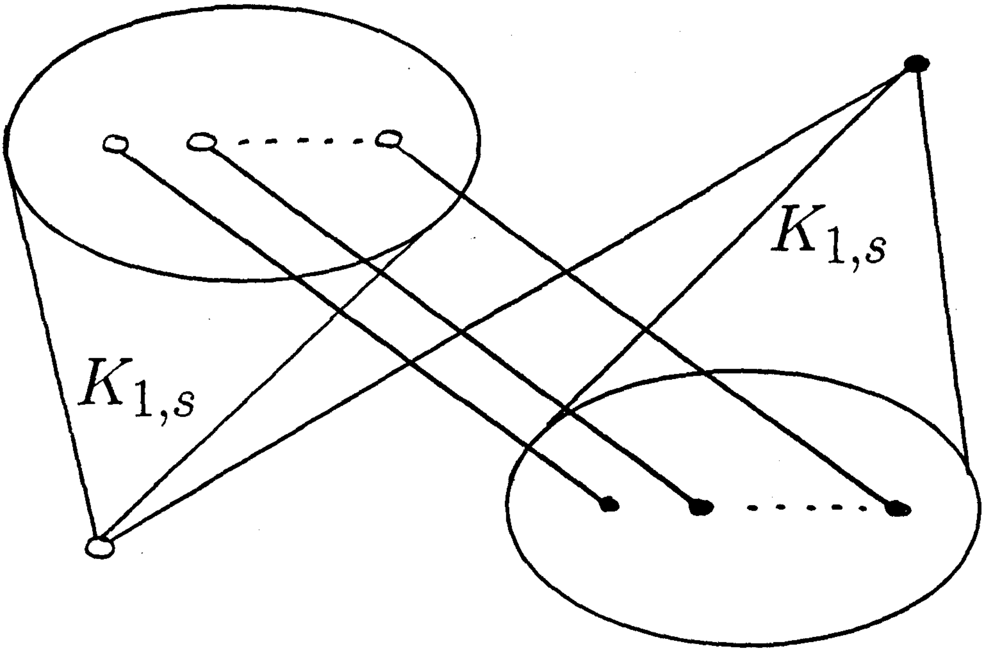


Figure 10.1: $G(s)$

$b = 3$. When $(t, x, y) = (1, 2, 1)$ we have $s = 10$ and $b \in \{2, 5\}$. We calculate the values for ρ as before and present the results in Table 10.2.

Example 10.11 Let G be a maximal graph with star complement $H \cong K_{1,s}$ corresponding to μ. Let $\mu = 1$ and $s = 5$. Then the star set X consists solely of vertices of type $(0, 2)$. This can be considered in terms of the family $\mathcal{F}$ of all 2-sets taken from a 5-set. These 2-sets intersect in either 0 or 1 point which is consistent with the values obtained for ρ in Table 10.2, the corresponding vertices being adjacent if and only if the 2-sets have no point in common. This is comparable to the situation in Theorem 9.4 where we

s	u, v of type:	ρ
5	$(0,2)$, $(0,2)$	$1 - a_{uv}$
9	$(1,3)$, $(1,3)$	$3 - a_{uv}$
10	$(1,2)$, $(1,2)$	$2 - a_{uv}$
	$(1,2)$, $(1,5)$	$3 - a_{uv}$
	$(1,5)$, $(1,5)$	$5 - a_{uv}$

Table 10.2: $\mu = 1$.

had vertices of type $(0,3)$ in the case where $H \cong K_{2,5}$. Thus we see that in this case the subgraph induced by X is the Petersen graph. Here G is the Clebsch graph (see [Row4, Thm 3.6].)

Example 10.12 Let G be a maximal graph with star complement $H \cong K_{1,s}$ corresponding to μ. Let $\mu = 1$ and $s = 9$. Then the star set consists solely of vertices of type $(1,3)$. Again this can be considered in terms of families of 3-sets taken from a 9-set and intersecting in either a 2-set or a 1-set. One such family is $\mathcal{F} = \{\{1,2,3\},\{3,4,5\},\{1,6,5\},\{1,7,4\},\{3,7,6\},\{2,7,5\},\{2,6,4\}\}$ which is the projective geometry $PG(2,2)$.

The graph where the S-neighbourhoods of the vertices in X are described by $PG(2,2)$ has 17 vertices and is the smallest maximal graph in this situation. There are 24 others, the largest of which has 38 vertices. The adjacency matrices and spectra for these graphs were found by M. Lepović using a computer.

