Preference Conditions for Linear Demand Functions

Theodoros M. Diasakos*

November 29, 2021

Abstract

The present study takes consumer preferences as the primitive and a most general formulation of a linear demand system as the desideratum. To investigate how the two are related, I take a novel approach to demand integrability that relies on some recent results in Diasakos and Gerasimou (2020). The methodology applies for either of the two possible price-normalization regimes (with respect to the price of a numeraire commodity or income); in either case, it leads to a complete characterization of linear demand systems in terms of the properties for the underlying rationalizing preference relation, and analytical solutions for the corresponding (direct) utility function. My results provide a proper microfoundation for linear demand systems - in a way that addresses knowledge gaps in the extant literature on linear demand that leave space for fundamental misunderstanding.

JEL Classifications: C02, D01, D11

Keywords: Linear demand; preference differentiability; strict convexity; strict monotonicity; law of demand.

*Economics Division, Stirling Management School, University of Stirling, U.K. (E-mail: theodoros.diasakos@stir.ac.uk) An earlier version was presented at a session of the 20th annual SAET conference; I wish to thank the participants for helpful comments.

1 Introduction

Linear demand functions have been used extensively in the economics literature as a convenient modelling tool to showcase important properties of market systems. The reliance on linear demand has been long standing in the modern theory of industrial organization (see Amir et al. (2017) but also Kopel et al. (2017) for insightful overviews) and important in the empirical estimation of consumer demand (see, for instance, Deaton (1974b)-(1978) but also Deaton (1974a) for aggregate demand) as well as of labour supply (see Stern (1984) for an overview). Linear demand functions are also ubiquitous in microeconomic textbooks for the purposes of demonstrating various properties of consumer or market demand. Given these observations, it is somewhat surprising that incomplete progress has been made with respect to a proper characterization of the underlying preferences which can rationalize linear demand systems.

It is well known that linear demand is not easily generated by rational preferences or market structures. With respect to preferences, the existing literature has looked at the problem from the classical perspective on demand integrability: the (Marshallian) demand function of interest is assumed to satisfy enough regularity conditions (e.g., being sufficiently smooth, satisfying the Law of Demand, being injective, or its Slutsky matrix being symmetric and negative semidefinite) for the corresponding system of partial differential equations to be solved by an appropriate expenditure function, which can lead then to a utility function via duality (see, for instance, Houthakker (1960), Epstein (1981), or Jackson (1986); see also Epstein (1982), LaFrance (1990) as well as Nocke and Schutz (2017) for incomplete demand systems). In this spirit, LaFrance (1985) established that individual linear demand places strong restrictions on the underlying preference: it requires a quadratic or Leontief quasi-direct utility function. In a similar spirit, Alperovich and Weksler (1996) solve for the underlying (direct) utility function for the two-commodities case with income-normalized prices. With respect to market structures, Jaffe and Weyl (2010) have shown that aggregate linear demand cannot result from (sufficiently smooth) rational discrete-choice models. More recently, Amir et al. (2017) investigated the required properties for a quasi-linear/quadratic utility function to generate a linear demand function satisfying the Law of Demand; as it turns out, these properties have important implications for some widely used theoretical frameworks in industrial organization.

Their important contributions notwithstanding, these studies fail to assign unambiguously the key desirable properties of linear demand to the requisite characteristics of rational choice. As a result, they fall short from actually characterizing the microfoundations of linear demand - an important desideratum as linear demand models are deployed mainly to obtain basic economic intuition to facilitate predictions and policy making (see, for instance, Berry and Haile (2021) for a discussion on the advantages of preference-based demand estimation). As it will become apparent in what follows, our current knowledge on linear demand functions is incomplete, leaving space for fundamental misunderstanding.

In contrast to the existing literature, the present study takes consumer preferences as the primitive and a most general formulation of a linear demand system as the desideratum. To analyse how the two are related, I take a novel approach to demand integrability that relies on some recent results in Diasakos and Gerasimou (2020). They refer to a (weak) notion of smooth preferences which admits geometric interpretation via the concept of a preference gradient and the associated notion of preference differentiability. Diasakos and Gerasimou (2020) establish that this notion of smooth preferences is fundamentally linked to the invertibility of the resulting demand function. The present study begins by showing that this notion of smooth preferences provides also theoretical underpinnings for a ubiquitous (albeit hitherto axiomatic) assumption in the literature on the microfoundations of linear demand; namely, that we refer to incomplete demand systems. More precisely, there are $k \in \mathbb{N} \setminus \{0\}$ commodities whose demand levels are observed, but also $m + 1$ commodities ($m \in \mathbb{N}$) with unobserved demands. The observed demand function is linear (i.e., exhibits constant coefficients) with respect to the prices of the k commodities.

Another feature that sets the present approach apart from the extant literature is that it applies for either of the two possible price-normalization regimes (with respect to the price of a numeraire commodity or income). I proceed to establish that, under either price-normalization regime, a linear demand function is generated by a differentiable preference relation if and only if (i) the unobserved part of the demand system comprises but *one* commodity (i.e., $m = 0$) while (ii) the matrix of constant coefficients on the prices of the observed commodities is non-singular (see Theorems 1 and 3 below). Combining preference differentiability with properties (i)-(ii) facilitates a straightforward integrability exercise via the preference gradient (the inverse demand function). This leads to analytical solutions for the underlying (direct) utility function (see Theorems 2 and 4 below).

Somewhat unexpectedly perhaps, when prices are normalized with respect to a numeraire, the combination of preference differentiability and properties (i)-(ii) above dictates that the linear demand *cannot* be function of income. As to be expected, on the other hand, the corresponding utility function is of the quasi-linear/quadratic form. Given this utility formulation, well-known arguments can be deployed to show that the Slutsky matrix of the total demand system must be symmetric and negative semi-definite. And as the linear demand varies only with the prices of the observed commodities, it follows that the (non-singular) matrix of constant coefficients itself must be symmetric and negative definite; hence, that the linear demand must obey also the strict Law of Demand.

This translates into important messages with respect to the quest for microfoundations of linear demand systems (such as multi-variate linear demand functions for differentiated products in oligopolistic markets). On the one hand, linear demand systems that do not satisfy the (strict) Law of Demand or are income dependent are *not* rationalizable by preferences that are smooth even in the least sense. On the other hand, linear demand systems that are income independent and satisfy the (strict) Law of Demand are fully consistent with continuous, strictly monotonic, strictly convex, and weakly smooth rationalizing preferences. Yet we should always keep in mind not only that these demand systems are incomplete, but more importantly that their unobserved part plays an integral role for the underlying preference relation: it depicts a single (numeraire) commodity whose quantity demanded is completely determined by what remains from income once the observed expenditure has been funded.

The next section introduces the notational and theoretical backdrop for our approach. Section 3 presents the analysis itself along with the underlying intuition. In Section 4, we compare our results with those in the relevant literature and discuss their implications for microfounding linear demand functions. The supporting results whose proofs are too long to be included in the main text can be found in the Appendix (Section A).

2 The theoretical framework

As our consumption set we consider an open and convex $X \subseteq \mathbb{R}^n_{++}$ where $n \in \mathbb{N} : n \geq 2$. The consumer's preferences are captured by a continuous weak order $\succsim$ on X (i.e. by a complete and transitive binary relation whose graph is a closed subset of $X \times X$). For $A \subseteq X$, we let

$$\max_{\succsim} A := \{x \in A : x \succsim y \text{ for all } y \in A\}$$

denote the set of all $\succsim$-greatest elements in A. Given some set $Y \subseteq \mathbb{R}^n_{++}$ of income-normalized strictly positive prices, the budget correspondence $B : Y \twoheadrightarrow X$ is defined by[1]

$$B(p) := \{x \in X : px \leq 1\}$$

We will say that $\succsim$ *generates* the demand correspondence $\xi : Y \twoheadrightarrow X$ if the latter is defined by

$$\xi(p) := \max_{\succsim} B(p)$$

We will refer to such a demand correspondence as *rational*. A rational demand correspondence is *onto* if, for all $x \in X$ there exists $p \in Y$ such that $x \in \xi(p)$. If $\xi(\cdot)$ is single-valued (hence a demand *function*), it is said to be *injective* if for all $p, p' \in Y$, $p \neq p'$ implies $\xi(p) \neq \xi(p')$. A demand function $\xi : Y \to X$ that is both injective and onto is *invertible*. If $\xi(\cdot)$ has this property, then the inverse demand given by

$$p(x) := \{p \in Y : \ x = \xi(p)\}$$

is itself a well-defined bijective function $p : X \to Y$.

Proposition 1 in Diasakos and Gerasimou (2020) establishes that, within the realm of continuous preferences, a rational demand function $\xi : Y \to X$ requires that the generating preference relation $\succsim$ is strictly convex and strictly monotone on X.[2] Their analysis proceeds to show that, within the realm of strictly convex, strictly monotone and continuous preferences, the generated demand function is invertible (in fact, an homeomorphism) if and only if the underlying preference relation satisfies a particular notion of smoothness, *weak smoothness*.

The first notion of smooth preferences in the literature was proposed in Debreu (1972), where a preference relation $\succsim$ on a consumption set X was defined to be *smooth of order r* (C^r for short) if the graph of the indifference relation (i.e., the set $\{(x, y) \in X \times X : x \sim y\} \subset X \times X$) is a C^r-manifold on $X \times X$.[3] It turns out that a monotone preference relation on X is C^r if and only if it is representable by a C^r (i.e., r-times continuously differentiable) utility function. Generalizing Debreu's notion, Neilson (1991) defined a preference relation on X as *weakly smooth of order r* if each of its *indifference sets* ($\mathcal{I}_x := \{z \in X : z \sim x\}$,

[1] Throughout the paper, for any $x, y \in \mathbb{R}^k$ and $1 < k \leq n$ the dot-product $p^\intercal x$ will be denoted simply by px.

[2] As usual, the preferences are said to be *convex* if, for all $x, y \in X$ and any $\alpha \in [0, 1]$, $x \succsim y$ implies $\alpha x + (1 - \alpha)y \succsim y$, and *monotonic* if $x \gg y$ implies $x \succ y$. They are *strictly convex* if, for all $x, y \in X$ and $\alpha \in (0, 1)$, $x \succsim y$ implies $\alpha x + (1 - \alpha)y \succ y$, and *strictly monotonic* if $x > y$ implies $x \succ y$.

[3] Let $A \subseteq \mathbb{R}^n$. A function $f : A \to \mathbb{R}^n$ is an *homeomorphism* if it is injective, continuous, and its inverse function is continuous on $f(A)$. Letting A be in addition open, a C^r function $f : A \to \mathbb{R}^n$ is a C^r *diffeomorphism* if it is an homeomorphism with a C^r inverse function. A set $M \subseteq \mathbb{R}^n$ is a C^r *k-dimensional* ($k \leq n$) *manifold* if for every $x \in M$ there is a C^r diffeomorphism $f : A \to \mathbb{R}^n$ ($A \subseteq \mathbb{R}^n$ open) which carries the open set $A \cap (\mathbb{R}^k \times \{\mathbf{0}^{n-k}\})$ onto an open neighborhood of x in M. For more details and some economic-theoretic examples, see Chapter 1.H in Mas-Colell (1985).

$x \in X$) is a C^r-manifold on X. In Diasakos and Gerasimou (2020), a preference relation that is weakly smooth of order 1 is referred to simply as *weakly smooth*.

More recently Rubinstein (2006) defined the preference relation $\succsim$ on X to be *differentiable* if for every $x \in X$ there exists $p_x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

$$\{z \in \mathbb{R}^n : p_x \cdot z > 0\} = \{z \in \mathbb{R}^n : \text{ there exists } \lambda_z^* > 0 \text{ such that } x + \lambda z \succ x \text{ for all } \lambda \in (0, \lambda_z^*)\} \quad (1)$$

To interpret this geometrically, for distinct bundles x and z in X, call z an *improvement direction at x* if there exists $\lambda^* > 0$ such that $x + \lambda z \succ x$ for all $\lambda \in (0, \lambda^*)$, assuming $(x + \lambda z) \in X$. In light of this definition, the right hand side of (1) defines the set of all improvement directions at x. The left hand side of (1) on the other hand defines the set of all directions z that are evaluated as strictly positive by some vector p_x (which depends on x). Preference differentiability of $\succsim$ at x requires the existence of a vector p_x such that the set of all directions that receive strictly positive valuations under p_x coincide with the set of all improvement directions of $\succsim$. Such a vector p_x will be referred to as a *preference gradient* at x.[4]

To relate these notions to the present investigation, for (arbitrary) $x \in X$ consider the projection of $\mathcal{I}_x$ along the (arbitrary) ith dimension of $\mathbb{R}^n_+$,

$$\mathcal{I}_x^i := \{z_i \in \mathbb{R}_+ : \text{ there exists } z_{-i} \in \mathbb{R}_+^{n-1} \text{ such that } z \in \mathcal{I}_x\},$$

and define the set

$$\mathcal{I}_x^{-i} := \{z_{-i} \in \mathbb{R}_+^{n-1} : \text{ there exists } z_i \in \mathbb{R}_+ \text{ such that } z \in \mathcal{I}_x\}$$

analogously, as the projection of $\mathcal{I}_x$ on $\mathbb{R}_+^{n-1}$ (the resulting subspace when the ith dimension is removed from $\mathbb{R}^n_+$). We can construct then the *indifference-projection correspondence* $l_i(\cdot|x) : \mathcal{I}_x^{-i} \twoheadrightarrow \mathcal{I}_x^i$ for good i by requiring

$$z_i \in l_i(z_{-i}|x) \iff z \in \mathcal{I}_x$$

whose graph is the indifference set $\mathcal{I}_x$. As established in Diasakos and Gerasimou (2020), for $\succsim$ continuous, strictly convex and strictly monotonic, the mapping $l_i(\cdot|x)$ is a locally convex and thus also continuous function. As a result, its local subdifferential $\partial l_i(z_{-i}|x)$, which comprises the collection of the function's local subgradients at z_{-i}, is non-empty and fundamentally linked to its smoothness: $l_i(\cdot|x)$ is differentiable at z_{-i} if and only if $\partial l_i(z_{-i}|x)$ is a singleton, in which case the unique local subgradient coincides with the gradient.

