ALGEBRAIC CONDITIONS ON BANACH ALGEBRAS

by

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CONTENTS

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DECLARATION

I hereby declare that this thesis has been composed by myself, that the work of which it is a record has been done by myself (unless indicated otherwise), and that it has not been accepted in any previous application for a higher degree. Each source of information has been acknowledged in the text, where appropriate, by means of a reference to the relevant publication.

CHAPTER 0

INTRODUCTION

The basic mathematical structure to be studied in this thesis is that of the complex Banach algebra. We shall impose certain algebraic conditions on this structure, in addition to those it already possesses, and investigate the properties which the resulting structure possesses.

Definition Let A be a linear associative algebra over the field, $\mathbb{C}$, of complex numbers. A is called a normed algebra if we can associate with each element $x \in A$ a real number, $\| x \|$, (called the norm of x) which satisfies

1) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$,
2) $\| x + y \| \leq \| x \| + \| y \|$ for each $x, y \in A$,
3) $\| \alpha x \| = | \alpha | \, \| x \|$ for each $\alpha \in \mathbb{C}$, $x \in A$,
4) $\| x y \| \leq \| x \| \, \| y \|$ for each $x, y \in A$.

Given a norm on A we have a natural metric determined by the norm, namely

$$d(x,y) = \| x - y \| \quad \text{for each } x, y \in A.$$

If A is complete with respect to this metric then it is called a (complex) Banach algebra. Unless otherwise stated, we shall always be considering non-commutative complex Banach algebras. Henceforth we shall omit the adjective "complex". If A has a unit we shall denote this unit by 1.

Most of the work in this thesis stemmed from a remark of Professor E. L. Stout of the University of Washington, made at a seminar at the University of Glasgow. Stout remarked that it ought to be possible to prove that a commutative Noetherian Banach algebra is finite-dimensional by using more elementary techniques than those employed in the proofs available at that time. He suggested a theorem due to Kaplansky ([21]) as a possible route. This theorem states that a semi-simple Banach algebra in which every element has a finite spectrum is necessarily finite-dimensional. Due to the problems in handling spectra this theorem is sometimes difficult to apply.

In Chapter 1 we prove a result which is more basic than the "finite spectrum" theorem, namely that a semi-prime Banach algebra which is all socle is finite-dimensional. By making use of this theorem we are able to prove a variety of results including the finite spectrum theorem.

In Chapter 2 we study the effect of putting chain conditions on a Banach algebra. It is well known that a semi-simple Artinian Banach algebra is finite-dimensional. We give an elementary proof of this fact, consider various ways of weakening the Artinian chain condition and then study the consequences. Next in this chapter we prove that a Noetherian Banach algebra is finite-dimensional. As corollaries of this result we obtain that any Artinian Banach algebra is finite-dimensional and that any Banach algebra in which every left ideal is closed is finite-dimensional.

Chapter 3 is concerned with Banach *-algebras. We give a simplified proof of the Shirali-Ford theorem. The techniques used to prove this theorem find further application in showing that the positive wedge in a Banach *-algebra is closed when the involution is Hermitian and continuous. Finally, with reference to the material in Chapter 1, we show that a B*-algebra is finite-dimensional if every self-adjoint element of the algebra has finite spectrum.

In Chapter 4 we collect together several miscellaneous results. Some of these are concerned with questions of existence for nilpotents and quasinilpotents in a Banach algebra. Others are concerned with conditions which force a Banach algebra to be commutative. We observe that the above two problems are very intimately connected in certain special cases. Also in this chapter, we consider some properties of the spectrum of an element of a Banach algebra.

Remark In the following, if a result holds for arbitrary (not necessarily normed) algebras we shall indicate this by stating simply "algebra" rather than "Banach algebra".

CHAPTER 1

In this chapter, we are concerned with algebraic conditions on Banach algebras which force finite dimensionality. In the main, these are conditions on the ideals in the algebra. To begin with, we shall assume the existence of an identity element and then later remove this assumption.

The basic theorem in this chapter states that a Banach algebra with identity which is topologically simple and has minimal one-sided ideals is necessarily finite dimensional. We then show that a semi-prime Banach algebra (with identity) which coincides with its own socle is a finite direct sum of simple Banach algebras with minimal one-sided ideals and so is finite dimensional. As an application of the "socle" theorem we prove Kaplansky's "finite spectrum" theorem. We remark that Kaplansky's theorem extends easily to the case of Banach algebras without identity and use this fact to deduce that the socle theorem also holds for Banach algebras without identity. Several consequences of the socle theorem are then considered.

Fundamental to the proofs of most of the theorems in this chapter is the concept of the idempotent element. In a given algebra we may look at families of orthogonal idempotents (assuming such exist) and by determining how large these families may be we obtain information about the dimensionality of the algebra. Our first result (Lemma 1) makes this statement more precise.

<u>Definition</u> An element e of an algebra A is idempotent if $e^2 = e$. Two idempotents $e, f \in A$ are said to be orthogonal if $ef = fe = 0$. A family of idempotents is pairwise orthogonal if for

each pair $\{e, f\}$ of distinct idempotents in the family e and f are orthogonal.

<u>Notation</u> We write $Sp(A, x)$ for the spectrum of x in A.

<u>Lemma 1</u> If A is a Banach algebra (not necessarily with an identity element) which contains an infinite sequence, (e_n), of pairwise orthogonal non-zero idempotents then there is an element $x \in A$ such that $Sp(A, x)$ is an infinite set.

<u>Proof:</u> Choose $(c_n) \subseteq$ {positive real numbers} such that there are an infinite number of distinct c_n and $\left(\sum_{n=1}^{N} c_n e_n\right)$ converges in A. Let $x = \sum_{n=1}^{\infty} c_n e_n$ then $x e_n = c_n e_n$ for each n and so $c_n \in Sp(A, x)$ (see for example [29] Theorem 1.6.9). Thus $Sp(A, x)$ is infinite.

<u>Definition</u> For a Banach algebra A, the carrier space of A is the set, Φ_A, of non-zero multiplicative linear functionals on A. Note that a multiplicative linear functional on a Banach algebra is automatically continuous so that $\Phi_A \subseteq A'$ - the dual of A. We take the topology on Φ_A to be the relative topology induced by the weak $*$ topology on A'.

The proof of our next result requires the following theorem due to Silov.

<u>Silov's Idempotent Theorem</u>

Let A be a commutative Banach algebra. Let Ψ be a non-empty open and closed subset of Φ_A. Then there is a non-zero idempotent $e \in A$ such that

$$\Psi = \{\phi \in \Phi_A : \phi(e) = 1\} .$$

<u>Note</u> If $\phi \in \Phi_A$ and $e \in A$ is idempotent then $\phi(e) \in \{0, 1\}$ so

$$\Phi_A \setminus \Psi = \{\phi \in \Phi_A : \phi(e) = 0\} .$$

<u>Notation</u> If a set is both open and closed we say it is clopen.

<u>Notation</u> We write rad(A) for the (Jacobson) radical of A. If rad(A) = 0 we say that A is semi-simple.

<u>Definition</u> A left (right, two-sided) ideal I of A is said to be (i) nil if every element of I is nilpotent;

(ii) nilpotent if there is a positive integer k such that for any elements $a_1, \ldots, a_k$ in I we have $a_1 \ldots a_k = 0$. Clearly, a nilpotent ideal is nil;

(iii) topologically nil if every element $a \in I$ is quasi-nilpotent i.e. $Sp(A, a) = 0$, $(a \in I)$; (Notice that a nilpotent element is quasi-nilpotent.)

(iv) quasi-regular if every element $a \in I$ is quasi-regular. Notice that a quasi-nilpotent element is quasi-regular.

We shall require the following property of the radical in a Banach algebra:

The radical is a topologically nil ideal which is equal to the sum of all the topologically nil left (right) ideals in the algebra.

In particular, every nil ideal is contained in the radical.

In an arbitrary algebra the following holds:

the radical is a quasi-regular ideal which is equal to the sum of all the quasi-regular left (right) ideals in the algebra. Note that since a non-zero idempotent cannot be quasi-regular the only idempotent in the radical of an algebra is 0.

<u>Remark:</u> If P, Q are disjoint clopen sets in Φ_C then the idempotents e,f (given by Silov's theorem) which correspond to P,Q are orthogonal. For,

$$\phi(ef) = 0 \qquad (\phi \in \Phi_C)$$

and so $ef \in rad(C)$. But 0 is the only idempotent in rad(C) so $ef = 0$.

Theorem 2 Let C be a commutative Banach algebra such that $Sp(C, x)$ is finite for each $x \in C$. Then

$$C \cong rad(C) \oplus \mathbb{C}^n \qquad \text{(for some } n \in \mathbb{P}\text{)} .$$

Proof: Let $S = \{ \mathcal{F} : \mathcal{F}$ is a family of pairwise disjoint clopen sets in Φ_C whose union is $\Phi_C\}$. By Lemma 1 and the remark preceding the theorem each $\mathcal{F} \in S$ is finite. Define a partial ordering on S as follows:

$\mathcal{F}_1 < \mathcal{F}_2$ if for every $F_2 \in \mathcal{F}_2$ there is $F_1 \in \mathcal{F}_1$ with $F_2 \subseteq F_1$.

Suppose $(\mathcal{F}_n)$ is a chain in S. $(\mathcal{F}_n)$ must have an upper bound. Otherwise, at least one of the $F \in \mathcal{F}_1$ must be an infinite union of pairwise disjoint clopen sets which contradicts our assumption that every spectrum is finite. We now apply Zorn's lemma to obtain a maximal element $\mathcal{F}_o$ in S. $\mathcal{F}_o$ is finite and, since it is maximal, each of its members is a connected set. Thus Φ_C has a finite number of components. Suppose Φ_1 is a component of Φ_C. If Φ_1 is not a singleton we may choose an element $c \in C$ such that $\hat{c}(\Phi_1) = \{\phi(c) : \phi \in \Phi_1\}$ is not a singleton. Since $\hat{c}$ is continuous (with respect to the weak $*$ topology), $\hat{c}(\Phi_1)$ is a connected subset of the complex numbers so is uncountable. But $\hat{c}(\Phi_1) \subseteq Sp(C, c)$ so this is impossible. Hence Φ_1 is a singleton. Thus Φ_c is finite,

$$\Phi_c = \{\phi_1 , \dots , \phi_n\} , \text{ say} .$$

Let $e_1 , \dots , e_n$ be the idempotents (given by Silov's theorem) which correspond to $\phi_1 , \dots , \phi_n$. By the remark immediately preceding the theorem these idempotents are pairwise orthogonal. Let

$$E = \text{linear span of } \{e_1 , \dots , e_n\} .$$

Then

$$E \cong \mathbb{C}^n .$$

For $x \in C$, $\phi_k(x - \sum_{j=1}^{n} \phi_j(x)e_j) = 0$ $\qquad (k = 1, \ldots, n)$.

So, $$x - \sum_{j=1}^{n} \phi_j(x)e_j \in \text{rad}(C) .$$

Thus $$C = \text{rad}(C) \dot{+} E .$$

Since $E \cap \text{rad}(C) = 0$, $C = \text{rad}(C) \oplus E$.

It follows that $C \cong \text{rad}(C) \oplus \mathbb{C}^n$.

Remark: Since C is commutative $\text{rad}(C) = \{x \in C : x \text{ is quasi-nilpotent}\}$. So every element of C may be expressed uniquely as the sum of a quasi-nilpotent and a linear combination of idempotents.

Corollary 3 If C is semi-simple then it is finite dimensional.

Corollary 4 Let A be a commutative Banach algebra in which every non-zero closed ideal of A can be expressed as a finite intersection of maximal modular ideals. Then

$$A \cong \text{rad}(A) \oplus \mathbb{C}^n \quad \text{for some } n \in \mathbb{P}.$$

Proof: Since a maximal modular ideal of A has codimension one, our assumption on closed ideals means that each closed ideal must be cofinite. As the kernel of any continuous representation of A is a closed ideal it follows that these representations must all be finite dimensional. In particular, the regular representation of A (on $X = A$ or $X = A \oplus \mathbb{C}$ depending on whether A has a unit or not) is finite dimensional. We have

$$Sp(A, a) = Sp(B(X), T_a) \qquad (a \in A)$$

where $T_a x = ax (x \in X)$ and $B(X)$ is the space of bounded operators on X. Write $B_1 = \{T_a : a \in A\}$. $Sp(B_1, T_a)$ is finite $(a \in A)$ so, since $Sp(B(X), T_a) \subseteq Sp(B_1, T_a)$ $(a \in A)$, $Sp(A, a)$ is finite $(a \in A)$ and the theorem applies.

<u>Remark:</u> If A is as in Corollary 4 and A has a non-zero nilpotent element then $(\mathrm{rad}(A))^2 = 0$ and $\mathrm{rad}(A)$ is one-dimensional. For we may suppose that there is $z \in A \setminus (0)$, $z^2 = 0$. Then Az is nil so $Az \subseteq \mathrm{rad}(A)$. If $Az = 0$ then the closed nil ideal $\{z \in A : Az = 0\}$ is non-empty so is equal to $\mathrm{rad}(A)$. Then $(\mathrm{rad}(A))^2 = 0$. If $Az \neq 0$ then $\overline{Az} = \mathrm{rad}(A)$ and hence $r^2 = 0$ $(r \in \mathrm{rad}(A))$. Thus if $r, s \in \mathrm{rad}(A)$, $rs = \frac{1}{2}(r + s)^2 = 0$ so $(\mathrm{rad}(A))^2 = 0$. $\mathrm{rad}(A)$ is one-dimensional as all its finite dimensional subspaces are closed ideals.