Example 10.13 Let G be a maximal graph with star complement $H \cong K_{1,s}$

corresponding to μ. Let $\mu = 1$, $s = 10$ and suppose that the star set consists solely of vertices of type $(1, 2)$. The number of 2-sets taken from a 10-set is ${}^{10}C_2 = 45$, the 2-sets intersecting in either a 1-set or a 0-set. This corresponds to the values for ρ obtained in Table 10.2 and so the maximal star set consists of 45 vertices. Thus G is a unique maximal graph with 56 vertices.

We will now give the remaining parameter sets for $|\mu| \leq 3$. We start with the case where μ is not an integer. In this case $a = 0$ and we can see from Equation 10.9 that $a_{uv} = 0$. It follows that the constructed star set will be an independent set. We present the results in Table 10.3. The parameter sets obtained when μ is an integer can be found in Table 10.4, Table 10.5 and Table 10.6.

We observe from Table 10.3 that, in the case where μ is not an integer, for a given μ, the smallest value obtained for s is $5\mu^2$. In this case the star set consists of a single type of vertex, namely $(0, 2\mu^2)$, and the value for ρ is μ^2. Further examination reveals another pattern. Let $\mu^4 = qr$ where q and r are integers. It is always the case that for a given s we have $\rho \in \{\mu^2, q, r\}$. This would seem to indicate that the number of different values of s for a given μ can be determined by the number of ways μ^4 can be expressed as a product of two integers. For example, if $\mu^2 = 8$, then

μ	s	u, v of type:	$\rho,\ u \not\sim v$
$\pm\sqrt{2}$	10	$(0,4)$, $(0,4)$	2
	11	$(0,3)$, $(0,6)$	2
		$(0,3)$, $(0,3)$	1
		$(0,6)$, $(0,6)$	4
$\pm\sqrt{3}$	15	$(0,6)$, $(0,6)$	3
	19	$(0,4)$, $(0,12)$	3
		$(0,4)$, $(0,4)$	1
		$(0,12)$, $(0,12)$	9
$\pm\sqrt{5}$	25	$(0,10)$, $(0,10)$	5
$\pm\sqrt{6}$	30	$(0,12)$, $(0,12)$	6
	31	$(0,10)$, $(0,15)$	6
		$(0,10)$, $(0,10)$	4
		$(0,15)$, $(0,15)$	9
	33	$(0,9)$, $(0,18)$	6
		$(0,9)$, $(0,9)$	3
		$(0,18)$, $(0,18)$	12
	38	$(0,8)$, $(0,24)$	6
		$(0,8)$, $(0,8)$	2
		$(0,24)$, $(0,24)$	18
	55	$(0,7)$, $(0,42)$	6
		$(0,7)$, $(0,7)$	1
		$(0,42)$, $(0,42)$	36
$\pm\sqrt{7}$	35	$(0,14)$, $(0,14)$	7
	71	$(0,8)$, $(0,56)$	7
		$(0,8)$, $(0,8)$	1
		$(0,56)$, $(0,56)$	49
$\pm\sqrt{8}$	40	$(0,16)$, $(0,16)$	8
	44	$(0,12)$, $(0,24)$	8
		$(0,12)$, $(0,12)$	4
		$(0,24)$, $(0,24)$	16
	58	$(0,10)$, $(0,40)$	8
		$(0,10)$, $(0,10)$	2
		$(0,40)$, $(0,40)$	32
	89	$(0,9)$, $(0,72)$	8
		$(0,9)$, $(0,9)$	1
		$(0,72)$, $(0,72)$	64

Table 10.3: $\sqrt{2} \leq |\mu| \leq \sqrt{8}$: $a = 0$

$(q, r) \in \{(8,8), (4,16), (2,32), (1,64)\}$ which is consistent with the fact that we found four values for s in this case.

Example 10.14 We give an example of a maximal graph G with star complement $H \cong K_{1,s}$ corresponding to μ where $\mu = \pm\sqrt{2}$, $s = 11$ and the star set consists solely of vertices of type $(0,3)$. From Table 10.3 we see that $\rho = 1$. Hence this can be considered in terms of families of 3-sets taken from an 11-set and intersecting in a 1-set.

Let $\mathcal{F} = \{\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{1,8,9\}, \{1,10,11\}\}$. In order to show that this yields a maximal graph we must show that it is not possible to add another 3-set. There are two cases to consider; a 3-set containing 1 and a 3-set which does not contain 1. We cannot add a 3-set containing 1 since such a set would have either a 2- or 3-set intersection with one of the sets in $\mathcal{F}$. We cannot add a 3-set which does not contain 1 since such a set would have a 1-set intersection with at most three sets in $\mathcal{F}$.