With regard to economic interpretation, when $l_i(\cdot|x)$ is differentiable at z_{-i} the jth entry $\partial l_i(z_{-i}|x)/\partial z_j$ of the gradient $\nabla l_i(z_{-i}|x)$ defines the marginal rate of substitution of good i for good $j \neq i$. Indeed, if $\succsim$ is representable by a utility function $u : X \to \mathbb{R}$ that is continuously differentiable at z, we have

$$\frac{\partial l_i\left(z_{-i}|x\right)}{\partial z_j} = -\frac{\dfrac{\partial u(z)}{\partial z_j}}{\dfrac{\partial u(z)}{\partial z_i}} \quad (2)$$

[4] An intuitive interpretation for the entries of p_x is that they represent the consumer's "subjective values" of the different goods relative to the reference bundle x: *"Starting from x, any small move in a direction that is evaluated by this vector as positive is an improvement"* (Rubinstein, 2006 p. 71). It is also noteworthy that the notion of preference gradient can also be viewed as a generalization of the notion of *valuation equilibrium* in Radner (1993).

The right-hand side of this equation depicts the textbook definition of the marginal rate of substitution of good i for good j. The definition rests upon invoking the Implicit Function Theorem; thus, upon assuming that $u(\cdot)$ is a C^1 function (equivalently, that $\succsim$ is itself C^1). By contrast, the left-hand side of (2) exists and is continuous in a more general environment: when $\succsim$ is differentiable - see Proposition 2 in Diasakos and Gerasimou (2020). And given that it is continuous, strictly convex and strictly monotonic, $\succsim$ being differentiable is equivalent to $\succsim$ being weakly smooth - see Theorem 1 in Diasakos and Gerasimou (2020).

More importantly for our purposes, $\succsim$ being differentiable is equivalent to $\succsim$ generating a unique, homeomorphic demand function $\xi : Y \to X$ with Y an open subset of R^n_{++} - see Proposition 3 in Diasakos and Gerasimou (2020). Specifically, letting $q_{-i}(x)$ denote the negative of the gradient $l_i(\cdot|x)$ at x, the preference gradient p_x coincides with $p(x)$, the value of the inverse demand at this bundle. Formally, we have

$$\begin{aligned} q_{-i}(x) &:= -\nabla l_i(x_{-i}|x) && (3)\\ q_i(x) &= \frac{1}{x_i + q_{-i}(x)\cdot x_{-i}} && (4)\\ p(x) &= q_i(x)\big(1, q_{-i}(x)\big) && (5) \end{aligned}$$

where $q_{-i}(x) \in \mathbb{R}^{n-1}_{++}$, $q_i(x) > 0$, and $p(x) \in \mathbb{R}^n_{++}$. Notice finally that, although taking distinct index goods i and j in the above system leads to distinct vectors $(q_i(x), q_{-i}(x))$ and $(q_j(x), q_{-j}(x))$, the preference gradient, $p(x)$, is invariant with respect to the choice of the index good. Moreover, that $q_i(x) = p_i(x)$ for the index good i is due to the fact that we normalize prices with respect to income.

3 Linear demand

The preceding overview of the key theoretical concepts was given in terms of prices that are normalized with respect to income. Yet most of the literature on linear demand concerns itself with the case where prices are normalized instead with respect to a numeraire commodity.

3.1 When prices are normalized with respect to a numeraire

Taking the nth commodity as the numeraire, we can deploy (5) above to define the functions $w : Y_n \to \mathbb{R}_{++}$ and $q_{-n} : Y \to \mathbb{R}^{n-1}_{++}$, respectively, by $w(p_n) := 1/p_n$ and $q_{-n}(p) = p_{-n}/p_n$. We then have a mapping between the income-normalized prices $p \in Y$ from the preceding section and the corresponding vector of numeraire-normalized prices and income, $(q_{-n}, w) \in Q \times W$ - where $q_n := q_{-n}(p)$ while $W := w(Y_n)$ and $Q := q_{-n}(Y)$. This mapping gives also the numeraire-normalized (i.e., Marshallian) demand $\widetilde{\xi} : Q \times W \to X$ as $\widetilde{\xi}(q_{-n}, w) := \xi((1, q_{-n}(p)/w)$. It is trivial moreover to check that, since $w(\cdot)$ is an homeomorphism, if $\succsim$ is continuous, strictly convex, strictly monotonic and differentiable on X then $\widetilde{\xi}(\cdot)$ is itself an homeomoprhism and thus $Q \times W$ is open in $\mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++}$.

We will restrict attention to demand functions $\widetilde{\xi} : Q \times W \to X$ that satisfy both of the following conditions.

(A) The domain $Q \times W$ has non-empty interior: $\exists (q, \varepsilon) \in (Q \times W) \times \mathbb{R}_{++}$ such that $\mathcal{B}_q(\varepsilon) \subset Q \times W$.[5]

[5]For $y \in \mathbb{R}^n$ and $\varepsilon > 0$, $\mathcal{B}_\varepsilon(y)$ denotes the open ball in $\mathbb{R}^n$ with center y and radius ε.

(B) For at least one of the non-numeraire commodities its quantity demanded responds to a change in its own relative price, other things being equal:

$$\exists\, (j, q, \delta) \in \{1, \ldots, n-1\} \times (Q \times W) \times \mathbb{R} \setminus \{0\} : \; q + \delta e_j \in Q \times W \;\wedge\; \widetilde{\xi}_j\left(q + \delta e_j\right) \neq \widetilde{\xi}_j\left(q\right) \quad (6)$$

Together conditions (A)-(B) above provide the theoretical underpinnings (see Claim 2 and Remark (ii) in Section A) for a key assumption in the literature on linear demand: namely, that the observed linear demand system is incomplete. Specifically, linear demand models always assume that, for some $k \in \mathbb{N}$: $1 \leq k < n$, the demands of the commodities indexed by $M := \{k+1, \ldots, n\}$ are unobserved. The observed linear functional form depicts the demands of the commodities indexed by $K := \{1, \ldots, k\}$; their demand exhibits constant coefficients with respect to the prices $q_1, \cdots, q_k$.

In what follows we depict the unobserved demands by the vector $\mathbf{z} \in X_M$ and the observed ones by $x \in X_K$. Moreover, for $M_0 := M \setminus \{n\}$ we denote the respective relative prices by $q_{M_0} \in Q_{M_0}$ and $q_K \in Q_K$.[6] Letting then $x(\cdot)$ denote the observed components of $\widetilde{\xi}(\cdot)$, a linear demand system is given by

$$x\left(q_K, q_{M_0}, w\right) := \alpha\left(q_{M_0}, w\right) + Bq_K \quad (7)$$

where B is a $k \times k$ matrix of constants while $a : Q_{M_0} \times W \to R^k$ is a continuous function.

As we have already pointed out, a theoretical justification for the formulation in (7) is given by the assumption that the total demand system $\widetilde{\xi}(\cdot)$ satisfies conditions (A)-(B) above simultaneously. With respect to (B), given condition (A), it suffices for (6) that the matrix B has a non-zero diagonal element or a symmetric principal minor (see Remarks (iii)-(iv) in Section A). With respect to condition (A), it suffices that the domain for the relative prices and income is open. A theoretical justification for the latter requirement is given by the discussion in the opening paragraph of this section. For if the demand system is generated by a continuous, strictly monotonic, strictly convex, and differentiable preference relation then $Q \times W$ is necessarily open.

In fact, differentiability of the underlying preference relation places additional restrictions not only on the formulation for the observed linear demand but also on the total commodity system itself.

Theorem 1 *Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X which generates the observed demand function in (7). The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). B is non-singular, $M_0 = \varnothing$, and $\alpha(\cdot)$ is a constant.

Proof. That (i) $\Rightarrow$ (ii) follows from Lemmas A.1, A.2, and A.3 (see Section A). To show that (ii) $\Rightarrow$ (i), observe first that, by Theorem 1 in Diasakos and Gerasimou (2020), $\succsim$ is differentiable if the total demand $\widetilde{\xi}(\cdot)$ is injective. To see that the latter property does hold under the hypotheses in (ii), let $\alpha(\cdot) := \alpha$ and take $(q^1, w_1), (q'', w_2) \in Q_K \times W$ with $(q', w_1) \neq (q'', w_2)$. There are two cases to consider. If $q' \neq q''$, we cannot have $\alpha + Bq' = x(q') = x(q'') = \alpha + Bq''$ given that B is non-singular; clearly, we must have

[6] Our analysis will use the following notation. Consider the index set $\mathcal{N} := \{1, \ldots, n\}$ and let $\mathcal{A} \subset \mathcal{N}$. For $y \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$, we let $y_{\mathcal{A}}$ and $S_{\mathcal{A}}$ denote, respectively, the projections of y and S on the subspace that results from $\mathbb{R}^n$ when the dimensions in $\mathcal{N} \setminus \mathcal{A}$ are removed.

$\widetilde{\xi}(q', w_1) \neq \widetilde{\xi}(q'', w_2)$. If, on the other hand, $q' = q = q''$ and $w_1 \neq w_2$, notice that $\widetilde{\xi}(q, w_1) = (z, x(q)) = \widetilde{\xi}(q, w_2)$ only if $w_1 - qx(q) = z = w_2 - qx(q)$; i.e., only if $w_1 = w_2$. ■

In terms of the analytical arguments to establish Theorem 1, the involved ones refer to the "only if" direction. Lemma A.1 shows that $\succsim$ is differentiable only if B is non-singular; it does so by an argument ad absurdum which can be outlined intuitively as follows. If B is singular, there must exist some $v \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ such that $Bv = \mathbf{0}$; hence, such that $x(q^0_{M_0}, q^0_K + \lambda v, w_0) = x(q^0_{M_0}, q^0_K, w_0)$ for some $(q^0_{M_0}, q^0_K, w_0) \in Q \times W$ and $\lambda \in \mathbb{R} \setminus \{0\}$ sufficiently small. It is straightforward to show that this leads to a violation of the Weak Axiom of Revealed Preference (WARP) when $vx(q^0_{M_0}, q^0_K, w_0) = 0$ or $vx(q^0_{M_0}, q^0_K, w_0) = vx(q^0_{M_0}, q^0_K, w')$ for some $w' \in W$ with $w' \neq w_0$. For the case where $vx(\widetilde{q}_{-i}, q, w) \neq vx(\widetilde{q}_{-i}, q, w')$ for all $(\widetilde{q}_{-i}, q) \in Q$ and all $w, w' \in W$ with $w' \neq w$, we fix the unobserved part of the demand at the bundle $\mathbf{z}^0 := z(q^0_{M_0}, q^0_K, w_0)$ and restrict attention to the relationship between the n-dimensional price-income space $Q \times W$ and the k-dimensional space of observed demand bundles $\{(\mathbf{z}^0, x), x \in X_K\}$. The latter space is open in $\mathbb{R}^k$ (recall that X is open in $\mathbb{R}^n$), and thus can be covered by a collection of hyperplanes $\{x \in X_K : vx = \rho, \rho \in L\}$ from some interval $L \subseteq \mathbb{R}$. Letting $x^0 := x(q^0_{M_0}, q^0_K, w_0)$, we show that the hyperplane $\{x \in X_K : vx = vx^0\}$ embeds the set $X_{(\mathbf{z}^0, x^0)} := \{x \in X_K : x = x(q^0_{M_0}, q_K, w_0), q_K \in V_{q^0_K}\}$ for some neighbourhood $V_{q^0_K}$ of q^0_K in Q_K. But this is absurd given that $\succsim$ is differentiable only if the demand system is an homeomorphism. For, on the one hand, being the image of $V_{q^0_K}$ under an homeomorphic demand, $X_{(\mathbf{z}^0, x^0)}$ must be open in $\mathbb{R}^k$. Yet, on the other hand, it must also lie within a hyperplane in $\mathbb{R}^k$.

Given this result, Lemma A.2 establishes that $\succsim$ is differentiable only if the set of commodities whose demands are unobserved is a singleton. To do so we exploit the fact that preference differentiability allows for direct demand integrability along the indifference sets of $\succsim$ via the preference gradient function, $q_K(\cdot)$ - recall equation (3) above. Under the functional form in (7) and as B is invertible, the integrability exercise leads to a quasi-indirect utility function which is quasi-linear in the unobserved demands. To complete the argument we show that the linear part of the utility function cannot admit a multi-dimensional consumption vector.