Our next objective in this chapter is to prove the theorem which allows us to determine the nature of the "building blocks" in the socle theorem - namely, the theorem which states that a topologically simple Banach algebra (with identity element) which has minimal one-sided ideals is necessarily finite dimensional.

For the proof of the theorem we require several standard algebraic results concerning minimal ideals which we shall state without proofs.

<u>Lemma 5</u> Let A be an arbitrary algebra and L a minimal left ideal in A such that $L^2 \neq 0$. Then there is an idempotent $e \in A$ such that $L = Ae$ and eAe is a division algebra with unit element e.

<u>Corollary 6</u> If A is a Banach algebra then L is closed and (by Mazur's theorem) $eAe = \mathbb{C}e$.

<u>Definition</u> An idempotent e in a Banach algebra A is said to be minimal if $eAe = \mathbb{C}e$.

<u>Definition</u> An algebra is said to be semi-prime if (0) is the only ideal (left or right) which has square equal to (0).

<u>Remark:</u> A semi-simple algebra is necessarily semi-prime.

<u>Lemma 7</u> Let A be a semi-prime Banach algebra. An idempotent $e \in A$ is minimal if and only if Ae and eA are minimal (left, right

respectively) ideals.

Notation We refer to a two-sided ideal as a bi-ideal.

Definition The sum of all the minimal left (right) ideals of A is called the left (right) socle of A. When the left and right socles exist and are equal the resulting bi-ideal is called simply the socle of A and is denoted by $soc(A)$.

Lemma 8 If A is a semi-prime algebra which contains minimal one-sided ideals then $soc(A)$ is defined.

Definition An algebra, A, of operators on a complex vector space, X, is said to be strictly dense on X if, given any positive integer k, and arbitrary vectors $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ where $x_1, \ldots, x_k$ are linearly independent, there is an operator $T \in A$ such that

$$Tx_j = y_j \qquad (j = 1, \ldots, k)$$

If an algebra of operators on X satisfies the above condition for $k = 1$ the algebra is said to be (strictly) irreducible on X.

We require the following result which was proved (independently) by Rickart and Yood (see for example [29] Theorem (2.4.6)).

Theorem 9 Let A be a strictly irreducible complex Banach algebra of operators on a complex vector space X. Then A is strictly dense on X. For the next theorem we require the following definitions.

(i) If S is a subset of an algebra A we define the left annihilator of S to be the set

$$lan(S) = \{x \in A : xS = 0\} .$$

The right annihilator of S is the set

$$ran(S) = \{x \in A : Sx = 0\} .$$

$lan(S)$ $(ran(S))$ is a left (right) ideal of A. If A is a normed algebra the left and right annihilators of any set are always closed. If L is a left ideal of A then lan(L) is a bi-ideal of A. Similarly, ran(R) is a bi-ideal if R is a right ideal.

(ii) An algebra A is said to be (algebraically) simple if the only bi-ideals of A are (0) and A. A normed algebra is said to be topologically simple if the only closed bi-ideals of A are (0) and A.

Notation If X is a normed linear space we denote by B(X) the space of all bounded operators on X.

Theorem 10 If A is a Banach algebra with unit which is topologically simple and contains minimal one-sided ideals then A must be finite-dimensional.

Proof: Suppose L is a minimal left ideal of A. Since $AL \neq 0$ and lan(L) is a closed bi-ideal we must have lan(L) = 0. Thus, the left regular representation of A on L is faithful. Since L is minimal this representation is also irreducible so we may regard A as an irreducible Banach algebra of operators on L.

It now follows (Theorem 9) that A is strictly dense on L so that soc(A) coincides with the set of finite rank operators in A (see for example Rickart [29] P.65). Since $A = \overline{soc(A)}$, 1 (the unit in A) is a limit of finite rank operators in the given norm on A and hence in the usual operator norm on B(L). Since L is closed (Corollary 6) the set of compact operators in B(L) is closed in the uniform operator topology. Thus 1 is a compact operator and so the unit ball in L is compact and hence L is finite dimensional. Since $A \subseteq B(L)$ we have that A is finite dimensional.

Remarks: (a) Theorem 10 fails if A has no unit. The algebra of compact operators on an infinite dimensional Hilbert space is a topologically simple Banach algebra (in the uniform norm) and contains minimal one-sided ideals.

(b) The converse of Theorem 10 is clearly false.

(c) We shall see later that if we strengthen the other conditions then the assumption that A has a unit may be removed.

<u>Notation</u> If $\{S_i : i \in I\}$ is a family of subspaces of a linear space X we denote by $\sum_{i\in I} S_i$ the sum of the subspaces S_i. If the sum is direct we write $\sum_{i\in I} \oplus S_i$.

<u>Definition</u> We say that two left ideals I, J of an algebra A are module-isomorphic if there is a linear bijective mapping $\phi : I \to J$ of I onto J such that

$$\phi(xa) = x\phi(a) \qquad (x \in A , a \in I) .$$

Suppose A is an algebra for which soc(A) is defined. Given a minimal left ideal L of A let

$M(L) = \{K : K$ is a minimal left ideal and K is module-isomorphic to $L\}$.

Let $H_L = \sum_{K\in M(L)} K$; then we call H_L the homogeneous component of the socle determined by L.

<u>Remark:</u> The homogeneous components of soc(A) are bi-ideals. We require the following purely algebraic results.

<u>Lemma 11</u> If A is an algebra such that A = soc(A) then A is a direct sum of its homogeneous components.

<u>Lemma 12</u> If A is a semi-prime algebra the homogeneous components of soc(A) are simple algebras.

These results may be found in Jacobson [18] P64, P65.

We are now ready to prove the main theorem in this chapter.

<u>Theorem 13</u> If A is a semi-prime Banach algebra with unit and A = soc(A), then A is finite-dimensional.

Proof: If L is a minimal left ideal of A then, since A is semi-prime, L = Ae for some minimal idempotent $e \in A$. Since $1 \in A$ and A = soc(A), there are minimal idempotents $e_1, \ldots, e_n$ (say) and elements $a_1, \ldots, a_n$ in A such that

$$1 = \sum_{j=1}^{n} a_j e_j$$

and hence

$$A = \sum_{j=1}^{n} Ae_j \ .$$

That is, we can express A as a finite sum of minimal left ideals. Thus, by Lemma 11,

$$A = \sum_{j=1}^{m} \oplus I_j$$

where $I_1, \ldots, I_m$ are the homogeneous components of A.

Since the homogeneous components of A are bi-ideals, (Lemma 12 asserts that they are in fact minimal bi-ideals) $I_j I_k = 0$ if $j \neq k$. Suppose $1 = \sum_{j=1}^{m} 1_j$ is the decomposition of 1 with respect to the direct sum $A = \sum_{j=1}^{m} \oplus I_j$ then it is clear that 1_k is the unit element of $I_k (k = 1, \ldots, m)$. Each I_k is closed ($I_k = \text{lan}(\sum_{j \neq k} \oplus I_j)$) is simple as an algebra (Lemma 12) and by its definition contains a minimal left ideal Ae of A. We have

$$\mathbb{C}e = eAe = e\left(\sum_{j=1}^{m} \oplus I_j\right)e = eI_k e \quad \text{so that}$$

$Ae = I_k e$ is a minimal left ideal of I_k.

By Theorem 10, each I_k is finite-dimensional so since A is a finite sum of I_k's it also must be finite-dimensional.

Remarks: (a) Theorem 13 gives us a characterisation of semi-prime finite-dimensional Banach algebras. That is,

A is finite-dimensional if and only if A = soc(A).

(If A is finite-dimensional then A = soc(A) is a consequence of the Wedderburn Structure Theorem).

(b) We shall see later that Theorem 13 holds without the assumption of a unit element.

The next lemma was proved by Kaplansky [21]. We include a proof of the result since the one given here differs from that given by Kaplansky.

Lemma 14 If A is a semi-simple Banach algebra such that $Sp(A, x)$ is a singleton for each $x \in A$, then A is one-dimensional.

Proof: First suppose that A is primitive; then A can be regarded as a strictly dense Banach algebra of operators on some Banach space, X. Thus, if x, y are two linearly independent vectors in X there is $T \in A$ such that

$$Tx = 0 \quad \text{and} \quad Ty = y .$$

This would mean that $\{0, 1\} \subseteq Sp(A, T)$ which contradicts our assumption that every spectrum is a singleton. It follows that X is one-dimensional and hence that A is one-dimensional.

Now suppose that A is semi-simple and that P is a primitive ideal of A. If $x + P \in A/P$ then, since $Sp(A/P, x + P) \subseteq Sp(A, x)$, the above gives that A/P is one-dimensional. This says that every irreducible representation of A is one-dimensional so A is commutative. The result now follows easily (for example, by Theorem 2).

The next two lemmas are well-known Banach algebra results so we omit their proofs.

Lemma 15 Let A be a Banach algebra and let e be a proper idempotent in A. Then

$$Sp(A, x) = Sp(eAe, x) \cup (0) \qquad (x \in eAe)$$

(By a proper idempotent we mean e is non-zero and $e \neq 1$).

<u>Lemma 16</u> A Banach algebra A will contain a proper idempotent if and only if there is at least one element of A whose spectrum is not connected.

(See for example [12] Theorem 5.5.2).

We are now ready to prove the finite spectrum theorem of Kaplansky [21]. Other proofs of this result are to be found in [6] and [13].

<u>Theorem 17</u> Let A be a semi-simple Banach algebra with unit. Suppose that, for each $x \in A$, $Sp(A, x)$ is a finite set. Then A is finite dimensional.

<u>Proof:</u> The proof falls naturally into three parts:

(i) If every spectrum is a singleton then Lemma 14 applies. Otherwise, by Lemma 16, A has proper idempotents.

(ii) A cannot contain an infinite sequence of pairwise orthogonal non-zero idempotents. This is clear from our assumption on the spectra and Lemma 1.

(iii) Suppose $e \in A$ is a proper idempotent. $e_1 = e$, $e_2 = 1 - e$ are orthogonal and $1 = e_1 + e_2$. Now consider the Banach algebra e_1Ae_1 which has e_1 as identity element. If $e_1^{(1)}$ is an idempotent in e_1Ae_1 other than 0 or e_1 then $e_1^{(1)}$ and $e_2^{(1)} = e_1 - e_1^{(1)}$ are orthogonal and $e_1 = e_1^{(1)} + e_2^{(1)}$. Similarly, in e_2Ae_2, we may obtain idempotents $e_3^{(1)}$, $e_4^{(1)}$ which are orthogonal and satisfy $e_2 = e_3^{(1)} + e_4^{(1)}$. Thus $1 = \sum_{j=1}^{4} e_j^{(1)}$ and $\{e_j^{(1)} : j = 1, \ldots, 4\}$ is a set of pairwise orthogonal idempotents. We repeat the process for $e_j^{(1)}Ae_j^{(1)}$ and so on.

If at any stage $e_j^{(n)}$ is the only non-zero idempotent in $e_j^{(n)}Ae_j^{(n)}$ then $Sp(e_j^{(n)}Ae_j^{(n)}, x)$ is connected for each $x \in e_j^{(n)}Ae_j^{(n)}$ so is a singleton (Lemmas 16, 15). Since $e_j^{(n)}Ae_j^{(n)}$ is semi-simple, it is one-dimensional (Lemma 14) so $e_j^{(n)}$ is a minimal idempotent and $Ae_j^{(n)}$ is a minimal left ideal. By (ii) and Lemma 1, this "splitting process" must terminate after a finite number of steps.

We now have a set $\{f_1, \ldots, f_k\}$ of pairwise orthogonal minimal idempotents such that $1 = \sum_{j=1}^{k} f_j$. Hence $A = \sum_{j=1}^{n} \oplus Af_j$ and so, by Theorem 13, A is finite-dimensional.

Remark: Theorem 17 holds without the assumption that A has a unit, for if A is semi-simple then so is $A \oplus \mathbb{C}$ and also

$$Sp(A, x) = Sp(A \oplus \mathbb{C}, x) \qquad (x \in A)$$

so that $A \oplus \mathbb{C}$ satisfies the conditions of the theorem.

Corollary 18 If A is a Banach algebra such that $Sp(A, x)$ is finite for each $x \in A$ then $rad(A)$ is cofinite.

Proof: $A/rad(A)$ is semi-simple and

$$Sp(A/rad(A), x + rad(A)) \subseteq Sp(A, x) \qquad (x \in a) .$$

Hence $A/rad(A)$ is finite-dimensional.

Corollary 19 Let X be a complex Banach space and suppose that each compact operator on X has finite rank. Then, X is finite-dimensional.

Proof: The Banach algebra of compact operators on X is semi-simple. The assumption above implies that each compact operator has finite spectrum so by the theorem the algebra of compact operators on X is finite-dimensional. In particular, the algebra of finite rank operators is finite-dimensional and the result follows. (Corollary 19 tells us that every infinite-dimensional Banach space has defined on it a compact infinite rank operator.)

Definition An arbitrary algebra A is said to be locally finite if every finitely generated subalgebra of A is finite-dimensional.

A is said to be algebraic if every singly generated subalgebra of A is finite-dimensional.

We require the following results on local finiteness (see for example Jacobson, [18]).

Lemma 20 Let A be an arbitrary algebra and let I be a bi-ideal of A such that I and A/I are locally finite. Then A is locally finite.

Lemma 21 The radical of a locally finite algebra is nil.

We also require the following result due to Grabiner, [9].

Lemma 22 A nil Banach algebra is nilpotent.

Corollary 23 For a Banach algebra A, the following are equivalent:

(i) A is locally finite

(ii) rad(A) is nilpotent and cofinite.