In preparation for our final example we consider the graph G constructed on the unique $S(3,6,22)$ design. This design has point set Y with $|Y| = 22$ and block set $\mathcal{B}$ with $|\mathcal{B}| = 77$. Let $V(G) = Y \cup \mathcal{B} \cup \{x\}$ where Y is an independent set and x is adjacent to every vertex in Y. A vertex in Y and a vertex in $\mathcal{B}$ are adjacent if and only if they are incident. Two vertices in $\mathcal{B}$ are adjacent if and only if the intersection of the blocks is empty. Then

μ	s	u, v of type:	ρ
2	20	$(0,8)$, $(0,8)$	$4 - 2a_{uv}$
	22	$(0,6)$, $(0,12)$	$4 - 2a_{uv}$
		$(0,6)$, $(0,6)$	$2 - 2a_{uv}$
		$(0,12)$, $(0,12)$	$8 - 2a_{uv}$
	28	$(1,10)$, $(1,10)$	$7 - 2a_{uv}$
	29	$(0,5)$, $(0,20)$	$4 - 2a_{uv}$
		$(0,5)$, $(0,5)$	$1 - 2a_{uv}$
		$(0,20)$, $(0,20)$	$16 - 2a_{uv}$
		$(1,8)$, $(1,13)$	$7 - 2a_{uv}$
		$(1,8)$, $(1,8)$	$5 - 2a_{uv}$
		$(1,13)$, $(1,13)$	$10 - 2a_{uv}$
		$(0,5)$, $(1,8)$	$2 - 2a_{uv}$
		$(0,5)$, $(1,13)$	$3 - 2a_{uv}$
		$(0,20)$, $(1,8)$	$8 - 2a_{uv}$
		$(0,20)$, $(1,13)$	$12 - 2a_{uv}$
	31	$(1,7)$, $(1,16)$	$7 - 2a_{uv}$
		$(1,7)$, $(1,7)$	$4 - 2a_{uv}$
		$(1,16)$, $(1,16)$	$13 - 2a_{uv}$
	36	$(1,6)$, $(1,22)$	$7 - 2a_{uv}$
		$(1,6)$, $(1,6)$	$3 - 2a_{uv}$
		$(1,22)$, $(1,22)$	$19 - 2a_{uv}$
	53	$(1,5)$, $(1,40)$	$7 - 2a_{uv}$
		$(1,5)$, $(1,5)$	$2 - 2a_{uv}$
		$(1,40)$, $(1,40)$	$37 - 2a_{uv}$
-2	12	$(1,6)$, $(1,6)$	$3 + 2a_{uv}$
	13	$(1,5)$, $(1,8)$	$3 + 2a_{uv}$
		$(1,5)$, $(1,5)$	$2 + 2a_{uv}$
		$(1,8)$, $(1,8)$	$5 + 2a_{uv}$
	20	$(0,8)$, $(0,8)$	$4 + 2a_{uv}$
	22	$(0,6)$, $(0,12)$	$4 + 2a_{uv}$
		$(0,6)$, $(0,6)$	$2 + 2a_{uv}$
		$(0,12)$, $(0,12)$	$8 + 2a_{uv}$
	29	$(0,5)$, $(0,20)$	$4 + 2a_{uv}$
		$(0,5)$, $(0,5)$	$1 + 2a_{uv}$
		$(0,20)$, $(0,20)$	$16 + 2a_{uv}$

Table 10.4: $\mu = \pm 2$

G is the Higman-Sims graph ([CvL, pg 107].) This graph has 100 vertices and is strongly regular of degree 22. Its spectrum contains the eigenvalue 2 with multiplicity 77. The subgraph induced by $\mathcal{B}$ is the so-called 77-graph and the subgraph induced by $Y \cup \{x\}$ is $K_{1,22}$.

Example 10.15 Let G be a maximal graph with star complement $H \cong K_{1,s}$ corresponding to μ. Let $\mu = 2$, $s = 22$ and suppose that the star set consists solely of vertices of type $(0,6)$. From Table 10.4 we have $\rho \in \{0,2\}$. Thus the H-neighourhoods can be regarded as blocks of size 6 taken from a 22-set such that any two blocks meet in either 0 or 2 points; the corresponding vertices being adjacent if and only if their intersection is empty. Such a design is the unique $S(3,6,22)$ design. It follows that G is the Higman-Sims graph. This result is also mentioned in [Row2].