Finally, Lemma A.3 shows that $\succsim$ is differentiable only if the function $\alpha(\cdot)$ does not vary with income - its only possible argument as M_0, and thus also Q_{M_0}, must be empty (Lemma A.2)). The argument here is once again ad absurdum and exploits the functional form in (7) in conjunction with the fact that B is invertible. Dropping now the subscript K from our notation, we fix again the unobserved part of the demand at the level $z^0 := z(q^0, w_0)$ and restrict attention to the relationship between the $(k+1)$-dimensional price-income space $Q_K \times W$ and the k-dimensional space of observed demand bundles $\{(z^0, x), x \in X_K\}$. A contradiction obtains now by considering the set $X_{(z^0, x^0)} := \{x \in X_K : x := x(q, w_0), (q, w) \in V_{(q^0, w_0)}\}$ for some neighbourhood $V_{(q^0, w_0)}$ of (q^0, w_0) in $Q_K \times W$. For as $V_{(q^0, w_0)}$ is open in $\mathbb{R}^{k+1}$, so should be its image under the homeomorphic demand. Yet the latter is contained in $X_{(\mathbf{z}^0, x^0)} \subset \mathbb{R}^k$.

In light of these results, under preference differentiability the expression in (7) above reduces to the following

$$x(q) := \alpha + Bq, \qquad q \in Q_K \tag{8}$$

where $\alpha \in \mathbb{R}^k$ is a constant and B is non-singular, while $M = \{n\}$. Preference differentiability means also that this formulation allows for integrability of the preference gradient function to trace out the indifference sets analytically. This leads to a complete characterization of the linear demand function in terms of the generating preference relation.

Theorem 2 *Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X which generates the observed demand function in (8). The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). B is non-singular.
(iii). $\succsim$ is represented by the utility function $u : X \to \mathbb{R}$ given by

$$u\left(z, x\right) := z - xB^{-1}\alpha + xB^{-1}x/2 \tag{9}$$

(iv). B is symmetric and negative definite.
(v). $x(\cdot)$ satisfies the strict Law of Demand:

$$(q' - q'')(x(q') - x(q'')) < 0 \qquad \forall q', q'' \in Q_K : \; q' \neq q''$$

Proof. That (i) $\Leftrightarrow$ (ii) is due to Theorem 1 while (iv) $\Rightarrow$ (v) holds trivially. Observe also that, Q_K being open, (v) necessitates that B is non-singular.[7] We only have to show thus that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
(ii) $\Rightarrow$ (iii). Let B be non-singular (and, thus, $\succsim$ be differentiable). We can write then $q = B^{-1}(x - \alpha)$ where, by the very choice of normalization, we have $q = -\nabla_x l_z(x|(z,x))$ - recall equation (3) above. For any given $(z^0, x^0) \in X$, therefore, we must obey the system of differential equations

$$\partial z/\partial x_i = \left(B^{-1}\left(\alpha - x\right)\right)_i, \qquad i = 1, \ldots, k \tag{10}$$

along the indifference curve $\mathcal{I}_{(z^0,x^0)}$. Integrating along this curve gives

$$z = xB^{-1}\alpha - xB^{-1}x/2 + c, \qquad (z,x) \in \mathcal{I}_{(z^0,x^0)}$$

where c remains constant along $\mathcal{I}_{(z^0,x^0)}$. The claim follows by setting $u(z,x) := c$.
(iii) $\Rightarrow$ (iv). The utility maximization problem (UMP) for the objective in (9) results in the inverse demand $q(x) = B^{-1}(\alpha - x)$. Observe also that, being represented by the C^1 utility function in (9), $\succsim$ is itself C^1 and thus differentiable. As a result, by Proposition 2 in Diasakos and Gerasimou (2020), $q(\cdot)$ must be injective; hence, B^{-1} must be non-singular. It follows then that the total demand is $\widetilde{\xi}(\cdot) := (z(\cdot), x(\cdot))$ where $x(\cdot)$ is given by (8) while $z(\cdot)$ is given by $z(q,w) := w - qx(q)$. It is trivial to check now that $\widetilde{\xi}(\cdot)$ satisfies the hypotheses of Theorem 1 in Hurwicz and Uzawa (1971). As a result, the Slutsky matrix of $\widetilde{\xi}(\cdot)$ must be symmetric and negative semidefinite. And as $x(\cdot)$ is given by the specification in (8), the kth principal minor of the Slutsky matrix for $\widetilde{\xi}(\cdot)$ - i.e., the Slutsky matrix for $x(\cdot)$ - coincides with B. Therefore, B must be symmetric and negative semidefinite; more precisely, symmetric and negative definite given that it is also non-singular.[8] ■

Remark. Within the realm of Theorem 2, the requirement that $\succsim$ be monotonic imposes the following restriction on its domain:[9]

$$X \subseteq \mathbb{R}_{++} \times \left\{x \in \mathbb{R}^{n-1}_{++} : B^{-1}\left(\alpha - x\right) \ll 0\right\}$$

[7] Let $q' \in Q_K$. If B is singular, there must exist some $v \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ such that $Bv = \mathbf{0}$; hence, such that $x(q' + \lambda v) = x(q')$ for any $\lambda \in \mathbb{R} \setminus \{0\}$ sufficiently small to give $q' + \lambda v \in Q_K$. Clearly, (v) above cannot hold if B is singular.

[8] Recall that a symmetric (square) matrix is positive semidefinite [resp. positive definite] if and only if all of its eigenvalues are nonnegative [resp. strictly positive], while a (square) matrix is non-singular if and only if all of its eigenvalues are non-zero.

[9] For two vectors $x, y \in \mathbb{R}^n$, we write $x \ll y$ whenever $x_i < y_i$ for all $i \in \{1, \ldots, n\}$.

3.2 When prices are normalized with respect to income

Returning to the case where the prices are normalized with respect to income, we will restrict attention to demand functions $\xi : Y \to X$ that satisfy both of the following conditions.

(A*) The domain Y has non-empty interior: $\exists(p, \varepsilon) \in Y \times \mathbb{R}_{++}$ such that $\mathcal{B}_p(\varepsilon) \subset Y$.

(B*) For at least one of the commodities its quantity demanded responds to a change in its own relative price, other things being equal:

$$\exists\,(i, p, \delta) \in \{1, \ldots, n\} \times Y \times \mathbb{R} \setminus \{0\} : \; p + \delta e_i \in Y \;\wedge\; \xi_i\,(p + \delta e_i) \neq \xi_i\,(p) \tag{11}$$

Similarly to the preceding case, conditions (A*)-(B*) provide the theoretical underpinnings for the key assumption in the literature that the observed linear demand system is incomplete (see Claim 1 and Remark (iii) in Section A). Letting again the demands of the commodities indexed by M be unobserved and indexing the commodities with observed demands by K, the respective relative prices will be depicted now by $p_M \in Y_M$ and $p_K \in Y_K$. The observed linear demand system is given by

$$x\,(p_M, p_K) := \alpha\,(p_M) + Bp_K \qquad (p_M, p_K) \in Y \tag{12}$$

where B is a $k \times k$ matrix of constants while $a : Y_M \to R^k$ is a continuous function.

A theoretical justification for the formulation in (12) is given by the assumption that the total demand system $\xi(\cdot)$ satisfies conditions (A*)-(B*) above simultaneously. Regarding condition (B*), given condition (A*), it suffices for (11) that the matrix B has a non-zero diagonal element or a symmetric principal minor (see Remarks (iii)-(iv) in Section A). With respect to condition (A*), on the other hand, it suffices that the domain for the relative prices is open. A theoretical justification for this assumption is given by the discussion in Section 2: if the demand system is generated by a strictly monotonic, strictly convex and differentiable preference relation then Y is necessarily open.

And as before, differentiability of the underlying preference relation places additional restrictions not only on the formulation for the observed linear demand but also on the total demand system itself.

Theorem 3 *Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X which generates the observed demand function in (12). The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). B is non-singular and $M = \{n\}$.

Proof. That (i) $\Rightarrow$ (ii) follows from Corollaries A.1 and A.2 (see Section A). The argument to show that (ii) $\Rightarrow$ (i) is trivially similar to the respective part in the proof of Theorem 1. ■

Given these results, under preference differentiability and assuming that $\alpha(\cdot)$ is also a linear function, the expression in (12) above reduces to the following

$$x\,(p_n, p_K) := \alpha + \gamma p_n + Bp_K, \qquad (p_n, p_K) \in Y \tag{13}$$

where $\alpha, \gamma \in \mathbb{R}^k$ are constants while $M = \{n\}$. In light of Theorem 3, preference differentiability allows for direct integrability of the formulation in (13) along the indifference sets via the preference gradient function. As before, this leads to a complete characterization of the linear demand function in terms of the properties of the underlying generating preference.

Theorem 4 *Let $\succsim$ be a continuous, strictly convex, and strictly monotonic weak order on X which generates the demand function in (13). The following are equivalent.*
(i). $\succsim$ is differentiable.
(ii). B is non-singular.
(iii). $\succsim$ is represented by the utility function $u : X \to \mathbb{R}$ given by

$$u(z,x) := \left(z + xB^{-1}\gamma\right) \exp\left(-\int \frac{xB^{-1}(x-\alpha)}{1 - xB^{-1}(x-\alpha)} dx\right) \tag{14}$$

Proof. That (i) $\Leftrightarrow$ (ii) is due to Theorem 3. Moreover, since $u(\cdot)$ is C^1 so must be $\succsim$. Hence, the preference is weakly smooth and that (iii) $\Rightarrow$ (i) is due to Proposition 2 in Diasakos and Gerasimou (2020). It remains to show that (ii) $\Rightarrow$ (iii).
(ii) $\Rightarrow$ (iii). Let B be non-singular (and, thus, $\succsim$ be differentiable). Recall also equations (3)-(5). We have $p = p_n q_K$ with $q_K = -\nabla_x l_z(x|(z,x))$ and $p_n = (z + q_K x)^{-1}$. The given demand schedule can be written therefore as follows

$$x = \alpha + p_n (Bq_K + \gamma)$$

or equivalently

$$\begin{aligned} q_K = B^{-1}\left(p_n^{-1}(x-\alpha) - \gamma\right) &= B^{-1}\left((z + q_K x)(x-\alpha) - \gamma\right) \\ &= B^{-1}(x-\alpha) q_K x + B^{-1}(z(x-\alpha) - \gamma) \end{aligned}$$

This implies though that

$$\left(1 - x^\intercal B^{-1}(x-\alpha)\right) q_K x - x^\intercal B^{-1}(x-\alpha) z = -x^\intercal B^{-1}\gamma$$

For any given $(z^0, x^0) \in X$, therefore, we must obey the differential equation

$$(1 - f(x)) \sum_{j=1}^{k} x_j \partial z / \partial x_j - f(x) z = -g(x)\gamma \tag{15}$$

along the indifference curve $\mathcal{I}_{(z^0,x^0)}$, and where $f(x) := x^\intercal B^{-1}(x-\alpha)$ while $g(x) := x^\intercal B^{-1}$.
To integrate now along the indifference curve, take any $j \in K$ and consider the parametrization $x_j := e^\tau h_j(s)$ with $(\tau, s) \in T \times S$ for some functions $h_j : S \to \mathbb{R}_{++}$ and where $T \subset \mathbb{R}$ while $S \subset \mathbb{R}^{k-1}$. Since $x_j = \partial x_j / \partial \tau$, the PDE in (15) can be transformed to the following ODE

$$(1 - f(x(\tau,s)))\, \mathrm{d}z/\mathrm{d}\tau - f(x(\tau,s))\, z = -g(x(\tau,s))\,\gamma$$

Restricting attention to the set $\left\{(z,x) \in X : x^\intercal B^{-1}(x-\alpha) \neq 1\right\}$, we can re-write this as[10]

$$\mathrm{d}z/\mathrm{d}\tau - \frac{f(x(\tau,s))\, z}{1 - f(x(\tau,s))} = -\frac{g(x(\tau,s))\,\gamma}{1 - f(x(\tau,s))}$$

[10]On the set $\left\{(z,x) \in X : x^\intercal B^{-1}(x-\alpha) = 1\right\}$ (15) gives $z = x^\intercal B^{-1}\gamma$. Our utility representation is such that $u(z,x) = 0$ along the indifference curve $\mathcal{I}_0 := \left\{(z,x) \in X : z = x^\intercal B^{-1}\gamma \wedge x^\intercal B^{-1}(x-\alpha) = 1\right\}$.