Proof: If A is locally finite then, for each $x \in A$, Sp(A, x) is finite so by Corollary 18 rad(A) is cofinite. By Lemma 21 rad(A) is nil so by Lemma 22 rad(A) is nilpotent. The converse is immediate by Lemma 20.

Corollary 24 If A is a semi-simple algebraic Banach algebra with unit then A is finite-dimensional.

Proof: Every spectrum is finite.

We now extend some of our previous results by removing the assumption of a unit element. We observed that Theorem 10 as it stands fails if A has no unit but if we strengthen our other assumptions, replacing topologically simple by (algebraically) simple, then we obtain a theorem which is true for A without unit.

Theorem 25 If A is a simple Banach algebra with minimal one-sided ideals then A is finite-dimensional.

Proof: Suppose that L is a non-zero minimal left ideal of A and that $L^2 = 0$. lan(L) is a non-zero bi-ideal and so lan(L) = A.

Thus $\operatorname{ran}(A) \neq 0$ so $\operatorname{ran}(A) = A$ that is, $A^2 = 0$. If $u \in A \setminus (0)$ then linear span of $\{u\}$ is a non-zero bi-ideal. Hence A is one-dimensional.

Now suppose that L is a non-nilpotent minimal left ideal of A. Since A is simple, $\operatorname{lan}(L) = 0$ and so the left regular representation of A on L is faithful. Since L is also minimal, the representation is strictly irreducible. (If $x \in L \setminus (0)$, $Ax \neq 0$ for otherwise $\operatorname{ran}(A) = A$ so that $A^2 = 0$. Hence $Ax = L$). Thus A can be regarded as a strictly dense Banach algebra of operators on L (Theorem 9) and hence

$$A = \operatorname{soc}(A) = \{\text{finite rank operators in } A\} .$$

This implies that every element of A has finite spectrum. Since A is certainly semi-simple, Theorem 17 gives that A is finite-dimensional.

We now prove Theorem 13 for Banach algebras without unit.

<u>Theorem 26</u> If A is a semi-prime Banach algebra and $A = \operatorname{soc}(A)$ then A is finite-dimensional.

<u>Proof</u>: By Lemma 11, $A = \sum_{\lambda \in \Lambda} \oplus I_\lambda$ where $\{I_\lambda : \lambda \in \Lambda\}$ are the homogeneous components of A. These are simple algebras by Lemma 12. As in Theorem 13 each I_{λ_0} is closed, being the left annihilator of $Z_{\lambda_0} = \sum_{\lambda \neq \lambda_0} \oplus I_\lambda$. This can be seen as follows:

Clearly $I_{\lambda_0} \subseteq \operatorname{lan}(Z_{\lambda_0})$ since $I_{\lambda_0} I_\lambda = 0$ if $\lambda \neq \lambda_0$.
Conversely, suppose $x_0 \in \operatorname{lan}(Z_{\lambda_0})$, $x_0 = x_{\lambda_0} + z_0$ where $x_{\lambda_0} \in I_{\lambda_0}$, $z_0 \in Z_{\lambda_0}$. Then $x_0 z = 0$ $(z \in Z_{\lambda_0})$ and so $z_0 z = 0$ $(z \in Z_{\lambda_0})$ since $x_{\lambda_0} Z_{\lambda_0} = (0)$. If $z_0 \neq 0$ then $L = \operatorname{lan}(Z_{\lambda_0}) \cap Z_{\lambda_0} \neq 0$. Thus $L^2 = 0$ which is impossible since A is semi-prime. Thus $x_0 = x_{\lambda_0} \in I_{\lambda_0}$.

By Theorem 25 each I_λ is finite dimensional and it remains to show that A has at most a finite number of homogeneous components.

Suppose, on the contrary, that Λ is infinite. Then we may choose an infinite sequence (e_n) of pairwise orthogonal non-zero idempotents such that each e_n belongs to a different I_λ. As in Lemma 1, we may choose a sequence $(c_n) \subseteq \mathbb{C} \setminus (0)$ such that $x_o = \sum_{n=1}^{\infty} c_n e_n$ belongs to A. But $x_o \notin soc(A)$ which contradicts our assumption that A is all socle. Hence Λ must be finite and the theorem is proved.

<u>Remark</u>: We cannot weaken the condition $A = soc(A)$ to $A = \overline{soc(A)}$. Suppose, for example, that A is an infinite dimensional semi-simple annihilator Banach algebra then A has dense socle (see for example Rickart [29]). However, if A is a semi-simple Banach algebra satisfying $A = \overline{soc(A)}$ then A is in some sense "nearly" finite-dimensional as the following discussion shows.

<u>Definition</u> A Banach algebra A is said to be finite rank if $x \to axa$ is a finite rank operator for each $a \in A$ and compact if $x \to axa$ is a compact operator for each $a \in A$. A finite rank algebra is therefore a compact algebra.

Now suppose also that A is semi-simple. Alexander [1] has shown that $x \to axa$ is finite rank if and only if soc(A) exists and $a \in soc(A)$. Thus, if A is a semi-simple finite rank Banach algebra then $A = soc(A)$ and so (Theorem 26) A is finite-dimensional. Since it is immediate that a finite-dimensional algebra is finite rank it follows that for semi-simple Banach algebras the two notions are equivalent. This is the content of the following theorem.

<u>Theorem 27</u> If A is a semi-simple Banach algebra then A is finite-dimensional if and only if A is finite rank.

Of interest in connection with the preceding Remark is the following theorem which is due to Alexander [1].

<u>Theorem 28</u> If A is a semi-simple Banach algebra and A has dense socle then A is compact.

<u>Notes</u> (i) Alexander [1] has also shown that a B*-algebra is compact if and only if it has dense socle.

(ii) We note in passing that an infinite dimensional compact Banach algebra can have no unit.

We now look at the problem of identifying Banach algebras in which each closed left ideal can be expressed as a finite intersection of maximal modular left ideals. We considered the commutative case of this problem in Corollary 4. Clearly this condition is fairly restrictive. In general, in the ~~non~~-commutative case we cannot obtain each closed ideal even as an infinite intersection of maximal modular ideals. For example, Malliavin [25] has shown that, for G a non-compact abelian group, $L^1(G)$ will always contain a closed ideal which is not an intersection of maximal modular ideals.

For A a non-commutative Banach algebra we start by considering the special case in which the zero ideal is a finite intersection of maximal modular left ideals.

<u>Theorem 29</u> If A is a Banach algebra such that for some finite set $\{L_j : j = 1 , \ldots , n\}$ of maximal modular left ideals of A ,

$$\bigcap_{j=1}^{n} L_j = 0 ,$$

then A is finite-dimensional.

<u>Proof:</u> We may suppose, without loss of generality, that

$$K_j = \bigcap_{i \neq j} L_i \neq 0 \qquad (j = 1 , \ldots , n)$$

in which case $\{K_j : j = 1 , \ldots , n\}$ is a family of minimal left ideals of A. (This follows easily from the fact that $A = L_j \oplus K_j$ for each j).

Let $a \in A$ and suppose that

$$a = l_j + k_j$$

is the expression for a with respect to the direct sum decomposition $A = L_j \oplus K_j$ $(j = 1, \ldots, n)$. Then,

$$a - \sum_{i=1}^{n} k_i = a - k_j - \sum_{i \neq j} k_i \qquad (j = 1, \ldots, n)$$

$$= l_j - \sum_{i \neq j} k_i \in L_j \qquad (j = 1, \ldots, n)$$

Hence $a - \sum_{i=1}^{n} k_n \in \bigcap_{i=1}^{n} L_j = 0$ so $a = \sum_{i=1}^{n} k_i$.

Thus $A = soc(A)$ so, since A is evidently semi-simple, it follows by Theorem 26 that A is finite-dimensional.

Corollary 30 If A is a Banach algebra whose radical is a finite intersection of maximal modular left ideals then $rad(A)$ is cofinite.

Proof: $A/rad(A)$ satisfies the conditions of Theorem 29.

Corollary 31 Let A be a Banach algebra in which every proper closed left ideal is a finite intersection of maximal modular left ideals and suppose that A contains a proper idempotent element. Then A is finite-dimensional and semi-simple.

Proof: Suppose $e \in A$ is a proper idempotent. Then Ae, $A(1 - e)$ are proper closed left ideals of A which have zero intersection. Hence $0 = \bigcap_{j=1}^{n} L_j$ for some finite set $\{L_j : j = 1, \ldots, n\}$ of maximal modular left ideals so A is finite-dimensional.

Corollary 32 If A is a semi-prime Banach algebra with minimal one-sided ideals which satisfies the intersection property given in the above corollary then A is finite-dimensional.

Proof: The result follows from Corollary 31 and Lemma 5.

Corollary 33 If A is a Banach algebra such that $R = rad(A) \neq 0$ and every proper left ideal is a finite intersection of maximal modular left ideals then $R^2 = 0$. If also A has a unit then A is finite-dimensional.

<u>Proof:</u> Suppose there is $x \in R$ such that $Ax = 0$. Then $\operatorname{ran} A \neq 0$. Since $\operatorname{ran} A \subseteq R$ it follows that $\operatorname{ran} A = R$ and hence $R^2 = 0$. Now suppose that $Ax \neq 0$ $(x \in R)$. Hence $\operatorname{lan} R \neq 0$. Otherwise, the left regular representation of A on R is faithful and irreducible which is impossible. If $\operatorname{lan} R \cap R = 0$ then (0) is a finite intersection of maximal modular left ideals which implies that $R = 0$ - contrary to assumption. Thus $\operatorname{lan} R \cap R \neq 0$ and hence $R \subseteq \operatorname{lan} R$ so that $R^2 = 0$. Now suppose A has a unit.

By Corollary 30, A/R is finite-dimensional so there are elements $u_1, \ldots, u_n$ of A, not in R, such that each $a \in A$ has an expression of the form

$$a = q(a) + \sum_{j=1}^{n} \alpha_i(a) u_i$$

where $q(a) \in R$ and $\alpha_j(a) \in \mathbb{C}$ $(j = 1, \ldots, n)$. Fix $x_o \in R$. Then

$$R = Ax_o = \text{linear span of } \{u_j x_o : j = 1, \ldots, n\} .$$

Hence R is finite-dimensional and so A is finite-dimensional.

<u>Remark:</u> It is not known whether topologically irreducible Banach algebras of operators need be semi-simple and so we are unable to apply the above technique to the case in which every closed left ideal is a finite intersection of maximal modular left ideals.

<u>Example</u>

$$\text{Let } A = \left\{ \begin{pmatrix} \alpha & \gamma & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\} . \quad \text{Then}$$

$R = \left\{ \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \gamma \in \mathbb{C} \right\}$ and the only ideals in A are the maximal ideals $I = \left\{ \begin{pmatrix} \alpha & \gamma & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \gamma \in \mathbb{C} \right\}$

and $J = \left\{ \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix} : \beta, \gamma \in \mathbb{C} \right\}$.

Thus A satisfies the conditions of Corollary 33.

Definition Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of Banach algebras. Write $\sum_{NF} \oplus A_\lambda$ for the class of all functions f on Λ with $f(\lambda) \in A_\lambda$, for each λ, and such that $\{\|f(\lambda)\|_\lambda : \lambda \in \Lambda\}$ is a bounded set (where $\|\cdot\|_\lambda$ is the given norm on A_λ). Define

$$|f| = \sup_{\lambda \in \Lambda} \|f(\lambda)\|_\lambda$$

Then, with the usual pointwise operations and $|\cdot|$ as norm, $\sum_{NF} \oplus A_\lambda$ is a Banach algebra which we call the normed full direct sum of the A_λ. If B is a subalgebra of $\sum_{NF} \oplus A_\lambda$ such that the point evaluation mappings are all surjections then B is called a normed subdirect sum of the A_λ. We denote a normed subdirect sum by $\sum_{NS} \oplus A_\lambda$. Note that these need not be complete with respect to $|\cdot|$.

We require the following result (see for example Rickart [29] Theorem (2.6.1)(i)).

Theorem 34 A semi-simple Banach algebra A is continuously isomorphic with a normed subdirect sum of primitive Banach algebras. (In fact, the normed subdirect sum of the theorem is just $\sum_{NS} \oplus A/P_\lambda$ where $\{P_\lambda : \lambda \in \Lambda\}$ is the set of primitive ideals of A.)

Suppose now that A is a semi-simple Banach algebra which is not primitive and that A satisfies our intersection property for closed left ideals; then -

<u>Proposition 35</u> A is a subdirect sum of full matrix algebras. That is,

$$A \cong \sum_{NS} \oplus M^{n_\lambda} \ .$$

<u>Proof:</u> If P is a primitive ideal of A then, by Theorem 29, A/P is finite-dimensional so, by Wedderburn's Structure Theorem, is isomorphic to M^n for some $n \in \mathbb{P}$ where M^n is the Banach algebra of all $n \times n$ complex matrices. Let $\{P_\lambda : \lambda \in \Lambda\}$ be the family of all primitive ideals of A and suppose $A/P_\lambda \cong M^{n_\lambda}$ then, by Theorem 34,

$$A \cong \sum_{NS} \oplus M^{n_\lambda} \ .$$

<u>Remark:</u> While we have not as yet determined whether the Banach algebras A of Proposition 35 need be finite-dimensional we note that any infinite-dimensional example must possess the following property:

If $x \in A$ and x is non-zero then x_λ is non-zero for λ in some infinite subset of Λ (x_λ is the λ'th coordinate of x with respect to $A \cong \sum_{NS} \oplus M^{n_\lambda}$).