We have seen that the problem of identifying graphs constructed from the parameter sets is often easier when the H-neighbourhoods of the vertices in X are thought of in terms of designs. This is especially true when the design is already related to a specific graph as in Theorem 7.3. However, even then it is sometimes necessary to restrict the star set to only one type of vertex as in Example 10.15. Examination of the parameters given in Tables 10.6 and 10.5 has yielded no readily-identifiable designs so far, but we by no means discount the possibility that further interesting graphs

μ	s	u, v of type:	ρ
-3	33	$(1,15)$, $(1,15)$	$7 + 3a_{uv}$
	34	$(1,13)$, $(1,18)$	$7 + 3a_{uv}$
		$(1,13)$, $(1,13)$	$5 + 3a_{uv}$
		$(1,18)$, $(1,18)$	$10 + 3a_{uv}$
	36	$(1,12)$, $(1,21)$	$7 + 3a_{uv}$
		$(1,12)$, $(1,12)$	$4 + 3a_{uv}$
		$(1,21)$, $(1,21)$	$13 + 3a_{uv}$
	41	$(1,11)$, $(1,27)$	$7 + 3a_{uv}$
		$(1,11)$, $(1,11)$	$3 + 3a_{uv}$
		$(1,27)$, $(1,27)$	$19 + 3a_{uv}$
	45	$(0,18)$, $(0,18)$	$9 + 3a_{uv}$
	57	$(0,12)$, $(0,36)$	$9 + 3a_{uv}$
		$(0,12)$, $(0,12)$	$3 + 3a_{uv}$
		$(0,36)$, $(0,36)$	$27 + 3a_{uv}$
	58	$(1,10)$, $(1,45)$	$7 + 3a_{uv}$
		$(1,10)$, $(1,10)$	$2 + 3a_{uv}$
		$(1,45)$, $(1,45)$	$37 + 3a_{uv}$
	109	$(0,10)$, $(0,90)$	$9 + 3a_{uv}$
		$(0,10)$, $(0,10)$	$1 + 3a_{uv}$
		$(0,90)$, $(0,90)$	$81 + 3a_{uv}$

Table 10.5: $\mu = -3$

will arise in the case where G is a graph with star complement $H \cong K_{1,s}$ corresponding to μ.

μ	s	u, v of type:	ρ
3	45	$(0,18)$, $(0,18)$	$9-3a_{uv}$
	57	$(0,12)$, $(0,36)$	$9-3a_{uv}$
		$(0,12)$, $(0,12)$	$3-3a_{uv}$
		$(0,36)$, $(0,36)$	$27-3a_{uv}$
		$(1,21)$, $(1,21)$	$13-3a_{uv}$
		$(0,12)$, $(1,21)$	$6-3a_{uv}$
		$(0,36)$, $(1,21)$	$18-3a_{uv}$
	58	$(1,18)$, $(1,25)$	$13-3a_{uv}$
		$(1,18)$, $(1,18)$	$10-3a_{uv}$
		$(1,25)$, $(1,25)$	$17-3a_{uv}$
	59	$(1,17)$, $(1,27)$	$13-3a_{uv}$
		$(1,17)$, $(1,17)$	$9-3a_{uv}$
		$(1,27)$, $(1,27)$	$19-3a_{uv}$
	63	$(1,15)$, $(1,33)$	$13-3a_{uv}$
		$(1,15)$, $(1,15)$	$7-3a_{uv}$
		$(1,33)$, $(1,33)$	$25-3a_{uv}$
	73	$(1,13)$, $(1,45)$	$13-3a_{uv}$
		$(1,13)$, $(1,13)$	$5-3a_{uv}$
		$(1,45)$, $(1,45)$	$37-3a_{uv}$
	84	$(1,12)$, $(1,57)$	$13-3a_{uv}$
		$(1,12)$, $(1,12)$	$4-3a_{uv}$
		$(1,57)$, $(1,57)$	$49-3a_{uv}$
	107	$(1,11)$, $(1,81)$	$13-3a_{uv}$
		$(1,11)$, $(1,11)$	$3-3a_{uv}$
		$(1,81)$, $(1,81)$	$73-3a_{uv}$
	109	$(0,10)$, $(0,90)$	$9-3a_{uv}$
		$(0,10)$, $(0,10)$	$1-3a_{uv}$
		$(0,90)$, $(0,90)$	$81-3a_{uv}$
	178	$(1,10)$, $(1,153)$	$13-3a_{uv}$
		$(1,10)$, $(1,10)$	$2-3a_{uv}$
		$(1,153)$, $(1,153)$	$145-3a_{uv}$

Table 10.6: $\mu = 3$

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