whose solution is straightforward:

$$z = \frac{1}{\mu\left(x\left(\tau, s\right)\right)}\left(c - \int \frac{\mu\left(x\left(\tau, s\right)\right) g\left(x\left(\tau, s\right)\right)\gamma}{1 - f\left(x\left(\tau, s\right)\right)} \mathrm{d}\tau\right) \tag{16}$$

where

$$\mu\left(x\left(\tau, s\right)\right) := \exp\left(-\int \frac{f\left(x\left(\tau, s\right)\right)}{1 - f\left(x\left(\tau, s\right)\right)} \mathrm{d}\tau\right)$$

while s is a vector of free parameters. Fix now these parameters and consider a change in τ. We have

$$\begin{aligned}
\Delta\left(e^{\tau}\mu\left(x\left(\tau, s\right)\right)\right) &= \mu\left(x\left(\tau, s\right)\right)\Delta e^{\tau} + e^{\tau}\Delta\mu\left(x\left(\tau, s\right)\right) \\
&= e^{\tau}\left(1 - \frac{f\left(x\left(\tau, s\right)\right)}{1 - f\left(x\left(\tau, s\right)\right)}\right)\mu\left(x\left(\tau, s\right)\right)\Delta\tau = \frac{e^{\tau}\mu\left(x\left(\tau, s\right)\right)}{1 - f\left(x\left(\tau, s\right)\right)}\Delta\tau
\end{aligned}$$

and thus

$$\begin{aligned}
\Delta\left(\mu\left(x\left(\tau, s\right)\right) g\left(x\left(\tau, s\right)\right)\right) &= \Delta\left(\mu\left(x\left(\tau, s\right)\right) x\left(\tau, s\right)^{\intercal} B^{-1}\right) \\
&= \Delta\left(\mu\left(x\left(\tau, s\right)\right)\sum_{j=1}^{k} B_j^{-1} e^{\tau} h_j\left(s\right)\right) \\
&= \Delta\left(e^{\tau}\mu\left(x\left(\tau, s\right)\right)\right)\sum_{j=1}^{k} B_j^{-1} h_j\left(s\right) \\
&= \frac{e^{\tau}\mu\left(x\left(\tau, s\right)\right)}{1 - f\left(x\left(\tau, s\right)\right)}\sum_{j=1}^{k} B_j^{-1} h_j\left(s\right)\Delta\tau \\
&= \frac{\mu\left(x\left(\tau, s\right)\right)}{1 - f\left(x\left(\tau, s\right)\right)}\sum_{j=1}^{k} B_j^{-1} x_j\left(\tau, s\right)\Delta\tau \\
&= \frac{\mu\left(x\left(\tau, s\right)\right)}{1 - f\left(x\left(\tau, s\right)\right)} B^{-1} x\left(\tau, s\right)\Delta\tau = \frac{\mu\left(x\left(\tau, s\right)\right) g\left(x\left(\tau, s\right)\right)}{1 - f\left(x\left(\tau, s\right)\right)}\Delta\tau
\end{aligned}$$

That is,

$$\int \frac{\mu\left(x\left(\tau, s\right)\right) g\left(x\left(\tau, s\right)\right)}{1 - f\left(x\left(\tau, s\right)\right)} \mathrm{d}\tau = \mu\left(x\left(\tau, s\right)\right) g\left(x\left(\tau, s\right)\right)$$

and (16) reads

$$z = \left(c - \mu\left(x\left(\tau, s\right)\right) g\left(x\left(\tau, s\right)\right)\gamma\right) / \mu\left(x\left(\tau, s\right)\right)$$

Clearly, we have that

$$\left(z + \mu\left(x\right) g\left(x\right)\gamma\right) = c, \qquad \left(z, x\right) \in \mathcal{I}_{\left(z^0, x^0\right)}$$

and the claim follows by setting $u\left(z^0, x^0\right) := c$. ■

Remark. Within the realm of Theorem 4, the requirement that $\succsim$ be monotonic imposes the following restriction on its domain:

$$X \subseteq \left\{\left(z, x\right) \in \mathbb{R}^n_{++} : B^{-1}\left(\frac{z + x^{\intercal}B^{-1}\gamma}{\left(1 - x^{\intercal}B^{-1}\left(x - \alpha\right)\right)^2}\left(2x - \alpha\right) - \gamma\right) \ll 0\right\}$$

To conclude our analysis, a comparison between the statements of Theorem 1 and Theorem 3 but also of Theorem 2 and Theorem 4 is noteworthy. Theorems 1 and 3 both establish that $\succsim$ being differentiable is fundamentally related to B being non-singular and M being singleton. There is a key feature in the proofs of Lemmas A.1-A.2 that renders Theorem 3 for its most part a corollary of Theorem 1. Namely, whenever we appeal to the linearity of the demand formulation in (7), we do so while holding income constant. We can do this also in the realm of the demand formulation in (12): $x(\cdot)$ is linear in q_K for a given p_n. Yet, in contrast to Theorem 1, Theorem 3 does not claim that $\alpha(\cdot)$ must be a constant. In the proof of Lemma A.3 we use that the linear part of $x(\cdot)$ in (7) is *independent* of income.[11] This property does not obtain under the formulation in (12).

The single discrepancy in the statements of Theorems 1 and 3 accounts in turn for the difference in scope between Theorems 2 and 4. The very fact that $\alpha(\cdot)$ is constant in (8) ensures that the kth principal minor of the Slutsky matrix for $\widetilde{\xi}(\cdot)$ coincides with B. Being also non-singular, the latter must be symmetric and negative definite; as a result, $x(\cdot)$ must also obey the strict Law of Demand. Needless to say, the formulation in (12) does not allow for an immediate mapping between the Slutsky matrix for $\xi(\cdot)$ and B.

As a final remark, notice that α, γ, and B in (13) are all scalars when $k = 1$. (15) reads then

$$(x(x-\alpha)-\beta)\,\mathrm{d}z/\mathrm{d}x-(x-\alpha)\,z=\gamma \tag{17}$$

For a closed-form solution for $u(\cdot)$ in this case, see Alperovich and Weksler (1996).[12]

4 Discussion and related literature

To compare the present analysis with the pertinent literature, we should note first that the studies most relevant for Section 3.1 above are those in LaFrance (1985) and Amir et al. (2017), while for Section 3.2 is that in Alperovich and Weksler (1996). LaFrance (1985) examines the demand formulation in (7) distinguishing between two cases: whether or not $a(\cdot)$ is function of income. For the case where $a(\cdot)$ is independent of income, he takes B to be symmetric and negative semidefinite and establishes that the underlying utility function must be quasi-linear/quadratic. The latter is a quasi-direct utility, conditional upon the relative prices of the unobserved commodities. For the case where $a(\cdot)$ does vary with income, LaFrance (1985) shows that the conditional (upon the relative prices of the unobserved commodities as well as income in this case) utility function must be Leontief. Amir et al. (2017) take the set of unobserved commodities to be a singleton and the demand formulation to be given by (8). They show that this can be generated by the utility function in (9) if B^{-1} is a symmetric, negative definite matrix with non-zero diagonal entries. Finally, Alperovich and Weksler (1996) investigate the demand formulation in (13) for the case where $n = 2$; they obtain a closed-form solution for the utility function in (14) for this special case.

As we established in the opening paragraphs of Sections 3.1-3.2, restricting attention to the incomplete demand systems in (7)-(8) or (12)-(13) can be justified by the conjunction of conditions (A)-(B) or

[11] More precisely, as the linear part of $x(\cdot)$ in (7) is separated from the part that varies with income, we can ensure that $\epsilon(\cdot)$ as defined by (32) below varies only with λ, not with q_K.

[12] Equation (17) is equivalent to equation (5) in Alperovich and Weksler (1996) - once a typo in their equations (3)-(5) has been corrected.

(A*)-(B*), respectively, for the complete demand system. With respect to conditions (A) and (A*), it suffices that the domains $Q \times W$ and Y, respectively, are open - an assumption to be found in all three studies above. Regarding conditions (B) and (B*), given conditions (A) and (A*), respectively, it suffices for either of (6) and (11) that the matrix B has a non-zero diagonal element or a symmetric principal minor (see Remarks (iii)-(iv) in the Appendix).[13] The former restriction is assumed in Amir et al. (2017), where the diagonal elements of B are all non-zero, but also in Alperovich and Weksler (1996) where B is a non-zero scalar. The latter restriction can be found in LaFrance (1985) where B itself is symmetric.

With respect to the restrictions placed on the matrix B, under the formulation in (8), the matrix being symmetric and negative semidefinite can be justified by assuming that the Slutsky matrix of the complete demand system itself is symmetric and negative semidefinite. Yet our analysis facilitates a direct connection with the underlying rationalizing preference. By our Theorem 1, as long as the preference relation is (at least weakly) smooth, B must also be non-singular; hence, B being negative semidefinite is equivalent to B being negative definite. Equally importantly, the possibility of more than one unobserved commodities in LaFrance (1985) is a vacuous generalization while his argument for the case where $a(\cdot)$ does vary with income should be read as ad absurdum, if the demand system is generated by (at least weakly) smooth preferences - to conclude that the latter must be Leontief is absurd. As for the analysis in Amir et al. (2017), Theorem 1 provides underpinnings for the theoretical framework itself. Their starting point is a continuously differentiable utility function; hence, a utility representation for preferences that are weakly smooth. A linear demand system generated by such preferences can only have the form in (8).

Our analysis in Section 3.1 relates also to the study in Nocke and Schutz (2017) who investigate the integrability of demand systems of the form $x(q)$ - i.e., the observed demands are independent of income - that satisfy the Law of Demand and for which there exists a function $v(\cdot)$ such that $\nabla_q v(q) = -x(q)$. Nocke and Schutz (2017) establish the existence of a rationalizing objective function, $z + \phi_x(q)$, where the convex function $\phi_x(q) := \inf_{q>>0}\{qx + v(q)\}$ is minimized at q_x: $x = x(q_x)$. It should be noted though that their objective lends itself to a direct utility function if and only if $x(\cdot)$ is invertible; for then we can define the inverse demand function $q(\cdot)$ and proceed to get the direct utility function $U(z + x) = z + \phi_x(q(x))$. When $x(\cdot)$ is in particular linear, the demand system investigated in Nocke and Schutz (2017) coincides with that in (8). In this case, $x(\cdot)$ is invertible if and only if B is non-singular (Theorem 1). And as the latter property requires that B is also symmetric (Theorem 2), we get that $v(q) = -\alpha q - q^\intercal Bq/2$ while $\phi_x(q) = q_x^\intercal x - q_x^\intercal Bq_x/2$ with q: $x = \alpha + Bq$. Moreover, B being in addition non-singular, we have $q_x = B^{-1}(x - \alpha)$ and $U(z, x)$ takes the form in (9) .

Our analysis bears also implications on the quest for microfoundations of demand estimation. Theorems 1-2 place strong restrictions on the functional form of quadratic utility the applied economist may appeal to. For instance, the form $(x - \alpha)A(x - \alpha)$ - see Deaton (1978) - is valid only if the $(n-1)$-th principal minor of A is symmetric and negative definite while $A_{nn} = 0 = A_{jn} + A_{nj}$ for $j = 1, \cdots, n-1$. Similarly, an additive utility function - see Houthakker (1960) - is consistent with linear demand only if it is of the form $u(z, x) = z + \sum_{j\in K}(\alpha_j x_j + b_j x_j^2)$ while for all $j = 1, \cdots, n-1$ we have $b_j = 0$ if $\alpha_j = 0$; the

[13]Remarks (iii)-(iv) in the Appendix highlight sufficient conditions for the hypothesis (C*) in Claim 2. The intuition for the latter result is that the gradient of the complete demand system with respect to the vector of relative prices cannot be constant everywhere. Our intuition agrees with that in Jaffe and Weyl (2010) which shows that the complete demand system cannot be linear under discrete choice.

constant-coefficients matrix of the corresponding linear demand is diagonal.

Most importantly perhaps, our results shed new light on the quest for microfoundations of linear demand systems in the context of applications in theoretical industrial organization. Amir et al. (2017) suggest that models of multi-variate linear demand functions for differentiated products ought to be regarded with some suspicion when the demand functions in question do not satisfy the Law of Demand. Their tone is understandably cautious given their key hypothesis of an underlying strictly concave quasi-linear/quadratic utility function. By contrast, as it offers a complete characterization of linear demand functions in terms of the underlying rationalizing preferences, the present analysis leads to far more commanding conclusions. Multi-variate linear demand functions for differentiated products that do not satisfy the (strict) Law of Demand or are income dependent are *not* rationalizable - at least not by rational preferences smooth enough to allow for tractable utility functions.

In the more specific context of duopoly, Bos and Vermeulen (2020) investigate whether a linear demand system can be rationalized by a representative agent.[14] They restrict attention to the quadratic part of the utility function in (9) keeping their analysis agnostic on the linear part. This leads to the conclusion that linear demand is inconsistent with strictly monotonic and strictly convex rationalizing preferences.[15] The present study paints a very different picture: linear demand systems are fully consistent with continuous, strictly monotonic, strictly convex, and weakly smooth rationalizing preferences. However, this microfoundation of linear demand cannot be agnostic about the unobserved part of the demand system. For it implies that linear demand systems depict general, not partial, equilibrium analysis.

The unobserved part of a rationalizable linear demand system cannot be ignored by appealing to the usual "other things being equal" assumption, nor by assuming some (sufficiently high) unspent income in the background. For the (representative) agent this is an integral part of her preference relation - one depicting a single (numeraire) commodity whose quantity demanded is determined by what remains from her income once her observed expenditure has been funded. If we want to interpret the unobserved part as some basket of goods and services (i.e, a Hicksian composite commodity), we have to accept that there are no substitution or income effects within this basket. The agent does not care about the composition of the basket; she cares only about its aggregate size. If we want to impose a ceteris paribus assumption on her (unobserved) income, we have to accept that her (equally unobserved) consumption of the numeraire commodity must vary along the observed demand hyperplane (modulo the zero-measure subset where observed expenditure remains constant).