For, if this were not so and x_λ is non-zero only finitely many times $\lambda_1, \cdots, \lambda_k$ (say) then

$$Ax \subseteq \sum_{1}^{k} \oplus M^{n_{\lambda_j}}$$

and so is finite-dimensional and hence closed. Thus $Ax = \bigcap_{\lambda \in \Lambda_o} L_\lambda$ for some finite set $\{L_\lambda : \lambda \in \Lambda_o\}$ of maximal modular left ideals.

Each L_λ is cofinite (since its quotient is cofinite) and so Ax is cofinite which means that A must be finite-dimensional.

Next, in this chapter, we give elementary proofs (using the socle theorem) that regular Banach algebras and semi-simple π-regular Banach algebras are finite-dimensional. Both results are due to Kaplansky ([19] and [20]).

Definition An algebra A is said to be regular if for each $a \in A$ there is some $x \in A$ such that

$$axa = a .$$

Note Such algebras are usually called "von Neumann regular" after J. von Neumann who first introduced the concept (see [26]). Here, however, we shall always refer to them simply as regular algebras.

A regular algebra is always semi-simple (see [16] for example) and we include a proof of this fact.

Remark: A non-trivial regular algebra A contains non-zero idempotents. For, if $x \in A$, $x \neq 0$, and $y \in A$ satisfies $xyx = x$ then xy and yx are both non-zero idempotents.

Lemma 36 A regular algebra is semi-simple.

Proof: Let A be a regular algebra and suppose $x \in \mathrm{rad}(A)$. Then there is $y \in A$ such that $xyx = x$. If $x \neq 0$ then xy is a non-zero idempotent in $\mathrm{rad}(A)$ which is impossible.

Lemma 37 A regular Banach algebra cannot contain an infinite sequence of pairwise orthogonal non-zero idempotents.

Proof (Kaplansky [19]) Suppose on the contrary that A is a regular Banach algebra containing an infinite sequence (e_n) of pairwise orthogonal idempotents.

Let $c_k = 2^{-k}\|e_k\|^{-1}$ $(k \in \mathbb{P})$ and put $x = \sum_1^\infty c_k e_k$. Then $x \in A$ and since A is regular we may choose $y \in A$ such that $xyx = x$.

We now have

$$c_k e_k = e_k x e_k = e_k xyx e_k = c_k e_k y c_k e_k = c_k^2 e_k y e_k$$

so that $c_k\|e_k\| \le c_k^2\|e_k\|^2\|y\|$ which leads to

$$1 \le c_k\|e_k\|\|y\|.$$

Thus $\|y\| \ge c_k^{-1}\|e_k\|^{-1} = 2^k \quad (k \in \mathbb{P})$ which is impossible.

Theorem 38 (Kaplansky [19]) A regular Banach algebra is finite-dimensional.

Proof: Let A be a regular Banach algebra and suppose, for the moment, that A contains only one non-zero idempotent element, e. From the remark immediately preceding Lemma 36 it is clear that e is the identity element of A and that A is a division algebra. Hence, by Mazur's Theorem $A = \mathbb{C}e$.

Now suppose that A has more than one non-zero idempotent. Let E be a maximal set of pairwise orthogonal non-zero idempotents of A. By Lemma 37, E is finite, $E = \{e_1, \cdots, e_m\}$ (say). Consider the algebra $e_1 A e_1$. This is a semi-simple regular Banach algebra with unit e_1. If e_1 is the only non-zero idempotent in $e_1 A e_1$ then $e_1 A e_1 = \mathbb{C}e_1$. Suppose that f_1 is a non-zero idempotent in $e_1 A e_1$ different from e_1. Then, f_1, $e_1 - f_1$ are orthogonal (non-zero) idempotents. Replace e_1 in E by f_1 and $e_1 - f_1$. This "idempotent-splitting" procedure may be continued until finally we obtain a finite (Lemma 37) maximal set of pairwise orthogonal minimal idempotents, $F = \{f_1, \cdots, f_n\}$ (say).

Let $f = \sum_1^n f_j$ and consider the algebra $(1 - f)A(1 - f)$. This is a (semi-simple) regular Banach algebra so, if it is non-zero, it must contain a non-zero idempotent f_{n+1} orthogonal to F. This contradicts the maximality of F so it must be that

$$(1 - f)A(1 - f) = 0 .$$

Hence,

$$(A(1 - f))^2 = 0 = ((1 - f)A)^2$$

so since A is semi-simple, $A(1 - f) = 0 = (1 - f)A$. That is, f is a unit element for A. Since $f = \sum_1^n f_j$ we now have $A = soc(A)$ and an application of the socle theorem completes the proof.

We now consider a condition which is weaker than regularity.

Definition An algebra A is said to be π-regular if for each $x \in A$ there is $y \in A$ and $n \in \mathbb{P}$ (depending on x) such that

$$x^n y x^n = x^n .$$

π-regularity is clearly preserved under homomorphisms.

Remark: Kaplansky ([20]) noted that any algebraic algebra is π-regular so π-regularity is a generalisation of the algebraic condition. As with regular algebras, the π-regular algebras (except of course in the nil case) have a plentiful supply of idempotents. For, if $x \in A$ is non-nilpotent and $y \in A$ satisfies $x^n y x^n = x^n$ for some $n \in \mathbb{P}$, then $x^n y$, yx^n are non-zero idempotents.

It is clear that a π-regular algebra need not be semi-simple. However, if x is in the radical of a π-regular algebra then x is necessarily nilpotent. Otherwise, by the same argument as used in Lemma 36, we would have a non-zero idempotent in the radical. Thus the radical of a π-regular algebra is nil. If A is a π-regular Banach algebra an application of Lemma 22 gives the following result.

Lemma 39 The radical of a π-regular Banach algebra is nilpotent.

In a similar way to that in which we proved Lemma 37 we may prove:

Lemma 40 A π-regular Banach algebra cannot contain an infinite sequence of pairwise orthogonal non-zero idempotents.

Theorem 41 A semi-simple π-regular Banach algebra is finite-dimensional.

Proof: First suppose that A is a semi-simple π-regular Banach algebra which contains only one non-zero idempotent element, e . Suppose $w, z \in A$ satisfy $wz = 0$, $w \neq 0$, $z \neq 0$. Since A is semi-simple, Aw and zA are non-zero, non-nil one-sided ideals of A so we may choose $a, b \in A$ such that aw, zb are not nilpotent. Since A is π-regular there are elements $x, y \in A$ such that xaw, zby are non-zero idempotents. Hence $xaw = e = zby$, so

$$e = e^2 = (xaw)(zby) = 0$$

which is a contradiction to the assumption $e \neq 0$. Thus one of z,w is zero and so 0 is the only zero divisor in A. If $x \in A\backslash(0)$ then since $e(ex - x) = 0 = (xe - x)e$ we have $ex = x = xe$ so that e is the identity of A. The π-regularity of A and the fact that A has no nilpotents (other than 0) show that every (non-zero) element of A has an inverse so, by Mazur's theorem, $A = \mathbb{C}e$.

The rest of the proof is the same as the proof of Theorem 38 except that we replace "regular" wherever it appears by "π-regular" and use Lemma 40 instead of Lemma 37.

Corollary 42 For Banach algebras, π-regularity is an equivalent condition to local finiteness.

Proof: Use Lemma 20, and remark on page 24.

We close this chapter with the following recent result due to T. J. Laffey [22].

Theorem 43 If A is an algebra in which every commutative subalgebra is finite-dimensional then A is finite-dimensional.

Remark: To prove the theorem for Banach algebras it is sufficient to show that the result holds for nilpotent Banach algebras. The assumption of the theorem implies that A is algebraic so, if A is a Banach algebra, rad(A) is nilpotent and cofinite.

CHAPTER 2

In this chapter, we study the effect of imposing chain conditions on the ideal structure of a Banach algebra. More precisely, we ask that every descending (or ascending) chain of left ideals of a certain type has at most finite length.

We begin the chapter by proving the (known) result that a semi-simple Artinian Banach algebra is finite-dimensional. Next, we consider a weakening of the Artinian condition and investigate some of the properties of such "weakly Artinian" Banach algebras. We then go on to look at a chain condition which is intermediate in strength between weakly Artinian and Artinian. We show that a semi-simple Banach algebra with this property is finite-dimensional.

Finally, we look at the consequences of imposing an ascending chain condition on a Banach algebra. It is already known that a commutative Noetherian Banach algebra must be finite-dimensional (see for example [11], [23]). Here we are able to obtain a generalisation of this result by removing the commutativity condition.

Definition An algebra A is said to be Artinian if every descending chain

$$L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n \supseteq \cdots$$

of left ideals becomes stationary. That is, there is some integer n_o such that $L_{n_o} = L_{n_o+1} = \cdots$.

Remark: In a non-zero algebra, this condition guarantees the existence of minimal left ideals and hence, in the case of a semi-prime algebra, the existence of minimal idempotents.

Theorem 1 Let A be a semi-simple Artinian Banach algebra. Then A is finite-dimensional.

Proof: Let $\mathcal{L} = \{L_\lambda : \lambda \in \Lambda\}$, where Λ is an indexing set, be the family of all maximal modular left ideals of A. Suppose that for every finite subset Λ_o of Λ

$$\bigcap_{\lambda \in \Lambda_o} L_\lambda \neq 0 .$$

If L_1, L_2 are distinct members of $\mathcal{L}$ then

$$J_1 = L_1 \supsetneq L_1 \cap L_2 = J_2 \text{ (say)} .$$

Further, there is $L_3 \in \mathcal{L}$ such that

$$J_2 = L_1 \cap L_2 \supsetneq L_1 \cap L_2 \cap L_3 = J_3 ,$$

otherwise

$$J_2 \subseteq \bigcap_{\lambda \in \Lambda} L_\lambda = 0 ,$$

which is contrary to our assumption. In general, once $J_{n-1} = \bigcap_{j=1}^{n-1} L_j$ has been obtained, we obtain $J_n \subsetneq J_{n-1}$ where $J_n = J_{n-1} \cap L_n$ for some $L_n \in \mathcal{L}$. Since J_{n-1} is never zero, such L_n will always exist. In this way we obtain a strictly decreasing infinite sequence (J_n) of left ideals of A. This contradicts the fact that A is Artinian. It follows that for some finite subset Λ_o of Λ we have $\bigcap_{\lambda \in \Lambda_o} L_\lambda = 0$ and hence, by Theorem 1.29, A is finite-dimensional.

Remark: It is well-known ([16], Theorem 19) that a semi-simple Artinian algebra is a direct sum of minimal left ideals. This fact, together with the socle theorem gives an alternative proof of Theorem 1.

Lemma 2 A homomorphic image of an Artinian algebra is Artinian.

This is clear since an infinite (strictly) descending chain of left ideals in the image will, on taking inverse images, give rise to an infinite (strictly) descending chain of left ideals in the pre-image.

Lemma 3 The radical of an Artinian algebra is nilpotent. (For a proof of this fact see [18], P.38.)

Corollary 4 An Artinian Banach algebra is locally finite.

Proof: By Lemma 2, $A/\mathrm{rad}(A)$ is Artinian so, by Theorem 1, is finite-dimensional. The result follows by Lemma 3 and Lemma 1.20.

We now consider a chain condition which is weaker than the Artinian condition.

Definition An algebra A is said to be weakly Artinian if for each $x \in A$, the chain of principal left ideals, $(Ax^k)_{k \in \mathbb{P}}$, terminates. That is, there is an integer k_o (depending on x) such that

$$Ax^{k_o} = Ax^{k_o+1} = \cdots .$$

Notice that $Ax^{k_o+r} = Ax^{k_o}$, for any integer r, implies that $Ax^{k_o} = Ax^{k_o+1} = \cdots$.

A stronger condition has been considered by Le Page ([24]) who showed that any Banach algebra with unit which satisfies $Ax^2 = Ax$ $(x \in A)$ is necessarily semi-simple and commutative. It had previously been shown by Arens and Kaplansky ([2]) that any such A must in fact be finite-dimensional so that the only complex Banach algebras with unit which satisfy Le Page's condition are (up to isomorphism) algebras of diagonal matrices.

Note: The condition $Ax^2 = Ax$ $(x \in A)$ is usually called strong regularity.

In the appendix of [20], Kaplansky discusses arbitrary weakly Artinian algebras but, so far as we know, very little is known about this general case.

We note that the weakly Artinian condition is genuinely weaker than strong regularity for, any semi-simple finite-dimensional (normed) algebra is weakly Artinian while if it is non-commutative it will not be strongly regular.

Remarks: (i) A homomorphic image of a weakly Artinian algebra is weakly Artinian.

(ii) If $e \in A$ is a non-zero idempotent then eAe is weakly Artinian whenever A is weakly Artinian.

Examples: (i) Any Artinian algebra is weakly Artinian.

(ii) Any locally finite Banach algebra is weakly Artinian.

Proof of (ii): Suppose A is a locally finite Banach algebra; then $\text{rad}(A)$ is nil and so (Grabiner, [9]) nilpotent and $A/\text{rad}(A)$ is finite-dimensional. Let $x \in A$. If x is nilpotent then clearly (Ax^k) terminates so we may suppose that x is non-nilpotent. Write $\bar{A} = A/\text{rad}(A)$ and $\bar{x} = x + \text{rad}(A)$. Since $\bar{A}$ is finite-dimensional, $(\bar{A}\bar{x}^k)$ terminates. Suppose

$$\bar{A}\bar{x}^{P+1} = \bar{A}\bar{x}^P$$

then

$$\bar{A}(\bar{x}^P)^3 = \bar{A}\bar{x}^P$$

so there is $a \in A$ such that $\bar{a}(\bar{x}^P)^3 = (\bar{x}^P)^2$.

That is,

$$ax^{3P} - x^{2P} \in \text{rad}(A) .$$

Writing $y = x^P$, this says $ay^3 - y^2 \in \text{rad}(A)$ —— (*) .

Thus $a(ay^3 - y^3)y = a^2y^4 - ay^3 \in \text{rad}(A)$ which gives —— when added with (*) —— $a^2y^4 - y^2 \in \text{rad}(A)$. Continuing the process we see that

$$a^k y^{k+2} - y^2 \in \text{rad}(A) \qquad (k \in \mathbb{P}) .$$

Since $\text{rad}(A)$ is nilpotent, $(\text{rad}(A))^n = 0$ for some $n \in \mathbb{P}$ so that

$$(a^k y^{k+2} - y^2)^n = 0 \qquad (k \in \mathbb{P}) \text{ —— } (\dagger).$$

Choose $k \in \mathbb{P}$ such that $2n < k + 2$; then from $(\dagger)$ we have

$$zy^{k+2} = y^{2n} \quad \text{for some } z \in A$$

and so (Ay^k) terminates which implies that (Ax^k) terminates. Hence A is weakly Artinian.

<u>Proposition 5</u> A weakly Artinian Banach algebra has nilpotent radical.

<u>Proof:</u> Let A be a weakly Artinian Banach algebra and suppose $x \in \text{rad}(A)$. Then x is quasi-nilpotent. If x is not nilpotent then (Grabiner, [10]), (Ax^k) is an infinite strictly decreasing chain of left ideals which contradicts our assumption that A is weakly Artinian. Thus, $\text{rad}(A)$ is a nil ideal and is therefore (Grabiner, [9]) nilpotent.

<u>Remark:</u> From the proof of the above proposition we note that a weakly Artinian Banach algebra cannot contain any properly quasi-nilpotent elements.

<u>Proposition 6</u> Let A be a weakly Artinian Banach algebra. Then A cannot contain an infinite sequence of pairwise orthogonal non-zero idempotents.

<u>Proof:</u> We suppose that A does contain such a sequence and derive a contradiction. Let (e_n) be such a sequence and choose $(c_n) \subseteq \mathbb{C} \setminus (0)$ such that $c_n \to 0$ as $n \to \infty$ and $x = \sum_{n=1}^{\infty} c_n e_n$ belongs to A. x is clearly not nilpotent. Since A is weakly Artinian there is an integer k such that $Ax^{k+1} = Ax^k$ so that $Ax^{3k} = Ax^k$ and hence for some $a \in A$ we have

$$ax^{3k} = x^{2k} .$$

Thus $\sum a c_n^{3k} e_n = \sum c_n^{2k} e_n$ and on multiplying each side by e_j we obtain

$$c_j^k a e_j = e_j \qquad (j \in \mathbb{P}) .$$

Hence $\|e_j\| \leq |c_j|^k \|a\| \|e_j\|$ $(j \in \mathbb{P})$ which implies that $\|a\| \geq |c_j|^{-k}$ $(j \in \mathbb{P})$ and this is impossible so the proposition is proved.

If we now assume that there is a uniform bound to the lengths of our descending chains we are able to prove the following result.

Theorem 7 Suppose A is a semi-simple weakly Artinian Banach algebra with unit and that there is some fixed integer n_o such that

$$Ax^{n_o+1} = Ax^{n_o} \qquad (x \in A) .$$

Then A is finite-dimensional.

Proof: Kaplansky, [20], has shown that if (as we have here assumed) n_o is independent of x then we can find an element $y \in A$ which commutes with x and satisfies $yx^{n_o+1} = x^{n_o}$. This gives that $y^{n_o}x^{n_o}$ is an idempotent. If the unit of A is the only idempotent in A we would therefore have that every non-nilpotent element of A is invertible and hence that

$$Sp(A,x) = \{\lambda \in \mathbb{C} : \lambda - x \text{ is nilpotent}\} \qquad (x \in A) .$$

Since the spectrum is always non-empty this means that A is algebraic and hence finite-dimensional by Corollary 1.2/4. By Wedderburn's structure theorem we now have $A = \mathbb{C}1$. (Alternatively, for the case in which the unit of A is the only idempotent in A we may prove that $A = \mathbb{C}1$ by the same method we used in the first part of the proof of Theorem 1.41).

Suppose that A has proper idempotents. We employ the "idempotent-splitting" technique and Proposition 6 to produce a finite set $\{e_1, \cdots, e_n\}$ of pairwise orthogonal idempotents satisfying $1 = \sum_{j=1}^{n} e_j$ and such that e_j is the only non-zero idempotent in e_jAe_j. Each e_j is therefore a minimal idempotent and so $A = soc(A)$. Thus, by Theorem 1.26, A is finite-dimensional.

Theorem 8 Let A be a weakly Artinian Banach algebra with unit and suppose that $rad(A)$ coincides with the set of nilpotent elements in A. Then $A/rad(A)$ is commutative and finite-dimensional.

Proof: $B = A/rad(A)$ is weakly Artinian and by assumption has no non-zero nilpotent elements. If P is a primitive ideal of B then $C = {}^B/_P$ is a primitive weakly Artinian Banach algebra. Suppose x is a non-zero element of C then for some $y \in C$, $k \in \mathbb{P}$ we have

$$yx^{2k} = x^k .$$

Thus $[x^k(yx^k - 1)]^2 = 0$ so since C has no properly nilpotent elements we must have $x^k yx^k = x^k$. Thus yx^k is an idempotent. Now, for any proper idempotent $e \in C$,

$$(ex - exe)^2 = 0 = (xe - exe)^2 \qquad (x \in C)$$

and so $ex = exe = xe$ so that e is central. Thus yx^k is a central idempotent. Since C is primitive this means that $yx^k = 1$. Thus every non-zero element of C has a left inverse so, by Mazur's theorem,

$$C = \mathbb{C}1 .$$

It follows that every irreducible representation of B is one dimensional and so since B is semi-simple it is therefore commutative.

Now suppose that u is a non-zero element of B. For some $v \in B$, $p \in \mathbb{P}$ we have

$$vu^{2p} = u^p$$

and as above, vu^p is idempotent. If 1 is the only non-zero idempotent in B then $B = \mathbb{C}1$. By Proposition 6, the fact that eBe is weakly Artinian for any idempotent $e \in B$, and the "idempotent-splitting" argument we see that B is finite-dimensional.

Corollary 9 If A is a commutative weakly Artinian Banach algebra with unit then A is locally finite.

Proof: Since A is commutative, $rad(A)$ and the set of nilpotent elements coincide and so the theorem applies to give $A/rad(A)$ finite-dimensional. The result follows by Proposition 5 and Lemma 1.20.

We may also prove Corollary 9 as follows:

Suppose $x \in A$ and $Sp(A,x)$ is infinite. Then $Sp(A,x)$ has a cluster point $p \in Sp(A,x)$. By considering a translation of x we may suppose that $p = 0$. Using the fact that $Sp(A,x) = \{\phi(x) : \phi \in \Phi_A\}$ we choose a sequence $(\phi_n) \subseteq \Phi_A$ such that

$$0 \neq \phi_n(x) \to 0 \quad \text{as} \quad n \to \infty .$$

Since A is weakly Artinian there is $y \in A$, $k \in \mathbb{P}$ such that

$$yx^{k+1} = x^k .$$

Hence

$$\phi_n(y)\phi_n(x)^{k+1} = \phi_n(x)^k$$

so $\phi_n(y) = \phi_n(x)^{-1} \to \infty$ as $n \to \infty$ which is impossible. It follows that $Sp(A,x)$ is finite $(x \in A)$ and hence

$$A = rad(A) \oplus \mathbb{C}^n \quad (\text{for some } n \in \mathbb{P}) \text{ by}$$

Theorem 1.2. By Proposition 5 and Lemma 1.20, A is locally finite.

The following result is an easy corollary of Theorem 8. It is due in part to Arens and Kaplansky ([2]) who proved finite-dimensionality and in part to Le Page ([24]) who proved semi-simplicity and commutativity.

<u>Theorem 10</u> Let A be a Banach algebra with unit such that

$$Ax^2 = Ax \quad \text{for each} \quad x \in A .$$

Then A is semi-simple, commutative and finite-dimensional.

<u>Proof:</u> The condition $Ax^2 = Ax$ $(x \in A)$ implies that A has no proper nilpotent elements.

<u>Remarks:</u> (i) The converse of Theorem 10 is immediate from Wedderburn's theorem.

(ii) The assertion of the theorem fails without the assumption of a unit element. Any Banach algebra, A, with the trivial multiplication (i.e. all products are zero) satisfies $Ax^2 = Ax$ $(x \in A)$.

(iii) Let $A = C[0,1]$, the algebra of continuous complex functions on $[0,1]$. Then $Ax^2 = Ax$ for each invertible element x, and the set of invertible elements is dense in A. In this example we have the weaker condition that

$$\overline{Ax^2} = \overline{Ax} \qquad (x \in A) \qquad (*)$$

It would be of interest to characterise those Banach algebras which satisfy (*).

(iv) Let A be an arbitrary Banach algebra and let $q \in A$ be a properly quasinilpotent element. Then (Grabiner, [10]) the sequence (Aq^n) is strictly decreasing.

<u>Proposition 11</u> If A is a semi-simple weakly Artinian Banach algebra then A cannot contain an infinite family of pairwise orthogonal non-zero bi-ideals.

<u>Proof</u>: Suppose $(I_j)_{j\in\mathbb{P}}$ is an infinite family of pairwise orthogonal non-zero bi-ideals of A. Since A is semi-simple none of the I_j's are nil and so, for each $j \in \mathbb{P}$, we may choose an $x_j \in I_j$ with x_j non-nilpotent and $\|x_j\| = 1$.

Let $(c_j) \in \ell^1$, $c_j > 0$ $(j \in \mathbb{P})$. Then $x = \sum_{j=1}^{\infty} c_j x_j \in A$ and x is non-nilpotent. As in the preceding proposition there is $k \in \mathbb{P}$ and $a \in A$ such that $ax^{3k} = x^{2k}$.

Thus $\sum_{j=1}^{\infty} c_j^{3k} a\, x_j^{3k} = \sum_{j=1}^{\infty} c_j^{2k} x_j^{2k}$ and multiplication of each side by x_j gives $c_j^{3k} a\, x_j^{3k+1} = c_j^{2k} x_j^{2k+1}$ $(j \in \mathbb{P})$.

Hence $c_j^{k} a\, x_j^{3k+1} = x_j^{2k+1}$ and so $|c_j|^k \|a\| \|x_j^{2k+1}\| \|x_j\|^k \geq \|x_j^{2k+1}\|$ which gives $\|a\| \geq |c_j|^{-k}$ $(j \in \mathbb{P})$ which is impossible.

Jacobson ([17]) has shown that in any algebra A with unit if $xy = 1$ while $yx \neq 1$ for some elements $x, y \in A$ then A contains an infinite sequence of pairwise orthogonal non-zero idempotents.

This, together with Proposition 6, shows that in a weakly Artinian Banach algebra with unit if an element is left (or right) invertible then it is invertible. That is, writing Inv(A) for the set of invertible elements in A,

$$\mathrm{Inv}(A) = \{x \in A : x \text{ has a left or right inverse}\}.$$

This leads to the following characterisation of Inv(A):

$$\mathrm{Inv}(A) = \{x \in A : \mathrm{lan}(x) = 0\} .$$

It is clear that $\mathrm{Inv}(A) \subseteq \{x \in A : \mathrm{lan}(x) = 0\}$. Conversely, suppose $\mathrm{lan}(x) = 0$. Then, in particular, x is not nilpotent so there is $y \in A \setminus (0)$ and $k \in \mathbb{P}$ such that $yx^{k+1} = x^k$. So, $(yx - 1)x^k = 0$ and hence $yx = 1$, $x \in \mathrm{Inv}(A)$.

For each $x \in A$ there is a smallest integer $k \in \mathbb{P}$ such that $yx^{k+1} = x^k$ for some $y \in A$. Denote this integer by ind(x) and define $R(x) = \{y \in A : yx^{\mathrm{ind}(x)+1} = x^{\mathrm{ind}(x)}\}$. If $x \in \mathrm{Inv}(A)$ then $R(x)$ is a singleton. The converse also is true for suppose $R(x)$ is a singleton, $\{y\}$, and that $zx = 0$. Then $zx^{\mathrm{ind}(x)+1} = 0$ and $(y + z)x^{\mathrm{ind}(x)+1} = x^{\mathrm{ind}(x)}$ so $y = y + z$ and hence $z = 0$. Therefore $\mathrm{lan}(x) = 0$ and $x \in \mathrm{Inv}(A)$.

Lastly, when A is semi-simple, $x \in \mathrm{Inv}(A)$ if and only if $R(x) \subseteq \mathrm{Inv}(A)$. It is clear that $x \in \mathrm{Inv}(A)$ implies $R(x) \subseteq \mathrm{Inv}(A)$. Suppose $R(x) \subseteq \mathrm{Inv}(A)$ and let $z \in \mathrm{lan}(x)$ then

$$(1 - z)x^{\mathrm{ind}(x)+1} = x^{\mathrm{ind}(x)+1}$$

so if $y \in R(x)$ we will have $y(1 - z)x^{\mathrm{ind}(x)+1} = x^{\mathrm{ind}(x)}$ which gives

$$y(1 - z) \in R(x) \subseteq \mathrm{Inv}(A) .$$

Since $y \in \mathrm{Inv}(A)$ this means that $1 - z \in \mathrm{Inv}(A)$, z is quasi-regular. Thus $\mathrm{lan}(x)$ is a quasiregular left ideal, $\mathrm{lan}(x) \subseteq \mathrm{rad}(A) = 0$

and so $x \in Inv(A)$. We collect these results together in

<u>Proposition 12</u> Let A be a weakly Artinian Banach algebra with unit. Then

$$Inv(A) = \{x \in A : x \text{ is left or right invertible}\}$$
$$= \{x \in A : lan(x) = 0\}$$
$$= \{x \in A : R(x) \text{ is a singleton}\} .$$

If also A is semi-simple then

$$Inv(A) = \{x \in A : R(x) \subseteq Inv(A)\} .$$

<u>Conjecture:</u> A semi-simple weakly Artinian Banach algebra is finite-dimensional.