References

ALPEROVICH, G. AND I. WEKSLER (1996): "A Class of Utility Functions Yielding Linear Demand Functions," *The American Economist*, 40, 20–23.

AMIR, R., P. ERICKSON, AND J. JIN (2017): "On the Microeconomic Foundations of Linear Demand for Differentiated Products," *Journal of Economic Theory*, 169, 641–665.

[14]See Kopel et al. (2017) for a detailed overview of the literature on duopoly models with linear demand.

[15]For instance, investigating the case $k = 2$, Bos and Vermeulen (2020) find that the "indifference curves'" of the quadratic part are elliptic violating the principle of diminishing MRS.

BERRY, S. T. AND P. A. HAILE (2021): "Foundations of Demand Estimation," *NBER Working Paper Series*, 29305.

BOS, I. AND D. VERMEULEN (2020): "On the Microfoundations of Linear Oligopoly Demand," *The B.E. Journal of Theoretical Economics*, 20190114.

DEATON, A. S. (1974a): "The Analysis of Consumer Demand in the United Kingdom, 1900-1970," *Econometrica*, 42, 341–367.

——— (1974b): "A Reconsideration of the Empirical Implications of Additive Preferences," *The Economic Journal*, 84, 338–348.

——— (1978): "Specification and Testing in Applied Demand Analysis," *The Economic Journal*, 88, 524–536.

DEBREU, G. (1972): "Smooth Preferences," *Econometrica*, 40, 603–615.

DIASAKOS, T. M. AND G. GERASIMOU (2020): "Preference Conditions for Invertible Demand Functions," *American Economic Journal: Microeconomics (accepted)*.

EPSTEIN, L. G. (1981): "Generalized Duality and Integrability," *Econometrica*, 49, 655–678.

——— (1982): "Integrability of Incomplete Systems of Demand Functions," *The Review of Economic Studies*, 49, 411–425.

HOUTHAKKER, H. S. (1960): "Additive Preferences," *Econometrica*, 28, 244–257.

HURWICZ, L. AND H. UZAWA (1971): "On the Integrability of Demand Functions," in *Preferences, Utility and Demand*, ed. by J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, New York: Harcourt Brace Jovanovich, 114–148.

JACKSON, M. O. (1986): "Integration of Demand and Continuous Utility Functions," *The Journal of Economic Theory*, 38, 298–312.

JAFFE, S. AND E. G. WEYL (2010): "Linear Demand Systems are Inconsistent with Discrete Choice," *The B.E. Journal of Theoretical Economics*, 10, Article 52.

KOPEL, M., A. RESSI, AND L. LAMBERTINI (2017): "Capturing Direct and Cross Price Effects in a Differentiated Products Duopoly Model," *The Manchester School*, 85, 282–294.

LAFRANCE, J. (1985): "Linear Demand Functions in Theory and Practice," *Journal of Economic Theory*, 37, 147–166.

——— (1990): "Incomplete Demand Systems and Semilogarithmic Denand Models," *Australian Journal of Agricultural Economics*, 34, 118–131.

MAS-COLELL, A. (1985): *The Theory of General Equilibrium: A Differentiable Approach*, Econometric Society Monographs, Cambridge University Press.

NEILSON, W. S. (1991): "Smooth Indifference Sets," *Journal of Mathematical Economics*, 20, 181–197.

NOCKE, V. AND N. SCHUTZ (2017): "Quasi-linear integrability," *Journal of Economic Theory*, 169, 603–628.

RADNER, R. (1993): "A note on the theory of cost-benefit analysis in the small," in *Capital, Investment and Development: Essays in Memory of Sukhamoy Chakravarty*, ed. by K. Basu, M. Majumdar, and T. Mitra, Blackwell, 129–141.

RUBINSTEIN, A. (2006): *Lecture Notes in Microeconomic Theory: The Economic Agent*, Princeton: Princeton University Press.

STERN, N. (1984): "On the specification of labour supply functions," in *Unemployment, search, and labour supply*, ed. by R. Blundell and I. Walker, Cambridge University Press, chap. 9.

A Supporting results

To make the exposition in this section less cumbersome, for $y \in \mathbb{R}^n$ and $i \in \mathcal{N}$ we will use the notation y_i and y_{-i} in lieu of $y_{\{i\}}$ and $y_{\mathcal{N}\setminus\{i\}}$, respectively. That is, y_i and y_{-i} will denote, respectively, the projections of y on the ith dimension of $\mathbb{R}^n$ and on the subspace that results from $\mathbb{R}^n$ when the ith dimension is removed. Taking also $j \in \mathcal{N} \setminus \{i\}$, we will use the notation $y_{-(i,j)}$ in lieu of $y_{\mathcal{N}\setminus\{i,j\}}$; i.e., $y_{-(i,j)}$ will denote the projection of y on the subspace that results from $\mathbb{R}^n$ when both the ith and jth dimensions are removed. Finally, as usual, $||y||$ denotes the Euclidean norm of y while e_i denotes the vector in $\mathbb{R}^n$ with 1 as its ith entry and zeroes everywhere else.

Claim 1 *Let the demand system $\xi : Y \to X$ be given by $\xi(p) := \alpha + Ap$, where α and A are, respectively, a constant n-dimensional real vector and an $n \times n$ real matrix. Suppose also that $\xi(\cdot)$ satisfies Walras' law. Then at least one of the following conditions*

(A) $\exists(p,\varepsilon) \in Y \times \mathbb{R}_{++}$ *such that* $\mathcal{B}_p(\varepsilon) \subset Y$

(C) $\exists\epsilon \in \mathbb{R}^n \setminus \{0\}$ *such that* $\epsilon A \epsilon \neq 0$,

cannot hold.

Proof. To establish the claim arguing ad absurdum, suppose that both conditions hold simultaneously. Letting $(p,\varepsilon) \in Y \times \mathbb{R}_{++}$ be as in (A) and $\epsilon \in \mathbb{R}^n \setminus \{0\}$ be as in (C), take $\lambda \in (0,1)$ sufficiently small so that $\lambda||\epsilon|| \leq 1$ and define the $(-\varepsilon,\varepsilon) \to Y$ function $p(\delta) := p + \delta\lambda\epsilon$. By Walras' law we ought to have

$$\begin{aligned} p\alpha + pAp = p\xi(p) = 1 = p(\delta)\,\xi(p(\delta)) &= p(\delta)\,\alpha + p(\delta)\,Ap(\delta) \\ &= p\alpha + pAp + \delta\lambda\epsilon\alpha + \delta\lambda\epsilon Ap + \delta^2\lambda^2\epsilon A\epsilon + \delta\lambda pA\epsilon \end{aligned}$$

As this implies in turn that

$$\delta = -\frac{\epsilon\alpha + \epsilon Ap + pA\epsilon}{\lambda\epsilon A\epsilon} \qquad \forall\delta \in (-\varepsilon,0) \cup (0,\varepsilon) \tag{18}$$

the desired contradiction obtains immediately. ■

Claim 2 *Let the demand system $\widetilde{\xi} : Q \times W \to X$ be given by $\widetilde{\xi}(q) := \alpha + Aq$, where α and A are, respectively, a constant n-dimensional real vector and an $n \times n$ real matrix. Suppose also that $\widetilde{\xi}(\cdot)$ satisfies Walra's law. Then at least one of the following conditions,*

(A)* $\exists (q, \varepsilon) \in Q \times W \times \mathbb{R}_{++}$ *such that* $\mathcal{A}_q(\varepsilon) \subset Q \times W$

(C)* $\exists \epsilon \in \mathbb{R}^n \setminus \{0\}$ *such that* $\epsilon_{-i} A_{-i} \epsilon \neq 0$ *- where* A_{-i} *denotes the* $(n-1) \times n$ *matrix that results from* A *when its ith row is removed, -*

cannot hold.

Proof. Letting $(q, \varepsilon) \in Y \times \mathbb{R}_{++}$ be as in (A*) and $\epsilon \in \mathbb{R}^n \setminus \{0\}$ be as in (C*), take $\lambda \in (0, 1)$ sufficiently small so that $\lambda ||\epsilon|| \leq 1$ and define the $(-\varepsilon, \varepsilon) \to Y$ function $q(\delta) := q + \delta\lambda\epsilon$. Using Walras' law again we now have

$$
\begin{aligned}
\delta\lambda &= w + \delta\lambda - w \\
&= \xi_i (q + \delta\lambda\epsilon) + (q_{-i} + \delta\lambda\epsilon_{-i}) \xi_{-i} (q + \delta\lambda\epsilon) - (\xi_i (q) + q_{-i}\xi_{-i} (q)) \\
&= \xi_i (q + \delta\lambda\epsilon) - \xi_i (q) + q_{-i} (\xi_{-i} (q + \delta\lambda\epsilon) - \xi_{-i} (q)) + \delta\lambda\epsilon_{-i}\xi_{-i} (q + \delta\lambda\epsilon) \\
&= \delta\lambda \left(A_i^r \epsilon + q_{-i} A_{-i}\epsilon + \epsilon_{-i}\alpha_{-i} + \epsilon_{-i} A_{-i} q + \delta\lambda\epsilon_{-i} A_{-i}\epsilon\right)
\end{aligned}
$$

where A_i^r denotes the ith row of A. As the last equality above means that

$$
\delta = \frac{1 - \left(A_i^r \epsilon + q_{-i} A_{-i}\epsilon + \epsilon_{-i}\alpha_{-i} + \epsilon_{-i} A_{-i} q\right)}{\lambda\epsilon_{-i} A_{-i}\epsilon} \quad \forall \delta \in (-\varepsilon, 0) \cup (0, \varepsilon)
$$

the claim follows. ■

Remarks

(i). Notice that (11) in the main text is a sufficient condition for hypothesis (C) in Claim 1. To see this, suppose that condition (C) above does not hold. We have then $\epsilon A \epsilon = 0$ for all $\epsilon \in \mathcal{B}_{\mathbf{0}}(1)$. Letting $\epsilon := e_i/2$ we get that $A_{ii} = 0$ for all $i \in \{1, \ldots, n\}$. But then (11) cannot hold.

(ii) Similarly, (6) in the main text is a sufficient conditon for hypothesis (C*) in Claim 2. To see this, suppose that (C*) above does not hold. We have then $\epsilon_{-i} A \epsilon = 0$ for all $\epsilon \in \mathcal{B}_{\mathbf{0}}(1)$. Letting $\epsilon := e_j/2$ we get that $A_{jj} = 0$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. But then (6) cannot obtain.

(iii). Condition (11) [resp. (6)] in the main text is equivalent to the requirement that *one of the diagonal elements of A [resp. A_{-i}] is not zero.*

(iv). For hypothesis (C) [resp. (C*)] to hold, it suffices that *one of the principal minors of A [resp. A_{-i}] is symmetric*.

To see this for hypothesis (C), suppose again that $\epsilon A \epsilon = 0$ for all $\epsilon \in \mathcal{B}_{\mathbf{0}}(1)$. Letting now $\epsilon := e_i + e_j$ for arbitrary $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we get that $A_{ii} + A_{ij} + A_{ji} + A_{jj} = 0$; i.e., that $A_{ij} + A_{ji} = 0$ (for, as observed above, we also have $A_{ii} = 0 = A_{jj}$).

For hypothesis (C*), suppose again that $\epsilon_{-i} A_{-i} \epsilon = 0$ for all $\epsilon \in \mathcal{B}_{\mathbf{0}}(1)$. Letting now $\epsilon := e_j + e_k$ for arbitrary $j, k \in \{1, \ldots, n\} \setminus \{i\}$ with $j \neq k$, we get that $A_{jj} + A_{jk} + A_{kj} + A_{kk} = 0$; i.e., that $A_{jk} + A_{kj} = 0$ (for we also have $A_{jj} = 0 = A_{kk}$).