<u>Remark:</u> To prove this using the "idempotent-splitting" technique one would necessarily require to have some method of constructing idempotents in A. We have so far been unable to do this using only the equations of the type $yx^{k+1} = x^k$ which the chain condition gives us.

However, by strengthening the chain condition slightly we are able to obtain idempotents.

<u>Definition</u> A is said to satisfy the descending chain condition (dcc) on principal left ideals if every descending chain of principal left ideals stabilises.

<u>Remarks</u> (i) Such an algebra necessarily contains minimal left ideals and hence, in the semi-prime case, minimal idempotents.

(ii) Any homomorphic image of A also satisfies the given chain condition.

(iii) If $e \in A$ is idempotent, eAe satisfies the d.c.c. on principal left ideals.

<u>Theorem 13</u> Let A be a semi-simple Banach algebra which satisfies the d.c.c. on principal left ideals. Then A is finite-dimensional.

Proof: We begin by remarking that since A is in particular weakly Artinian it cannot contain an infinite sequence of pairwise orthogonal non-zero idempotents (Proposition 6).

Let E be a maximal set of pairwise orthgonal idempotents in A. By the above remark, E is finite, $E = \{e_1, \cdots, e_m\}$, say. Consider e_1Ae_1. This is a semi-simple Banach algebra which satisfies the d.c.c. on principal left ideals so contains a minimal idempotent f_1. Since e_1 is the unit for e_1Ae_1 we have

$$f_1 A f_1 = f_1 e_1 A e_1 f_1 = \mathbb{C} f_1$$

so f_1 is minimal with respect to A. f_1 and $e_1 - f_1$ are orthogonal idempotents and if $e_1 - f_1 \neq 0$ we replace e_1 in E by $e_1 - f_1$ and f_1. The rest of the proof is identical to the corresponding part of the proof of Theorem 1.38 except that we replace "regular Banach algebra" by "Banach algebra satisfying the d.c.c. on principal left ideals".

Corollary 14 For Banach algebras, the d.c.c. on principal left ideals implies local finiteness.

Proof: If A satisfies the d.c.c. on principal left ideals, A is weakly Artinian so has nilpotent radical. By Theorem 13 and Remark (ii) the radical is cofinite. The result follows by Lemma 1.20.

Remark: In view of Corollary 14 and the fact that a locally finite Banach algebra is weakly Artinian, local finiteness appears as a condition which, for Banach algebras, is intermediate in strength between the weakly Artinian chain condition and the d.c.c. on principal left ideals.

We are able to prove one further result on weakly Artinian Banach algebras.

Definition An algebra A is two-sided weakly Artinian if, for each $x \in A$, the chains (Ax^k), (x^kA) terminate.

Theorem 15 A two-sided weakly Artinian semi-simple Banach algebra, A , is finite-dimensional.

Proof: We show that A is π-regular. Let $x \in A$. We may suppose that x is not nilpotent. There are integers $p, q \in \mathbb{P}$ such that

$$Ax^{p+1} = Ax^p \ , \ x^{q+1}A = x^qA \ .$$

Let $k = \max(p,q)$ so that

$$Ax^{k+1} = Ax^k \ , \ x^{k+1}A = x^kA$$

and in particular

$$Ax^{4k} = Ax^k \ , \ x^{4k}A = x^kA \ .$$

Write $a = x^{2k}$ then there are $y, z \in A$ such that

$$ya^2 = a = a^2z \ .$$

Thus

$$ya = ya^2z = az$$

and hence

$$aya = a^2z = a \ .$$

That is, $x^{2k}yx^{2k} = x^{2k}$ so A is π-regular. By Theorem 1.41, A is finite-dimensional.

Corollary 16 A two-sided weakly Artinian Banach algebra is locally finite. The converse is also true.

The last section of this chapter consists of a discussion of the effect an ascending chain condition has on a Banach algebra.

Definition. An algebra is said to be Noetherian if every ascending chain of left ideals becomes stationary.

Remark: The other chain conditions which we have studied in this thesis have all been descending chain conditions. Most of these have had the effect of guaranteeing the existence of minimal ideals. The ascending chain condition guarantees the existence of maximal ideals but since we are working in a Banach algebra we already know that these exist so, in this respect at least, the Noetherian condition tells us nothing new.

It is well known that a commutative Noetherian Banach algebra is finite-dimensional ([11], [23]). So far, there has been no elementary proof of this fact. Here we prove that any Noetherian Banach algebra is finite-dimensional. The main tools we shall use are the open mapping theorem for bounded operators and Kaplansky's finite spectrum theorem. We start with the following result which may be found in [11]. (The proof given here is essentially the same as that in [11]).

Theorem 17 Let A be a Noetherian Banach algebra. Then all left ideals in A are closed.

Proof: Let L be a non-zero left ideal of A. We show that $\overline{L} \subseteq L$.

Since A is Noetherian, $\overline{L}$ is finitely generated. In fact, there are elements $a_1, a_2 \cdots, a_n \in \overline{L}$ such that

$$\overline{L} = Aa_1 + Aa_2 + \cdots + Aa_n .$$

Define $\phi : A^n \to \overline{L}$ by

$$\phi(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i a_j .$$

With norm $\|(x_1, \cdots, x_n)\| = \max\{\|x_i\| : i = 1, \cdots, n\}$, A^n is a Banach space and ϕ is a bounded linear operator from A^n onto $\overline{L}$. If $\varepsilon > 0$, write $B(\varepsilon) = \{x \in A : \|x\| < \varepsilon\}$. Then, by the open mapping theorem, $L + \sum_{i=1}^{n} B(\varepsilon)a_i$ is all of $\overline{L}$. Thus there are elements $b_i \in L$, $c_{ij} \in B(\varepsilon)$ $(i, j = 1, 2, \cdots, n)$ such that

$$a_i = b_i + \sum_{j=1}^{n} c_{ij} a_j \qquad (i = 1, \cdots, n)$$

so

$$b_i = a_i - \sum_{j=1}^{n} c_{ij}\, a \qquad (i = 1, \cdots, n) .$$

Define $\Psi : A^n \to A^n$ by

$$(\Psi x)_i = x_i - \sum_{j=1}^{n} c_{ij}\, x_j \qquad (i = 1, \cdots, n) .$$

Then Ψ is a bounded linear operator on A^n and if $\|x\| \le 1$

$$\|(I - \Psi)x\| \le n\,\varepsilon \qquad (I = \text{identity operator}) .$$

Thus, if $n\,\varepsilon < 1$, Ψ is invertible with inverse Ψ^{-1} given by

$$\Psi^{-1} = \sum_{k=0}^{\infty} (I - \Psi)^k = \sum_{k=0}^{\infty} \Theta^k \qquad (\Theta = I - \Psi) .$$

Now $(\Theta x)_i = \sum_{j=1}^{n} c_{ij}\, x_j$ so that

$$(\Theta^k x)_i = \sum_{j=1}^{n} c_{ij}^{(k)} x_j \quad \text{for some elements} \quad c_{ij}^{(k)} \in A .$$

Thus $\left(\left(\sum_{k=0}^{N} \Theta^k\right) x\right)_i = \sum_{j=1}^{n} \left(\sum_{k=0}^{N} c_{ij}^{(k)}\right) x_j$ where $c_{ij}^{(0)} = 1$, $c_{ij}^{(1)} = c_{ij}$ $(i, j = 1, 2, \cdots, n)$. Next,

$$\sum_{k=0}^{N} \|c_{ij}^{(k)}\| \le 1 + \|c_{ij}\| + \left\|\sum_{p=1}^{n} c_{ip} c_{pj}\right\| + \left\|\sum_{p,q=1}^{n} c_{ip} c_{pq} c_{qj}\right\| + \cdots$$

$$\le 1 + \varepsilon + n\varepsilon^2 + n^2\varepsilon^3 + \cdots$$

$$= 1 + \frac{\varepsilon}{1 - n\varepsilon} .$$

Hence $\left(\sum_{k=0}^{N} c_{ij}^{(k)}\right)$ converges in A. Write $c'_{ij} = \sum_{k=0}^{\infty} c_{ij}^{(k)}$ $(i, j = 1, 2, \cdots, n)$.

Then, as $N \to \infty$,

$$\left(\left(\sum_{k=0}^{N} \Theta^k\right) x\right)_i \to \left(\Psi^{-1} x\right)_i \qquad (i = 1, \cdots, n)$$

and

$$\sum_{j=1}^{n} \left(\sum_{k=0}^{N} c_{ij}^{(k)}\right) x_j \to \sum_{j=1}^{n} c'_{ij} x_j \qquad (i = 1, \cdots, n) .$$

So, $(\Psi^{-1}x)_i = \sum_{j=1}^{n} c'_{ij} x_j$ and in particular

$$a_i = (\Psi^{-1}b)_i = \sum_{j=1}^{n} c'_{ij} b_j \in L .$$

Hence $\bar{L} \subseteq L$ and L is closed.

<u>Lemma 18</u> Let A be a normed algebra and π be the left (right) regular representation of A on A. Then

$Sp(A,x) \cup (0) = Sp(B(A), \pi(x)) \cup (0)$ $(x \in A)$ and in particular

$\partial Sp(A,x) \cup (0) = \partial Sp(B(A), \pi(x)) \cup (0)$ $(x \in A)$ where "∂" means topological boundary.

<u>Lemma 19</u> Let X be a Banach space and $T \in B(X)$. Suppose that TX is closed and $0 \in \partial Sp(B(X),T)$. Then 0 is an eigenvalue of T.

<u>Proof:</u> Since $0 \in \partial Sp(B(X),T)$ there is a sequence $(x_n) \subseteq X$ such that $\|x_n\| = 1$ and $Tx_n \to 0$ as $n \to \infty$ (see [29], P. 278). Since TX is closed Banach's Isomorphism Theorem shows that $\operatorname{Ker} T \neq 0$.

<u>Lemma 20</u> Let A be a Noetherian Banach algebra. Then $rad(A)$ has finite codimension in A.

<u>Proof:</u> The proof is by contradiction. Suppose there is an element $x \in A$ such that $\partial Sp(A,x)$ is infinite. Choose a sequence $(\lambda_n) \subseteq \partial Sp(A,x) \setminus (0)$ of distinct elements, λ_n. Let

$L_n = \{a \in A : a(\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x) = 0\}$.

Then (L_n) is a non-decreasing sequence of closed left ideals of A. By Theorem 17 and Lemmas 18, 19 there is $a \in A \setminus (0)$ such that

$$a(\lambda_{n+1} - x) = 0$$

while

$$a(\lambda_1 - x)(\lambda_2 - x) \cdots (\lambda_n - x) = (\lambda_1 - \lambda_{n+1}) \cdots (\lambda_n - \lambda_{n+1}) a \neq 0 .$$

Thus (L_n) is a strictly increasing sequence which contradicts our assumption that A is Noetherian. Hence $\partial Sp(A,x)$ is finite. So $Sp(A,x)$ is finite for each $x \in A$. The result follows by Kaplansky's finite spectrum theroem.

Theorem 21 Let A be a Noetherian Banach algebra. Then A is finite-dimensional.

Proof: By Lemma 20 it is sufficient to show that rad(A) is finite-dimensional. We show first that rad(A) is nilpotent.

Let $x \in rad(A)$. The sequence, $(lan(x^k))$, of left ideals of A must become stationary since A is Noetherian;

$$lan(x^N) = lan(x^{N+1}) = \cdots, \text{ say} .$$

Let $B = A/lan(x^N)$ and define $T : B \to B$ by

$$T(a + lan(x^N)) = ax + lan(x^N) .$$

Then T is a well-defined bounded operator on B. Furthermore, T is quasinilpotent because $x \in rad(A)$ and one-to-one because $lan(x^{N+1}) = lan(x^N)$. Also, $TB = (Bx + lan(x^N))/lan(x^N)$ is closed in B since $Bx + lan(x^N)$ is closed in A (Theorem 17). Thus, by Lemma 19, 0 is an eigenvalue of T. This contradicts the fact that T is one-to-one. It follows that $lan(x^N) = A$ and hence that x is nilpotent. Thus rad(A) is nil and so nilpotent; $R^m = 0$, say, where $R = rad(A)$.

Suppose R^{m-1} is infinite-dimensional. If $x \in R^{m-1}$ then, since A/R is finite-dimensional and $R.R^{m-1} = 0$, Ax is finite-dimensional. There are two cases to consider:

(i) AR^{m-1} is infinite-dimensional,

(ii) AR^{m-1} is finite-dimensional.

If case (i) obtains then we may choose a sequence $(x_n) \subseteq R^{m-1}$ such that $(Ax_1 +\cdots+ Ax_n)$ is a strictly increasing sequence of left ideals of A. This contradicts the fact that A is Noetherian. If case (ii) obtains then A annihilates an infinite-dimensional subspace Z of R^{m-1}. Then each subspace of Z is also a left ideal of A which is impossible since A is Noetherian. Thus R^{m-1} is finite-dimensional. $A_1 = A/R^{m-1}$ is a Noetherian algebra with radical $R_1 = R/R^{m-1}$ and $A_1/R_1 \cong A/R$ is finite-dimensional. Since $R_1^{m-1} = 0$ the above argument, applied to A_1 and R_1, shows that R_1^{m-2} is finite-dimensional. Thus R^{m-2} is finite-dimensional. A finite induction completes the proof.

Lemma 22 Let A be a Banach algebra in which every left ideal is closed. Then A is Noetherian.

Proof (S.J. Sidney): Let (L_n) be an increasing chain of left ideals in A. By assumption, each L_n and $L = \bigcup_{n=1}^{\infty} L_n$ is closed. By Baire's category theorem some L_{n_0} has non-empty interior in L and hence $L_{n_0} = L$ and A is Noetherian.

Theorem 23 A Banach algebra in which every left (right) ideal is closed is finite-dimensional.

Proof: The result follows immediately from Lemma 22 and Theorem 21.

We complete this chapter by proving the following extension of Theorem 1.

Theorem 24 An Artinian Banach algebra is finite-dimensional.

Proof: Let R be the radical of an Artinian Banach algebra A. Then ([18], p. 261) A/R is Noetherian and therefore finite-dimensional. R is nilpotent ([18], p. 38 Theorem 1) ; $R^m = 0$ (say).

R^{m-1} is a unital A/R-module ([18], p. 46 Theorem 1) so is completely reducible([18], p. 47 Theorem 2(1)). Thus R^{m-1} is the direct sum of all the irreducible A/R-modules which it contains.

From the way in which the module multiplication is defined, it can be seen that these are in fact minimal left ideals of A. Since A is Artinian, R^{m-1} can be written as a finite direct sum of these of these minimal left ideals ; $R^{m-1} = L_1 \oplus L_2 \oplus \ldots \oplus L_k$. Each L_j is finite-dimensional since A/R is finite-dimensional. So, R^{m-1} is finite-dimensional and closed. The Banach algebra $A_1 = A/R^{m-1}$ has radical $R_1 = R/R^{m-1}$ where $R_1^{m-1} = 0$. It follows that R^{m-2} is finite-dimensional. The process is repeated until, after a finite number of steps, we obtain that R is finite-dimensional.

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The results on Noetherian Banach algebras in Chapter 2 appear in a paper, [31], which Allan Sinclair and myself have recently submitted to the journal Mathematische Annalen.

CHAPTER 3

The aim of this chapter is to prove two results in the theory of Banach *-algebras. The first is the well-known Shirali-Ford theorem ([30]) which states that if the involution on a Banach *-algebra is Hermitian then it is symmetric. The second result is proved using the same technique as is used to prove the Shirali-Ford theorem. This time we are working in a Banach *-algebra in which the involution is assumed to be both Hermitian and continuous. The result is that the positive wedge in such an algebra is a closed set.

<u>Definition</u> Let A be an arbitrary algebra. An involution on A is a conjugate-linear, anti-automorphism of A of period two. That is, a mapping $x \to x^*$ of A into A with the following properties

1) $(\lambda x)^* = \bar{\lambda} x^*$ $(\lambda \in \mathbb{C},\ x \in A,$ where $\bar{\lambda}$ is the complex conjugate of $\lambda)$,
2) $(x + y)^* = x^* + y^*$ $(x, y \in A)$,
3) $(x y)^* = y^* x^*$ $(x, y \in A)$,
4) $(x^*)^* = x$ $(x \in A)$.

An algebra with involution is often called simply a *-algebra. A Banach algebra which has an involution defined on it is called a Banach *-algebra. The image x^* of an element x under the involution is called the adjoint of x. An element x in A is said to be self-adjoint if $x^* = x$, and non-negative if

$$Sp(A, x) \subseteq \mathbb{R}^+ = \{ \alpha \in \mathbb{R} : \alpha \geqslant 0 \}.$$

The involution is said to be Hermitian if for each self-adjoint element $x \in A$ we have $Sp(A, x) \subseteq \mathbb{R}$. The involution is said to be symmetric if for each $x \in A$ we have $Sp(A, x^*x) \subseteq \mathbb{R}^+$.

The equivalence of the preceding two notions for Banach *-algebras was demonstrated by Ford and Shirali in [30]. It is almost immediate that a symmetric Banach *-algebra is Hermitian. Here we intend to give a simple proof of the converse. Basic to the proof is the following square root lemma which is due to Ford, [8]. We denote by $r(x)$ the spectral radius of an element x in A where

$$r(x) = \sup\{|\lambda| : \lambda \in Sp(A, x)\}.$$

<u>Lemma 1</u> Let A be a Banach *-algebra with unit. If x is a self-adjoint element of A and $r(1-x) < 1$ then there is an element w in A such that w is self-adjoint and $w^2 = x$.

<u>Definition</u> For each $x \in A$ we define

$$P(x) = [r(x^*x)]^{\frac{1}{2}}.$$

Lemmas 2 and 3 are due to Pták, [28].

<u>Lemma 2</u> If A is a Banach *-algebra with a Hermitian involution then

$$r(x) \leq P(x) \qquad (x \in A).$$

<u>Proof</u>: (Pták) (The lemma is proved under the assumption that A has a unit but the result is extended easily to the case in which A has no unit.)

We show that if $P(x) < 1$ then $1 \notin Sp(A, x)$. Then since $P(\lambda a) = |\lambda| P(a)$ $\quad (\lambda \in \mathbb{C},\ a \in A)$ the result will follow.

Suppose $P(x) < 1$ then $1 - x^*x \in Inv(A)$ and so, by Lemma 1, there is a self-adjoint element w in A such that $1 - x^*x = w^2$.

Thus

$$\begin{aligned}(1 + x^*)(1 - x) &= (1 - x^*x) + (x^* - x)\\ &= w^2 + (x^* - x)\\ &= w[\,1 + w^{-1}(x^* - x)w^{-1}\,]w\end{aligned}$$

Now $iw^{-1}(x^* - x)w^{-1}$ is self-adjoint so has real spectrum. Thus $-1 \notin Sp(A, w^{-1}(x^* - x)w^{-1})$ and so $1 - x$ has a left inverse ~~1 + x*~~.

Similarly, we may show that $1 - x^*$ ~~is~~ has a right inverse ~~for $1 - x$~~.
Hence $1 - x$ is invertible and $1 \notin Sp(A, x)$.

Lemma 3 Let A be a Banach *-algebra with a Hermitian involution.
Then

1) $r(\cdot)$ is submultiplicative on the set of self-adjoint elements,

2) a sum of non-negative elements is non-negative.

Proof: (Ptak) 1) Let $u, v \in A$ be self-adjoint then

$$r(uv) \leq P(uv) = r(vuuv)^{\frac{1}{2}} = r(u^2v^2)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned} r(uv) &\leq r(u^{2^n} v^{2^n})^{2^{-n}} && (n \in \mathbb{P}) \\ &\leq \|u^{2^n}\|^{2^{-n}} \|v^{2^n}\|^{2^{-n}} && (n \in \mathbb{P}) \\ &\rightarrow r(u)\, r(v) \quad \text{as} && n \rightarrow \infty. \end{aligned}$$

2) It is sufficient to show that if u, v are non-negative then $-1 \notin Sp(A, u + v)$. Thus, suppose $u, v \in A$ and $Sp(A, u) \subseteq \mathbb{R}^+$, and $Sp(A, v) \subseteq \mathbb{R}^+$. Then

$$\begin{aligned} 1 + u + v &= (1 + u)(1 + v) - uv \\ &= (1 + u)[1 - hk](1 + v) \end{aligned}$$

where $h = (1 + u)^{-1}u$, $k = (1 + v)^{-1}v$ and $r(h) < 1$, $r(k) < 1$.
Thus, by 1), $r(hk) < 1$ so $1 + u + v \in Inv(A)$ and $-1 \notin Sp(A, u + v)$.

We are now ready to prove the main theorem of this chapter. Throughout the proof of this theorem, since we shall be concerned only with spectra of elements relative to the whole algebra A, we shall abbreviate $Sp(A, x)$ to simply $Sp(x)$.

Theorem 4 Let A be a Banach *-algebra with unit. If the involution on A is Hermitian then it is symmetric.

Proof: Let $x \in A$. We have to show that $Sp(x^*x) \subseteq \mathbb{R}^+$. The proof is by contradiction. Write $h = (x + x^*)/2$ and $k = (x - x^*)/2i$ so that h, k are self-adjoint and $x = h + ik$. Since the involution on A is Hermitian and x^*x is self-adjoint, $Sp(x^*x) \subseteq \mathbb{R}$. Let $\alpha = \inf Sp(x^*x)$, $\beta = \sup Sp(x^*x)$ so that $\alpha \leq x^*x \leq \beta$.

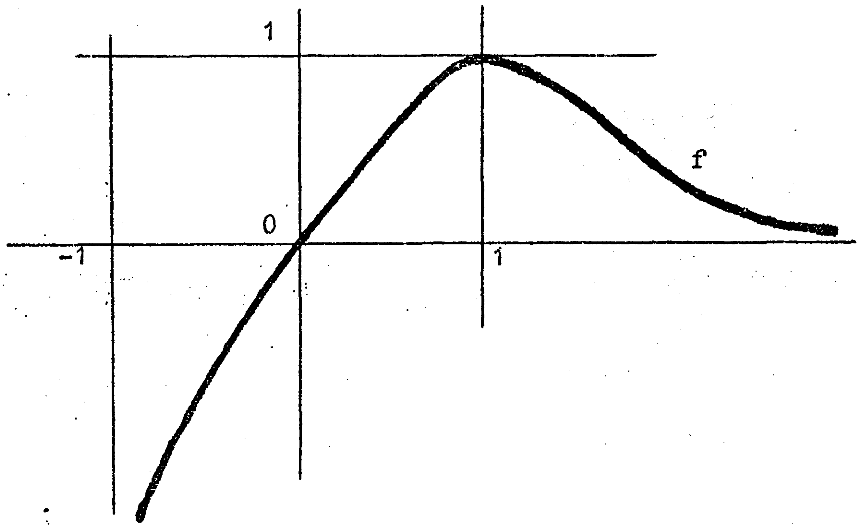
Since a sum of non-negative elements is non-negative, we have

$$xx^* = 2(h^2 + k^2) - x^*x \geq -\beta.$$

Since $Sp(x^*x)\backslash(0) = Sp(xx^*)\backslash(0)$ this shows that $-\beta \leq \alpha$, also that β cannot be strictly negative, and that if $\beta = 0$ then $\alpha = 0$. We may assume without loss of generality that $Sp(x^*x) \subseteq (-1,1)$.

Suppose now that $\alpha < 0$ and put $y = 2x(1 + x^*x)^{-1}$. Then $y^*y = 1 - (1 - x^*x)^2(1 + x^*x)^{-2}$ so, by the Spectral Mapping Theorem, $\inf Sp(y^*y) = f(\alpha)$, $\sup Sp(y^*y) = f(\beta)$ where $f: (-1,\infty) \to \mathbb{R}$ is given by $f(\lambda) = 1 - (1 - \lambda)^2(1 + \lambda)^{-2}$. It is clear that

$$f(\alpha) < 0 < f(\beta) < 1.$$



If $f(\alpha) < -1$ then we are finished for $f(\beta) \geq -f(\alpha)$, $f(\beta) < 1$ will provide a contradiction. If not, we repeat the process with $z = 2y(1 + y^*y)^{-1}$. Observe that $f(\gamma) < 4\gamma$ for any $\gamma \in (-1,0)$ so that

$$\inf Sp(z^*z) = f(f(\alpha)) < 4^2\alpha.$$

It is clear that eventually $(fofo\ldots of)(\alpha) \leq -1$ while we always have $(fofo\ldots of)(\beta) < 1$ and this gives the desired contradiction.

Corollary 5 The theorem holds even if A has no unit.

Proof: $A \oplus \mathbb{C}1$ is a Banach *-algebra with a Hermitian involution.

Theorem 6 Let A be a Banach *-algebra with unit and suppose that the given involution on A is Hermitian and continuous. Then the positive wedge, $P = \{h \in A: h^* = h,\ h \geq 0\}$, is closed.

Proof: Since the involution is continuous the set of self-adjoint elements is closed. Suppose $(h_n) \subseteq P$, $0 \leq h_n \leq \alpha < 1$ and $h_n \to h \notin P$ so that $\beta = \inf \mathrm{Sp}(h) < 0$. Since the involution is Hermitian, $r(\cdot)$ is subadditive on the set of self-adjoint elements [See [28]] and so $|r(h_n) - r(h)| \leq r(h_n - h) \leq \|h_n - h\| \to 0$ as $n \to \infty$. Hence $-1 < -\alpha \leq h \leq \alpha < 1$.

Define $g: (-1,\infty) \to R$ by $g(\lambda) = 2\lambda/(1 + \lambda)$. Then $g(h_n) \to g(h)$ as $n \to \infty$ so, since $0 \leq g(h_n) \leq g(\alpha) < 1$ we have $-1 < -g(\alpha) \leq g(h) \leq g(\alpha) < 1$. Now, $g(\beta) = \inf \mathrm{Sp}(g(h))$ so if $g(\beta) \leq -1$ we have a contradiction. If not, we consider $(g o g)(h_n)$ and so on. For $-1 < \gamma < 0$ we have $g(\gamma) < 2\gamma$ and so, as in the proof of Theorem 4, we see that eventually $(gogo\ldots og)(\beta) \leq -1$ while we always have $(gogo\ldots og)(h) > -1$. This proves the theorem.