Lemma A.1 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\widetilde{\xi} : Q \to X$ whose projection on the dimensions in K, $x : Q_{M_0} \times Q_K \times W \to X_K$, is given by $x(q_{M_0}, q_K, w) := \alpha(q_{M_0}, w) - Bq_K$ for some function $a : Q_{M_0} \times W \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if B is non-singular.*

Proof. To establish the contrapositive statement, let B be singular; that is, let there be $v \in \mathbb{R}^n \setminus \{0\}$ such that $Bv = 0$. Take an arbitrary $x^0 \in X_K$. Since the complete demand system $\widetilde{\xi}(\cdot)$ generated by $\succsim$ is onto - see Proposition 1 in Diasakos and Gerasimou (2020) - there exists $\left(q^0_{M_0}, q^0_K, w_0\right) \in Q_{M_0} \times Q_K \times W$ such that $x^0 = x\left(q^0_{M_0}, q^0_K, w_0\right)$. To argue ad absurdum, suppose also that $\succsim$ is differentiable. As this implies that $Q_{M_0} \times Q_K \times W$ is open - see Theorem 1 in Diasakos and Gerasimou (2020) - we may take $\lambda_0 \in \mathbb{R}_{++}$ sufficiently small so that $\left(q^0_{M_0}, q^0_K + \lambda v, w_0\right) \in Q_{M_0} \times Q_K \times W$ for all $\lambda \in (-\lambda_0, \lambda_0)$. Define then the function $q : (-\lambda_0, \lambda_0) \to Q_K$ by $q(\lambda) := q^0_K + \lambda v$. This gives $x\left(q^0_{M_0}, q(\cdot), w0\right) = x^0$. Moreover, letting $z^0 := z\left(q^0_{M_0}, q^0_K, w_0\right)$, we have

$$z^0_n + q^0_{M_0} z^0 + q(\lambda) x^0 = w_0 + \left(q(\lambda) - q^0_K\right) x^0 = w_0 + \lambda v x^0 \tag{19}$$

and

$$z_n\left(q^0_{M_0}, q(\lambda)\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q(\lambda)\right) + q^0_K x\left(q^0_{M_0}, q(\lambda)\right) = w_0 + \left(q^0_K - q(\lambda)\right) x\left(q^0_{M_0}, q(\lambda)\right) = w_0 - \lambda v x^0 \tag{20}$$

If $vx^0 = 0$, the desired contradiction obtains immediately. For, on the one hand, the bundle $\left(z^0, x^0\right)$ is affordable at the price vector $\left(q^0_{M_0}, q(\lambda), w_0\right)$ while at the same time $\left(z\left(q^0_{M_0}, q(\lambda)\right), x\left(q^0_{M_0}, q(\lambda)\right)\right)$ is affordable at $\left(q^0_{M_0}, q^0_K, w_0\right)$. Yet, on the other hand, $\succsim$ is differentiable only if the demand system is injective - see again Theorem 1 in Diasakos and Gerasimou (2020). The two bundles being thus distinct, we have a violation of the WARP.[16]

Suppose next that $vx(q_{M_0}, q_K, w) \neq 0$ for all $(q_{M_0}, q_K, w) \in Q_{M_0} \times Q_K \times W$. We must consider the following cases.

Case I: There exists $w' \in W \setminus \{w_0\}$ such that $vx\left(q^0_{M_0}, q^0_K, w'\right) = vx^0$.

Observe first that, letting $\Delta\alpha\left(q^0_{M_0}, w_0\right) := \alpha\left(q^0_{M_0}, w'\right) - \alpha\left(q^0_{M_0}, w_0\right)$, we have

$$x\left(q^0_{M_0}, q^0_K, w'\right) - \Delta\alpha\left(q^0_{M_0}, w_0\right) = x\left(q^0_{M_0}, q^0_K, w_0\right) = x^0 = x\left(q^0_{M_0}, q^0_K + \lambda v, w_0\right) \tag{21}$$

Letting also $\Delta w := w' - w_0$ and $\lambda := -\Delta w / vx^0$ we get that

$$\begin{aligned} w_0 - \lambda v x^0 = w' &= z_n\left(q^0_{M_0}, q^0_K, w'\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q^0_K, w'\right) + q^0_K x\left(q^0_{M_0}, q^0_K, w'\right) \\ &= z_n\left(q^0_{M_0}, q^0_K, w'\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q^0_K, w'\right) + q^0_K\left(x^0 + \Delta\alpha\left(q^0_{M_0}, w_0\right)\right) \\ &= z_n\left(q^0_{M_0}, q^0_K, w'\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q^0_K, w'\right) + w_0 - z^0_n - q^0_{M_0} z^0 + q^0_K \Delta a\left(q^0_{M_0}, w_0\right) \end{aligned}$$

and thus

$$z_n\left(q^0_{M_0}, q^0_K, w'\right) + q^0_{M_0} z_{-n}\left(q^0_{M_0}, q^0_K, w'\right) = z^0_n + q^0_{M_0} z^0 - \lambda v x^0 - q^0_K \Delta a\left(q^0_{M_0}, w_0\right) \tag{22}$$

[16] The demand system $\widetilde{\xi}(\cdot)$ results from the maximization of the rational and strictly convex preference $\succsim$. It is well known that $\widetilde{\xi}(\cdot)$ must satisfy the Weak Axiom of Revealed Preference.

However, (21) and (22) together imply that

$$\begin{aligned}
& z_n\left(q_{M_0}^0, q_K^0, w'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, w'\right) + \left(q_K^0 + \lambda v\right) x\left(q_{M_0}^0, q_K^0, w'\right) \\
= \; & z_n^0 + q_{M_0}^0 z^0 - \lambda v x^0 - q_K^0 \Delta\alpha\left(q_{M_0}^0, w_0\right) + \left(q_K^0 + \lambda v\right)\left(x^0 + \Delta\alpha\left(q_{M_0}^0, w_0\right)\right) \\
= \; & z_n^0 + q_{M_0}^0 z^0 + q_K^0 x^0 + \lambda v \Delta\alpha\left(q_{M_0}^0, w_0\right) \\
= \; & w_0 + \lambda v\left(x\left(q_{M_0}^0, q_K^0, w'\right) - x\left(q_{M_0}^0, q_K^0, w_0\right)\right) = w_0
\end{aligned}$$

as well as that

$$\begin{aligned}
& z_n\left(q_{M_0}^0, q_K^0 + \lambda v, w_0\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0 + \lambda v, w_0\right) + q_K^0 x\left(q_{M_0}^0, q_K^0 + \lambda v, w_0\right) \\
= \; & w_0 + \left(q_K^0 - \left(q_K^0 + \lambda v\right)\right) x\left(q_{M_0}^0, q_K^0 + \lambda v, w_0\right) = w_0 - \lambda v x^0 = w'
\end{aligned}$$

another violation of the WARP.

Case II: $vx\left(q_{M_0}, q_K, w'\right) \neq vx\left(q_{M_0}, q_K, w\right)$ for all $\left(q_{M_0}, q_K, w\right), \left(q_{M_0}, q_K, w'\right) \in Q_{M_0} \times Q_K \times W$.
Consider the sets

$$\begin{aligned}
Q^0 &:= \left\{\left(q_{M_0}, q_K, w\right) \in Q_{M_0} \times Q_K \times W : \; z\left(q_{M_0}, q_K, w\right) = z^0\right\} \\
X_K^0 &:= \left\{(z, x) \in X : \; z = z^0\right\}
\end{aligned}$$

Since X is open in $\mathbb{R}_{++}^n$, the set X_K^0 is open in $\mathbb{R}_{++}^k$. Since the total demand is an homeomorphism so is its restriction $x : Y_{z^0} \to X_K^0$; hence, Y_{z^0} is also open in $\mathbb{R}_{++}^k$. Moreover, the hyperplane

$$X_K^* := \left\{x \in X_K^0 : \; vx = vx^0\right\}$$

being open in $\mathbb{R}_{++}^{k-1}$, so is the preimage

$$Q^* := \left\{\left(q_{M_0}, q_K, w\right) \in Q^0 : \; x\left(q_{M_0}, q_K, w\right) \in X_K^*\right\}$$

Observe now that, for any $x\left(q_{M_0}^1, q_K^1, w_1\right), x\left(q_{M_0}^2, q_K^2, w_2\right) \in X_K^*$, we have

$$\begin{aligned}
0 &= vx\left(q_{M_0}^1, q_K^1, w_1\right) - vx\left(q_{M_0}^2, q_K^2, w_2\right) \\
&= vx\left(q_{M_0}^1, q_K^1, w_1\right) - vx\left(q_{M_0}^2, q_K^2, w_1\right) + vx\left(q_{M_0}^2, q_K^2, w_1\right) - vx\left(q_{M_0}^2, q_K^2, w_2\right) \qquad (23)
\end{aligned}$$

As a result, $x\left(q_{M_0}^2, q_K^2, w_1\right) \in X_K^*$ renders the first difference on the right-hand side of (23) above zero, necessitating in turn that $vx\left(q_{M_0}^2, q_K^2, w_1\right) = vx\left(q_{M_0}^2, q_K^2, w_2\right)$. Yet the latter equality contradicts the very hypothesis that defines the case under consideration. Clearly, for any $w_1 \neq w_2$, we have $x\left(q_{M_0}^2, q_K^2, w_1\right) \notin X_K^*$ if $x\left(q_{M_0}^2, q_K^2, w_2\right) \in X_K^*$. Similarly, we have that

$$\begin{aligned}
0 &= vx\left(q_{M_0}^1, q_K^1, w_1\right) - vx\left(q_{M_0}^2, q_K^2, w_2\right) \\
&= vx\left(q_{M_0}^1, q_K^1, w_1\right) - vx\left(q_{M_0}^1, q_K^1, w_2\right) + vx\left(q_{M_0}^1, q_K^1, w_2\right) - vx\left(q_{M_0}^2, q_K^2, w_2\right) \qquad (24)
\end{aligned}$$

And as the first difference on the right-hand side of (24) cannot be zero, for any $\left(q_{M_0}^1, q_K^1\right) \neq \left(q_{M_0}^2, q_K^2\right)$, we must have $x\left(q_{M_0}^1, q_K^1, w_2\right) \notin X_K^*$ if $x\left(q_{M_0}^2, q_K^2, w_2\right) \in X_K^*$.
Let now Q_n^* and $Q_{M_0 \cup K}^*$ be, respectively, the projections of Q^* along the income and the remaining $n-1$ price dimensions. The preceding argument means that there must exist a bijection $f : Q_n^* \to Q_{M_0 \cup K}^*$ such

that any $x(q_{M_0}, q_K, w) \in X_K^*$ can be written as $x(f(w), w)$. However, since $Q^*_{M_0\cup K}$ is open in $\mathbb{R}^{k-2}_{++}$ while X_K^* is open in $\mathbb{R}^{k-1}_{++}$, this is absurd. For, on the one hand, the homeomorphism $f(\cdot)$ necessitates that $k-2=1$. On the other hand, the graph of $f(\cdot)$ being open in $\mathbb{R}_{++}$, the homeomorphism $x(\text{graph} f(\cdot))$ on X_K^* necessitates that $k-1=1$.

Given the preceding contradiction, we conclude that income remains constant (at w_0) along the hyperplane X_K^*. We will show now that, for any $q_K^1, q_K^2 \in Q_K$ with $q_K^1 \neq q_K^2$, we cannot have $vx\left(q^0_{M_0}, q^1_K, w_0\right) \neq vx\left(q^0_{M_0}, q^2_K, w_0\right)$. To argue by contradiction, let

$$vB\left(q_K^2 - q_K^1\right) = vx\left(q^0_{M_0}, q^2_K, w_0\right) - vx\left(q^0_{M_0}, q^1_K, w_0\right) \neq 0$$

Choose $(\kappa, w) \in (0,1) \times \mathcal{B}_{w_0}$ such that $v\alpha\left(\widetilde{q}^0, w\right) = v\alpha\left(\widetilde{q}^0, w_0\right) - \kappa vB\left(q_K^2 - q_K^1\right)$.[17] Letting $q_K^3 := \kappa q_K^2 + (1-\kappa) q_K^1$, we now have

$$\begin{aligned} v\left(x\left(q^0_{M_0}, q^1_K, w_0\right) - x\left(q^0_{M_0}, q^3_K, w\right)\right) &= v\left(B\left(q_K^1 - q_K^3\right) + \alpha\left(\widetilde{q}^0, w_0\right) - \alpha\left(\widetilde{q}^0, w\right)\right) \\ &= \kappa vB\left(q_K^1 - q_K^2\right) + v\left(\alpha\left(\widetilde{q}^0, w_0\right) - \alpha\left(\widetilde{q}^0, w\right)\right) = 0 \end{aligned}$$

This contradicts though that income remains constant along X_K^*.