We conclude this chapter on Banach *-algebras with a theorem which is a B*-algebra version of Kaplansky's finite spectrum theorem.

Definition A Banach *-algebra A is called a B*-algebra if the norm and the involution on A are related by the formula

$$\| x \|^2 = \| x^*x \| \qquad (x \in A).$$

Remark A B*-algebra is semi-simple. (see e.g. [29], P.188).

Theorem 7 Let A be a B*-algebra in which every self-adjoint element has a finite spectrum. Then A is finite-dimensional.

Proof: We may suppose without loss that A has a unit ([29], Lemma(4.1.13)). Provided $A \neq \mathbb{C}1$ we may obtain self-adjoint minimal idempotents in A as follows. Let C be a maximal commutative

*-subalgebra of A and let $x \in C$. Suppose $x = h + ik$ where h, k are self-adjoint. Then $h, k \in C$ and $Sp(C,x) = \{\phi(h)+i\phi(k): \phi \in \Phi_c\}$. But $\{\phi(h): \phi \in \Phi_c\} = Sp(C, h) = Sp(A, h)$ and

$\{\phi(k): \phi \in \Phi_c\} = Sp(C, k) = Sp(A, k)$ are finite and therefore so is $Sp(C, x)$. Since C is semi-simple we have, by Theorem 1.2, $C = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \dots \oplus \mathbb{C}e_n$ where $e_1, e_2, \dots, e_n$ are pairwise orthogonal minimal (in C) idempotents. In fact, each e_j is self-adjoint and minimal in A. For,

$$e_j e_j^* e_j \in \mathbb{C}e_j$$

$$= e_j^* e_j e_j^* \in \mathbb{C}e_j^* \quad \text{so that} \quad e_j^* = e_j \quad \text{while}$$

if $y = u + iv \in A$ then $e_j y e_j = e_j u e_j + ie_j v e_j \in \mathbb{C}e_j$ since $e_j u e_j$, $e_j v e_j$ lie in C by maximality.

Let E be a maximal set of pairwise orthogonal self-adjoint minimal idempotents in A. By our assumption on spectra and Lemma 1, E is finite, $E = \{f_1, f_2, \dots, f_m\}$ say. We have $f_1 + f_2 + \dots + f_m = 1$. For if not, then $f = 1 - \sum_{j=1}^{m} f_j$ is a self-adjoint idempotent and fAf is a B*-algebra satisfying the condition of the theorem so that, by the process already described, we may find a self-adjoint minimal idempotent in fAf which is orthogonal to E thereby contradicting the maximality of E. It follows that $A = soc(A)$ and so A is finite-dimensional.

<u>Note</u> In [27], Ogasawara considers certain other conditions which force finite-dimensionality in Banach *-algebras. In particular, Theorem 7 (above) is an easy consequence of Theorem 1 in [27].

CHAPTER 4

In this chapter we collect together several miscellaneous results on Banach algebras most of which have appeared in [7].

We consider some conditions which are sufficient to ensure the existence of proper nilpotent elements in a Banach algebra and then go on to construct a non-commutative Banach algebra in which there are no quasinilpotent elements - and hence no nilpotent elements. A slight alteration in the construction of this algebra produces a non-commutative radical Banach algebra which has no divisors of zero other than 0.

Next we consider conditions on a Banach algebra which imply that the Banach algebra is commutative. In certain special cases we see that the commutativity question and the existence of nilpotents question are closely related.

Finally, we look at the spectrum of an element of a Banach algebra and show by an example that it is not in general possible to remove all (or indeed any) of the interior points of the spectrum by enlarging the algebra which contains the element.

Our first result is entirely algebraic.

<u>Theorem 1</u> An algebra of operators on a complex vector space which contains a non-central operator of finite rank also contains a non-zero nilpotent operator.

<u>Proof:</u> Let A be an algebra of operators on a complex vector space and let b be a non-central finite rank operator in A. Since b has finite rank, the subalgebra bAb is finite-dimensional and hence its radical consists of nilpotents. Suppose therefore that bAb is

semi-simple. By Wedderburn's Structure Theorem bAb is isomorphic to a finite direct sum of full matrix algebras over $\mathbb{C}$ and hence contains non-zero nilpotents unless it is commutative. Thus we may suppose that bAb is commutative. Let $\{ e_i : i = 1, \ldots ,k \}$ be a spanning subset of minimal idempotents of bAb, and let $t \in A$. For each i, $e_i t - e_i t e_i$ and $t e_i - e_i t e_i$ are nilpotent. If these are all zero then $e_i t = t e_i$ $(i = 1, \ldots ,k)$, and so

(1) $ct = tc$ $(t \in A,\ c \in bAb)$ which implies that

(2) $(bt)^n = (tb)^n$ $(t \in A,\ n = 2,3,\ldots,)$

By using (1) and (2) we can show that each term in the expansion of $(bt - tb)^3$ is precisely $(bt)^3$ and hence we have $(bt - tb)^3 = 0$. Since b is non-central, $bt - tb$ is non-zero for some $t \in A$ and thus A always has a non-zero nilpotent.

Corollary 2 Let A be an irreducible Banach algebra of operators and suppose that A contains a non-zero finite rank operator. Then A contains a non-zero nilpotent operator.

Proof: The centre of A is either (0) or the scalar multiples of the identity.

Corollary 3 Let H be a Hilbert space and suppose that $T \in B(H)$ is a compact Hermitian operator. Then any closed subalgebra A of B(H) which contains T and in which T is non-central also contains a non-zero nilpotent operator.

Proof: For each $\lambda \in Sp(A, T)\backslash(0)$ there is a corresponding finite rank spectral projection P_λ. Each P_λ is a uniform limit of polynomials in T and so belongs to A. In particular, if every P_λ is central in A then so is T. Thus, for some $\lambda \in Sp(A, T)\backslash(0)$, P_λ is non-central and the theorem applies.

Remark In general, we cannot drop the "non-central" condition. For example, let X be a complex vector space and choose any two linearly independent vectors $u, v \in X$. Now choose $f, g \in X'$ (the dual of X) such that $f(u) = g(v) = 1$ and $f(v) = g(u) = 0$. For $x \in X$, $h \in X'$ define

$$(x \otimes h)(z) = h(z)x \qquad (z \in X).$$

Then $x \otimes h$ is a rank one operator on X. Let $S = u \otimes f + v \otimes g$ and $T = u \otimes g + v \otimes f$. Then $S^2 = T^2 = S$ and $ST = TS = T$. Clearly, the algebra generated by S and T is simply the linear span of $\{S, T\}$ and a simple calculation shows 0 to be its only nilpotent. (A non-trivial example is given by Theorem 8).

The following theorem which is due to Behncke, [3], gives a necessary and sufficient condition for the existence of non-zero nilpotents in a certain class of L' algebras.

Theorem 4 Let G be a locally compact group. Then $L'(G)$ has non-zero nilpotent elements if and only if G is non-abelian.

Remark Any algebra which contains a non-central idempotent element also contains a non-zero nilpotent (see proof of Theorem 1).

In general, non-commutativity alone is not sufficient to guarantee the existence of non-zero nilpotents (or even quasinilpotents) in a Banach algebra. This is made clear by our next theorem.

Theorem 5 There exists a non-commutative Banach algebra in which 0 is the only quasinilpotent element.

Proof: Let F_2 be the free algebra on two symbols u, v. That is, the algebra of all finite linear combinations of words in u and v. The set of all such words, $\{w_n\}$, is countable and we take the standard enumeration given by

$$u,\ v,\ u^2,\ uv,\ vu,\ v^2,\ u^3,\ u^2v,\ uvu,\ \ldots$$

Let B be the algebra $l^1(F_2)$ with pointwise multiplication. That is, B is the algebra of all infinite series $x = \sum \alpha_n w_n$ where $\|x\| = \sum |\alpha_n| < \infty$. Then B is a non-commutative Banach algebra. Let $x \in B$, $x \neq 0$, and let α_p be the first non-zero coefficient in the series $\sum \alpha_n w_n$. Then the coefficient of w_p^m in x^m is precisely α_p^m and so $\|x^m\| \geq |\alpha_p|^m$ $(m = 1, 2, \ldots)$.

Hence $$r(x) \geq |\alpha_p| > 0.$$

<u>Remark</u> B is an infinite dimensional non-commutative Banach algebra in which the set of quasinilpotents coincides with the set of nilpotents.

With F_2 as in Theorem 5, let $\nu(w_n)$ denote the length of the word w_n, and let C be the algebra of all infinite series $\sum \alpha_n w_n$ where $\|x\| = \sum |\alpha_n| / \nu(w_n)! < \infty$. It is straightforward to verify that C is a non-commutative Banach algebra under $\|\cdot\|$.

Let $x \in C$, $k \in \mathbb{P}$. Then

$$\|x^k\| \leq \sum_{n_j} \frac{|\alpha_{n_1}| \ldots |\alpha_{n_k}|}{\nu(w_{n_1} \ldots w_{n_k})!}$$

$$= \sum_{n_j} \frac{\nu(w_{n_1})! \ldots\ldots \nu(w_{n_k})!}{(\nu(w_{n_1}) + \ldots + \nu(w_{n_k}))!} \frac{|\alpha_{n_1}| \ldots\ldots |\alpha_{n_k}|}{\nu(w_{n_1})! \ldots\ldots \nu(w_{n_k})!}$$

$$\leq (1/k!) \|x\|^k.$$

Thus $r(x) = 0$. It is clear that C has no divisors of zero. We have proved the following result.

<u>Theorem 6</u> There exists a non-commutative radical Banach algebra which has no divisors of zero.

<u>Remark</u> Hirschfeld and Rolewicz, [14], have constructed a class of Banach algebras without divisors of zero. In fact, given a commutative Banach algebra with no divisors of zero, they construct an associated non-commutative Banach algebra which has the same property.

We now consider conditions which force commutativity. Our first result concerns a condition on the ideals in the algebra.

Theorem 7 If A is a complex normed algebra such that $Ax = xA$ for each x in A then $A/rad(A)$ is commutative.

Proof: Note first that every left or right ideal of A is in fact a bi-ideal.

Any quotient of A by a bi-ideal satisfies the given condition so we may suppose that A is semi-simple. It is therefore enough to show that every irreducible representation of A is one-dimensional.

Suppose P is a primitive ideal of A. Then $B = A/P$ is a primitive normed algebra. For any modular ideal M of B we have $M = (M:B)$ - the quotient of M in B. Since B is primitive there is a maximal modular ideal M such that $(M:B) = 0$. Hence (0) is a maximal modular ideal of B. Thus B is a division algebra so, by Mazur's theorem, B is one-dimensional. It follows that every irreducible representation of A is one-dimensional.

Theorem 2.10 gives a sufficient algebraic condition for a Banach algebra with unit to be commutative - namely that A satisfy the strong regularity condition, $Ax^2 = Ax \quad (x \in A)$.

In [24], Le Page gives a variety of conditions (including strong regularity) which force Banach algebras with a unit element to be commutative. Most of these are conditions on the norm structure of the Banach algebra. For example,

(1) $\| x^2 \| = \| x \|^2$ for each $x \in A$

(2) $\| ab \| = \alpha \| ba \|$ for some $\alpha > 0$, each $a, b \in A$

A slight variation of the proof which Le Page gives for (1) yields the following sufficient condition for commutativity.

(1)' $r(x) \geq k \| x \|$ for some $k > 0$, each $x \in A$

(See for example [4], Theorem 4.10)

The next theorem which is due to Kaplansky (see [5], P.58) gives a necessary and sufficient condition for commutativity in C*-algebras. Recall that a C*-algebra is just a closed self-adjoint subalgebra of $B(H)$ for some Hilbert space H.

Theorem 8 A C*-algebra is commutative if and only if 0 is its only nilpotent element.

The above theorem is obviously of interest also in connection with the question of existence for nilpotent elements which we discussed earlier.

Finally in this chapter we take a brief look at how the spectrum of an element of a Banach algebra behaves when we shrink or enlarge the algebra to which the element belongs.

Let A be a Banach algebra with unit and let B be a closed subalgebra of A with $1 \in B$. For $x \in B$ it is well-known that $Sp(A, x) \subseteq Sp(B, x)$, $\partial Sp(A, x) \supseteq \partial Sp(B, x)$. In particular, if $Sp(A, x)$ is finite then $Sp(A, x) = Sp(B, x)$. Zelazko, [32], has shown that for commutative A $Sp(A, x) = Sp(B, x)$ ($x \in B$, B any closed subalgebra of A) if and only if $Sp(A, x)$ is totally disconnected for each $x \in A$. In general, for non-commutative A, it is known (see for example [12], Theorem) that if $x \in B$ where B is a closed subalgebra of A then $Sp(A, x) = Sp(B, x)$ if and only if $Sp(A, x)$ fails to separate the plane. In the opposite direction we may ask if it is possible to remove the topological interior of $Sp(A, x)$ by considering $Sp(C, x)$ for some (sufficiently large) superalgebra C of A. The next example shows that this is not always possible and that in fact the worst possible case can occur. That is, we may not be able to remove any of the topological interior.

<u>Example 9</u> Let A be the Banach algebra of all bounded operators on l^2, and let s be the unilateral shift operator. Then

$$Sp(A, s) = \{\lambda \in C: |\lambda| \leq 1\}.$$

Since every singular element of A is a topological divisor of zero, $\lambda - s$ is a topological divisor of zero and hence permanently singular for each $\lambda \in Sp(A, s)$. (See [29], pps. 185 and 20.)

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