Clearly, we have

$$\left\{x \in X_K^0 : \ x = x\left(q^0_{M_0}, q_K, w_0\right),\ q_K \in Q_K\right\} \subseteq X_K^* \tag{25}$$

Take now $\varepsilon \in \mathbb{R}_{++}$ sufficiently small so that $\mathcal{B}_{q_K^0}(\varepsilon) \subset Q_K^0$ - where Q_K^0 is the projection of Q^0 on Q_K. Consider also the budget sets

$$B(q_K) := \left\{x \in X : \ q_K x = w_0 - q^0_{M_0} z^0\right\}, \qquad q_K \in Q_K$$

The restriction of the preference relation $\succsim$ on $X \times \{z^0\}$ being strictly convex, strictly monotonic, and continuous, we obtain an homeomorphic demand function $\widetilde{x} : Q_K^* \to X$ where Q_K^* is an open subset of the set $\left(w_0 - q^0_{M_0} z^0\right)^{-1} Q_K$. Letting now $\kappa_1 = \min\left\{1, 1/\left(w_0 - q^0_{M_0} z^0\right)\right\}$ and comparing $\widetilde{x}(\cdot)$ with $x^0(\cdot) := x\left(q^0_{M_0}, \cdot, w_0\right)$ on $\mathcal{B}_{q_K^0}(\kappa_1\varepsilon)$ reveals the desired contradiction. For we must have $\left(z^0, x^0(q_K)\right) \succsim \left(z^0, \widetilde{x}(q_K)\right)$ everywhere on $\mathcal{B}_{q_K^0}(\kappa_1\varepsilon)$. Yet, $\widetilde{x}(\cdot)$ being an homeomorphism, the image set $\widetilde{x}\left(\mathcal{B}_{q_K^0}(\kappa_1\varepsilon)\right)$ is an open neighbourhood of x^0 in $\mathbb{R}^k_{++}$ while (25) necessitates that $x^0\left(\mathcal{B}_{q_K^0}(\kappa_1\varepsilon)\right) \subseteq X_K^*$, which is open in $\mathbb{R}^{k-1}_{++}$. The contradiction is due to the monotonicity of $\succsim$. ■

Corollary A.1 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\xi : Y \to X$ whose projection on the dimensions in K, $x : P_M \times P_K \to X_K$ given by $x(p_M, p_K) := \alpha(p_M) - Bp$ for some function $\alpha : P_M \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if B is non-singular.*

Proof. Recall how the two sets of normalized prices are related: $(p_M, p_K) = p_n((1, q_{M_0}), q_K)$ and $p_n = 1/w$. The argument in the preceding proof remains valid once we replace w_0, w', w, w_1, and

[17] By hypothesis, in this case, $\alpha\left(\widetilde{q}^0, \cdot\right)$ is a non-constant function; hence, by continuity, $v\alpha\left(\widetilde{q}^0, \cdot\right)$ is one-to-one on a sufficiently small neighbourhood of w_0. Observe also that, choosing κ sufficiently small, brings w arbitrarily close to w_0. The existence of w follows from the continuity of $\alpha\left(\widetilde{q}^0, \cdot\right)$.

w_2, respectively, by $1/p_n^0$, $1/p_n'$, $1/p_n$, $1/p_n^1$, and $1/p_n^2$.
A slight adjustment must be in Case I. Letting now $\Delta\alpha\left(q_{M_0}^0, p_n^0\right) := \alpha\left(q_{M_0}^0, p_n'\right) - \alpha\left(q_{M_0}^0, p_n^0\right)$, we have

$$x\left(q_{M_0}^0, q_K^0, p_n'\right) - \Delta\alpha\left(q_{M_0}^0, p_n^0\right) - \Delta p B q_K^0 = x\left(q_{M_0}^0, q_K^0, p_n^0\right) = x^0 = x\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) \tag{26}$$

where $\Delta p_n^0 := p_n' - p_n^0$ and $\lambda := \Delta p_n^0 / \left(p_n^0 p_n' v x^0\right)$. That is,

$$\begin{aligned}
1/p_n - \lambda v x^0 = 1/p_n' &= z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) + q_K^0 x\left(q_{M_0}^0, q_K^0, p_n'\right) \\
&= z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) + q_K^0\left(x^0 + \Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right) \\
&= z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) \\
&\quad + 1/p_n^0 - q_{M_0}^0 z^0 + q_K^0\left(\Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right)
\end{aligned}$$

and thus

$$z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p'\right) = z_n^0 + q_{M_0}^0 z_{-n}^0 - \lambda v x^0 - q_K^0\left(\Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right) \tag{27}$$

Now (26)-(27) imply that

$$\begin{aligned}
& z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) + \left(q_K^0 + \lambda v\right) x\left(q_{M_0}^0, q_K^0, p_n'\right) \\
= \; & z_n^0 + q_{M_0}^0 z_{-n}^0 - \lambda v x^0 - q_K^0\left(\Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right) \\
& + \left(q_K^0 + \lambda v\right)\left(x^0 + \Delta\alpha\left(q_{M_0}^0, p_n^0\right) + \Delta p B q_K^0\right) \\
= \; & z_n^0 + q_{M_0}^0 z_{-n}^0 + q_K^0 x^0 + \lambda v\left(\Delta a\left(q_{M_0}^0, p_n^0\right) + \Delta p_n^0 B q_K^0\right) \\
= \; & 1/p_n^0 + \lambda v\left(x\left(q_{M_0}^0, q_K^0, p_n'\right) - x^0\right) = 1/p_n^0
\end{aligned}$$

as well as

$$\begin{aligned}
& z_n\left(q_{M_0}^0, q_K^0\right) + q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) + q_K^0 x\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) \\
= \; & 1/p_n^0 + \left(q_K^0 - \left(q_K^0 + \lambda v\right)\right) x\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) \\
= \; & 1/p_n^0 - \lambda v x^0 = 1/p_n'
\end{aligned}$$

Yet

$$\begin{aligned}
& p_n^0 z_n\left(q_{M_0}^0, q_K^0, p_n'\right) + p_n^0 q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0, p_n'\right) + p_n^0\left(q_K^0 + \lambda v\right) x\left(q_{M_0}^0, q_K^0, p_n'\right) \\
= \; & 1 = p_n' z_n\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) + p_n' q_{M_0}^0 z_{-n}\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right) + p_n' q_K^0 x\left(q_{M_0}^0, q_K^0 + \lambda v, p_n^0\right)
\end{aligned}$$

is a violation of the WARP. ■

Lemma A.2 *Let the continuous, strictly convex and strictly monotonic weak order* $\succsim$ *on* X *generate the demand function* $\widetilde{\xi} : Q \to X$ *whose projection on the dimensions in* K, $x : Q_{M_0} \times Q_K \times W \to X_K$, *is given by* $x\left(q_{M_0}, q_K, w\right) := \alpha\left(q_{M_0}, w\right) - B q_K$ *for some function* $a : Q_{M_0} \times W \to \mathbb{R}^n$. *Then* $\succsim$ *is differentiable only if* $M_0 = \varnothing$.

Proof. To argue ad absurdum, let $j \in M_0 \neq \varnothing$. Recall first that $\succsim$ is differentiable at (z, x) if and only if the vector of relative prices $\left(q_{M_0}, q_K\right)$ is the unique subgradient of $l_n\left(\cdot \mid (z, x)\right)$ at (z_{-n}, x). Hence, $\succsim$ being differentiable, we have

$$l_n\left(\left(z_{-n}, x\right) \mid (z, x)\right) + q_{M_0} z_{-n} + q_K x \leq l_n\left(\left(\widetilde{z}_{-n}, \widetilde{x}\right) \mid (z, x)\right) + q_{M_0}\widetilde{z}_{-n} + q_K \widetilde{x}$$

for any $(\widetilde{z}_{-n}, \widetilde{x}) \in \mathcal{I}^{-i}_{(z,x)}$. And as $l_n(\cdot | (z,x))$ is differentiable everywhere along the latter set, this necessitates in fact that

$$0 = \nabla_{z_{-n}} l_n((z_{-n}, x) | (z,x)) + q_{M_0} \tag{28}$$
$$0 = \nabla_x l_n((z_{-n}, x) | (z,x)) + q_K \tag{29}$$

Take now $\left(q^0_{M_0}, q^0_K, w_0\right) \in Q_{M_0} \times Q_K \times W$, and let $z^0 := z\left(q^0_{M_0}, q^0_K, w_0\right)$ and $x^0 := x\left(q^0_{M_0}, q^0_K, w_0\right)$. Obviously, the system (28)-(29) must hold everywhere on the indifference set $\mathcal{I}_{(z^0,x^0)}$. Given this, if we restrict attention to relative price changes in the set $\left\{q^0_{M_0}\right\} \times Q_K$, we move along $\mathcal{I}_{(z^0,x^0)}$ as long as we obey the following system of partial differential equations

$$\begin{aligned} \partial z_n / \partial z_j &= -q^0_j, \qquad j \in M_0 \\ \partial z_n / \partial x_j &= -q_j(w, x) = \left(B^{-1}\left(\alpha\left(q^0_{M_0}, w\right) - x\right)\right)_j, \qquad j \in K \end{aligned}$$

where the last equality above uses the fact that B is non-singular (recall Lemma A.1). Integrating then along $\mathcal{I}_{(z^0,x^0)}$, we have

$$z_n = xB^{-1}\alpha\left(q^0_{M_0}, w\right)B^{-1} - xB^{-1}x/2 - q^0_{M_0} z_{-n} + c_0, \qquad (z,x) \in \mathcal{I}_{(z^0,x^0)}$$

where c_0 remains constant along $\mathcal{I}_{(z^0,x^0)}$. We can define thus a quasi-indirect utility function $v : X \times M \to \mathbb{R}$ by setting $v\left(z^0, x^0, q^0_{M_0}\right) := c_0$; that is, by letting

$$\begin{aligned} v(z, x, q_{M_0}, w) &:= -xB^{-1}\alpha(q_{M_0}, w)B^{-1} + xB^{-1}x/2 + q_{M_0} z_{-n} + z_n \\ &= -xB^{-1}x/2 + q(x, q_{M_0}, w)x + q_{M_0} z_{-n} + z_n \end{aligned}$$

Notice now that, as X is open in $\mathbb{R}^n_{++}$, taking $\varepsilon_0 > 0$ sufficiently small, the hyperplane

$$X^*_M := \left\{(z, x^0) \in X : \ z := \left(z^0_n + q^0_j \varepsilon, z^0_j - \varepsilon, z^0_{-(n,j)}\right), \ \varepsilon \in (0, \varepsilon_0)\right\}$$

lies in X and is open in $\mathbb{R}^{n-k-1}_{++}$. Consider also renormalizing the prices relative to income. As $p^0_n := 1/w_0$ and $p^0_j = q^0_j / w_0$, we get that

$$p^0_M z = p^0_M z^0 + \left(p^0_n q^0_j - p^0_j\right)\varepsilon = p^0_M z^0 \tag{30}$$

or equivalently $p^0_M z^0 + p^0_K x^0 = 1 = p^0_M z + p^0_K x^0$. Clearly, $(z^0, x^0) \succ (z, x^0)$ for any $(z, x^0) \in X^*_M$.
Observe next that the hyperplane

$$Y^*_{-(n,j)} := \left\{\left(p_{M_0 \setminus j}, p_K\right) \in Y_{M_0 \setminus \{j\}} \times Y_K : \ p_{M_0 \setminus j} z^0_{M_0 \setminus j} + p_K x^0 = 1 - \left(p^0_n z^0_n + p^0_j z^0_j\right)\right\}$$

is open in $\mathbb{R}^{n-3}_{++}$, and restrict the homeomorphic total demand $\xi(\cdot)$ to the domain $Y_j \times Y_n \times Y^*_{-(n,j)}$. The restriction itself being homeomorphic, the image set $\xi(Y_j \times Y_n \times Y^*_{-(n,j)})$ must be also open in $X_j \times X_n \times \mathbb{R}^{n-3}_{++}$. Moreover, since $(p^0_M, p^0_K) \in Y_j \times Y_n \times Y^*_{-(n,j)}$, $\xi(Y_j \times Y_n \times Y^*_{-(n,j)})$ must include a neighbourhood of (z^0, x^0) in $X_n \times X_j \times \mathbb{R}^{n-3}_{++}$. That is, $\xi(Y_j \times Y_n \times Y^*_{-(n,j)}) \cap X^*_M \neq \varnothing$.
Choosing, therefore, a sufficiently small $\varepsilon_1 \in (0, \varepsilon_0)$, we can find $(p^1_M, p^1_K) \in Y_j \times Y_n \times Y^*_{-(n,j)}$ such that $z^1 = z(p^1_M, p^1_K)$ and $x^0 = x(p^1_M, p^1_K)$ where $z^1 := \left(z^0_n + q^0_j \varepsilon_1, z^0_j - \varepsilon_1, z^0_{-(n,j)}\right)$. Now, since $(z^1, x^0) \in X^*_M$, we must have $(z^0, x^0) \succ (z^1, x^0)$. Taking $w_1 := 1/p^1_n$ and $q^1_j = w - 1p^1_j$, this necessitates that

$$0 < p^1_M\left(z^0 - z^1\right) + p^1_K\left(x^0 - x^0\right) = -\left(p^1_n q^0_j - p^1_j\right)\varepsilon_1 = -\left(q^0_j - q^1_j\right)\varepsilon_1 / w_1$$

i.e., that $q_j^1 > q_j^0$. Yet as we also have

$$\begin{aligned}
p_n^0 z_n^0 + p_j^0 z_j^0 = 1 - \left(p_{M\setminus\{j\}}^1 z_{-(n,j)}^0 + p_K^1 x^0\right) &= p_n^1 z_n^1 + p_j^1 z_j^1 \\
&= \left(z_n^1 + q_j^1 z_j^1\right)/w_1 \\
&> \left(z_n^1 + q_j^0 z_j^1\right)/w_1 \\
&= \frac{w_0}{w_1}\left(p_n^0 z_n^1 + p_j^0 z_j^1\right) = \frac{w_0}{w_1}\left(p_n^0 z_n^0 + p_j^0 z_j^0\right)
\end{aligned}$$

we get in fact that $w_0 < w_1$. This implies in turn that

$$\begin{aligned}
z_n^0 + q_{M_0}^0 z_{-n}^0 + q_K^0 x^0 = w_0 &< w_1 \\
&= z_n^1 + q_{M_0}^1 z_{-n}^1 + q_K^1 x^0 \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + q_{M_0}^0 z_{-n}^1 \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + w_0 p_{M\setminus\{n\}}^0 z_{-n}^1 \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 \\
&\quad + w_0\left(p_{M\setminus\{n\}}^0 z_{-n}^0 + p_n^0\left(z_n^0 - z_n^1\right)\right) \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + w_0\left(p_{M\setminus\{n\}}^0 z_{-n}^0 - p_n^0 \varepsilon_1\right) \\
&< z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + w_0 p_{M\setminus\{n\}}^0 z_{-n}^0 \\
&= z_n^1 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 + q_K^0 x^0 + q_{M_0}^0 z_{-n}^0
\end{aligned}$$

where the penultimate equality above follows from (30). Clearly, we have that

$$z_n^1 - z_n^0 + \left(q_{M_0}^1 - q_{M_0}^0\right) z_{-n}^1 + \left(q_K^1 - q_K^0\right) x^0 > 0 \tag{31}$$

But then we must have

$$\begin{aligned}
v\left(z^1, x^0, q_{M_0}^1, w_1\right) &= -xB^{-1}x/2 + q\left(x^0, q_{M_0}^1, w_1\right) x^0 + q_{M_0}^1 z_{-n}^1 + z_n^1 \\
&= -xB^{-1}x/2 + q_K^1 x^0 + q_{M_0}^1 z_{-n}^1 + z_n^1 \\
&> -xB^{-1}x/2 + q_K^0 x^0 + q_{M_0}^0 z_{-n}^0 + z_n^0 \\
&= -xB^{-1}x/2 + q\left(x^0, q_{M_0}^0, w_0\right) x^0 + q_{M_0}^0 z_{-n}^0 + z_n^0 = v\left(z^0, x^0, q_{M_0}^0, w_0\right)
\end{aligned}$$

the inequality above due to (31). And as this means that $\left(z^1, x^0\right) \succ \left(z^0, x^0\right)$, the desired contradiction follows from the absurdity $\left(z^1, x^0\right) \succ \left(z^0, x^0\right) \succ \left(z^1, x^0\right)$. ■

Corollary A.2 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\xi : Y \to X$ whose projection on the dimensions in K, $x : P_M \times P_K \to X_K$ given by $x(p_M, p_K) := \alpha(p_M) - Bp$ for some function $\alpha : P_M \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if $M_0 = \varnothing$.*

Proof. Recall again how the two sets of normalized prices are related: $(p_M, p_K) = p_n\left((1, q_{M_0}), q_K\right)$. The preceding proof applies as is - with the slight adjustment that B above should be replaced by $p_n B$. ■

Lemma A.3 *Let the continuous, strictly convex and strictly monotonic weak order $\succsim$ on X generate the demand function $\widetilde{\xi} : Q \to X$ whose projection on the dimensions in K, $x : Q_{M_0} \times Q_K \times W \to X_K$, is given by $x(q_{M_0}, q_K, w) := \alpha(q_{M_0}, w) - Bq_K$ for some function $a : Q_{M_0} \times W \to \mathbb{R}^n$. Then $\succsim$ is differentiable only if $\alpha(\cdot)$ is a constant.*

Proof. Let $\succsim$ be differentiable. As $M_0 = \varnothing$ (Lemma A.2), $\alpha(\cdot)$ can be a function only of income. In what follows, we will drop the subscript K from the members of Q_K and write (7) as $x(q, w) := \alpha(w) + Bq$. To argue ad absurdum, suppose that $\alpha(\cdot)$ is not constant around the arbitrary point $w_0 \in W$. Letting then $\lambda_0 \in \mathbb{R}_{++}$ be sufficiently small, we must have $\alpha(w) \neq \alpha(w_0)$ for all $w \in (w_0 - \lambda_0, w_0 + \lambda_0) \setminus \{w_0\}$. Take also an arbitrary $q^0 \in Q_K$ and let $x^0 := x(q^0, w_0)$ and $z_0 := z(q^0, w_0)$. Consider also the sets

$$\begin{aligned} Q_{Kz_0} &:= \{q \in Q_K : \ z(q, w_0) = z_0\} \\ X_{z_0} &:= \{(z, x) \in X : \ z = z_0\} \end{aligned}$$

Since X is open in $\mathbb{R}^n_{++}$, the set X_{z_0} is open in $\mathbb{R}^{n-1}_{++}$. Since the total demand is an homeomorphism so is the mapping $x : Y_{z_0} \to X_{z_0}$; hence, Q_{Kz_0} is also open in $\mathbb{R}^{n-1}_{++}$. And as $(z_0, x^0) \in X_{z_0}$, taking $\varepsilon_0, \rho_0 \in \mathbb{R}_{++}$ both sufficiently small ensures that $\mathcal{B}_{x^0}(\varepsilon_0) \subset X_{z_0}$ and $\mathcal{B}_{q^0}(\rho_0) \subset Q_{Kz_0}$.
Recall now that, $\succsim$ being differentiable, B is non-singular (Lemma A.1). As a result, the function

$$x^0(q) := \alpha(w_0) + Bq$$

defines an homeomorphism $x^0 : \mathcal{B}_{q^0}(\rho_0) \to \mathcal{B}_{x^0}(\varepsilon_0)$. Moreover, since $\alpha(w_0 + \lambda) \neq \alpha(w_0)$, we have $x(q, w_0) \neq x(q, w_0 + \lambda)$ for all $(\lambda, q) \in (-\lambda_0, \lambda_0) \times \mathcal{B}_{q^0}(\rho_0)$. In fact, letting $\lambda_1 \in (0, \lambda_0)$ and $\rho_1 \in (0, \rho_0)$ both sufficiently small so that $||\alpha(w_0 + \lambda) - \alpha(w_0)|| < \varepsilon_0/2$ for all $\lambda \in (-\lambda_1, \lambda_1)$ and $x(q, w_0) \in \mathcal{B}_{x^0}(\varepsilon/2)$ for all $q \in \mathcal{B}_{q^0}(\rho_1)$, we have

$$\begin{aligned} ||x(q, w_0 + \lambda) - x^0|| &\leq ||x(q, w_0 + \lambda) - x(q, w_0)|| + ||x(q, w_0) - x^0|| \\ &= ||\alpha(w_0 + \lambda) - \alpha(w_0)|| + ||x(q, w_0 + \lambda) - x^0|| < \varepsilon_0 \end{aligned}$$

That is, $x(q, w_0 + \lambda) \in \mathcal{B}_{x^0}(\varepsilon_0)$ for all $(\lambda, q) \in (-\lambda_1, \lambda_1) \times \mathcal{B}_{q^0}(\rho_1)$. And as $x^0(\cdot)$ is an homeomorphism, we have that

$$\exists! \ q^\lambda \in \mathcal{B}_{q^0}(\rho_0) : \qquad x(q, w_0 + \lambda) = x^0(q^\lambda)$$

Define then the $(-\lambda_1, \lambda_1) \to \mathcal{B}_{q^0}(\rho_0)$ function $\epsilon(q, \lambda) := q^\lambda - q$, and observe that the last relation above can be also written as

$$x(q, w_0 + \lambda) = x(q + \epsilon(q, \lambda), w_0)$$

Clearly, for all $(\lambda, q) \in (-\lambda_1, \lambda_1) \times \mathcal{B}_{q^0}(\rho_1)$, we have

$$\epsilon(q, \lambda) = \epsilon(\lambda) := B^{-1}(\alpha(w_0 + \lambda) - \alpha(w_0)) \tag{32}$$

This implies in turn that

$$\begin{aligned} x(q - \epsilon(\lambda), w_0 + \lambda) &= B(q - \epsilon(\lambda)) + \alpha(w_0 + \lambda) \\ &= Bq - (\alpha(w_0 + \lambda) - \alpha(w_0)) + \alpha(w_0 + \lambda) = x(q, w_0) \end{aligned}$$

and thus

$$\begin{aligned}
w_0 + \lambda &= z(q-\epsilon(\lambda), w_0+\lambda) + (q-\epsilon(\lambda))\, x(q-\epsilon(\lambda), w_0+\lambda) \\
&= z(q-\epsilon(\lambda), w_0+\lambda) + (q-\epsilon(\lambda))\, x(q, w_0) \\
&= z(q-\epsilon(\lambda), w_0+\lambda) + w_0 - z(q, w_0) - \epsilon(\lambda)\, x(q, w_0)
\end{aligned}$$

or, equivalently,

$$z(q-\epsilon(\lambda), w_0+\lambda) = z(q, w_0) + \lambda + \epsilon(\lambda)\, x(q, w_0) \tag{33}$$

Recall now the quasi-indirect utility function we obtained in the proof of Lemma A.2. As $M_0 = \varnothing$, this reads here

$$v(z, x, w) = xB^{-1}x/2 + q(x, w)\, x + z$$

That is,

$$\begin{aligned}
v(q-\epsilon(\lambda), w_0+\lambda) &= x(q-\epsilon(\lambda), w_0+\lambda)\, B^{-1} x(q-\epsilon(\lambda), w_0+\lambda)/2 \\
&\quad + (q-\epsilon)\, x(q-\epsilon(\lambda), w_0+\lambda) + z(q-\epsilon(\lambda), w_0+\lambda) \\
&= x(q, w_0)\, B^{-1} x(q, w_0)/2 + (q-\epsilon)\, x(q, w_0) \\
&\quad + z(q, w_0) + \lambda + \epsilon(\lambda)\, x(q, w_0) \\
&= v(q, w_0) + \lambda
\end{aligned} \tag{34}$$

which implies in turn that $\lambda \mapsto z(q, \lambda) := z(q-\epsilon(\lambda), w_0+\lambda)$ is an injective function.[18] Hence, for any $q \in \mathcal{B}_{q^0}(\rho_1)$, the image of $z(q, \lambda)$ on $(-\lambda_1, \lambda_1)$ is an open neighbourhood around the point $z(q, w_0)$.
Take now $\delta_0 \in \mathbb{R}_{++}$ such that the neighbourhood $\mathcal{B}_{z^0}(\delta_0)$ lies within the domain. Let also $z^0(\lambda) := z(q^0, \lambda)$. By the preceding argument, taking $\lambda_2 \in (0, \lambda_1)$ sufficiently small, $z^0(\cdot)$ on $(-\lambda_2, \lambda_2)$ maps onto $\mathcal{B}_{\mathbf{z}^0}(\delta_1)$ for some $\delta_1 \in (0, \delta_0)$.
Fix now some $\lambda \in (-\lambda_2, \lambda_2)$ and consider the sets

$$\begin{aligned}
Q_{Kz^0(\lambda)} &:= \left\{(q, w_0+\lambda) \in Q_K :\ z(q-\epsilon(\lambda), w_0+\lambda) = z^0(\lambda)\right\} \\
X_{z^0(\lambda)} &:= \left\{(z, x) \in X :\ z = z^0(\lambda)\right\}
\end{aligned}$$

By the same argument as in the opening paragraph above, $X_{z^0(\lambda)}$ and $Q_{Kz^0(\lambda)}$ are open in $\mathbb{R}^{n-1}_{++}$. And as $q^0 \in Q_{Kz^0(\lambda)}$, choosing $\rho_\lambda \in (0, \rho_1)$ sufficiently small, we get $\mathcal{B}_{q^0}(\rho_\lambda) \subset Q_{Kz^0(\lambda)}$. Moreover, using (33) above, we have that

$$\begin{aligned}
0 &= z(q-\epsilon(\lambda), w_0+\lambda) - z^0(\lambda) \\
&= \left(z(q, w_0) + \lambda + \epsilon(\lambda)\, x(q, w_0) - z(q^0, w_0) - \lambda - \epsilon(\lambda)\, x(q^0, w_0)\right) \\
&= \left(z^0 + \lambda + \epsilon(\lambda)\, x(q, w_0) - \mathbf{z}^0 - \lambda - \epsilon(\lambda)\, x^0\right) \\
&= \epsilon(\lambda)\left(x^0(q) - x^0\right) \qquad \forall q \in \mathcal{B}_{q^0}(\rho_\lambda)
\end{aligned}$$

[18] To see first that $z(q, \cdot)$ is a function, notice that we cannot have $z', z'' \in z(q, \lambda)$ with $z' \neq z''$. For (34) would imply then that $(z', x(q, w_0)) \sim (z'', x(q, w_0))$, an absurdity under monotonicity. To see now that $z(q, \cdot)$ must be injective, observe that we cannot have $z(q, \lambda') = z(q, \lambda'')$ with $\lambda' \neq \lambda''$. For, letting $z' := z(q, \lambda_1)$, (34) would imply now that $(z', x(q, w_0)) \succ (z', x(q, w_0))$.

As though $x^0(\cdot)$ is an homeomorphism, it maps $\mathcal{B}_{q^0}(\rho_\lambda)$ onto $\mathcal{B}_{x^0}(\varepsilon_\lambda)$ for some $\varepsilon_\lambda \in (0, \varepsilon_0)$. We have established thus that $\epsilon(\lambda)\left(x - x^0\right) = 0$ for every $x \in \mathcal{B}_{x^0}(\varepsilon_\lambda)$; equivalently, that $\epsilon(\lambda) = 0$. To complete the argument, recall (32). Since B^{-1} is non-singular, $\epsilon(\lambda) = 0$ implies that $\alpha(w_0 + \lambda) = \alpha(w_0)$. And as λ above was chosen arbitrarily, $\alpha(\cdot)$ must remain constant on $(w_0 - \lambda_2, w_0 + \lambda_2)$. ■