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Stability condition for the nonlinear McKendrick equation *

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Abstract

Nonlinear McKendrick equation with age-dependent mortality and fertility is considered. In [1] the author deduced the characteristic equation whose roots determine the stability. We are able to give sufficient conditions for the stability of the stationary solutions of the system in some cases.

Keywords: Age-structured population dynamics; Stability

1 Introduction

Non-linear age-dependent models have a recent history. Many biological phenomenons can be modelled better by non-linear differential equations. Age-specific mortality and fertility are among the most basic parameters of the

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theory of population dynamics and demography. In the present paper we shall consider the following McKendrick-type equation [2],[5] .

$$\frac{\partial p(t,a)}{\partial t} + \frac{\partial p(t,a)}{\partial a} = -\mu(a, P(t))p(t,a) \qquad (1.1)$$

$$p(0,a) := p_0(a),$$

where $p(t,a)$ denotes the density of individuals of age a at time t,

$$P(t) = \int_0^M p(t,a)da \qquad (1.2)$$

is the quantity of the total population at time t; $\mu(a, P(t))$ denotes the intrinsic mortality and $\beta(a, P(t))$ the fertility.

The density of newborns at time t is given by

$$p(t,0) = \int_0^M \beta(a, P(t))p(t,a)da. \qquad (1.3)$$

We assume a finite maximal age denoted by M.

This type of models has a wide literature. There exists results about stability in special cases. In [4] Sect. IV an Allee-logistic model is treated where the main assumption is that the mortality does not depend on the size of the population, and a factorization property for the fertility rate $\beta(a, P(t))$ is assumed. In the next section we will able to give sufficient condition for stability under the same assumptions.

Our motivation is that the characteristic equation deduced by the author in [1] seems to be very useful deciding stability as an example in [1] presented first by Gurtin and MacCamy in [3] shows.

2 μ does not depend on $P(t)$

Recall the characteristic equation corresponding to the stationary solution of (1.1)-(1.3), $p_1(a) = p_1(0)e^{-\int_0^a \mu(s,P_1)ds}$ from [1]

$$K(\lambda) = A_{11}(\lambda)A_{22}(\lambda) - A_{12}(\lambda)A_{21}(\lambda) + A_{12}(\lambda) + A_{21}(\lambda) = 1$$

where

$$A_{11}(\lambda) = \int_0^M e^{-\lambda a} e^{-\int_0^a \mu(s,P_1)ds} da,$$

$$A_{12}(\lambda) = -p_1(0) \int_0^M e^{-\lambda a} e^{-\int_0^a \mu(s,P_1)ds} \left(\int_0^a \mu'_P(s,P_1) e^{\lambda s} ds \right) da,$$

$$A_{21}(\lambda) = \int_0^M e^{-\lambda a} \beta(a,P_1) e^{-\int_0^a \mu(s,P_1)ds} da,$$

$$A_{22}(\lambda) = p_1(0) \left(\int_0^M \beta'_P(a,P_1) e^{-\int_0^a \mu(s,P_1)ds} da - \int_0^M e^{-\lambda a} \beta(a,P_1) e^{-\int_0^a \mu(s,P_1)ds} \int_0^a \mu'_P(s,P_1) e^{\lambda s} ds da \right)$$

where $P_1 = \int_0^M p_1(a)da$ is the quantity of the total population.
Recall the following notations:

$$\pi(a,P(t)) = e^{-\int_0^a \mu(s,P(t))ds}, \quad R(P(t)) = \int_0^M \beta(a,P(t))\pi(a,P(t))da$$

where the second term denotes the so called net reproductive number and $R(P_1) = 1$ at any stationary solution, obviously.

Now suppose that $\mu(a,P(t)) = m(a)$ and $\beta(a,P(t)) = b(a)f(P)$ where $b(.),m(.),f(.) \in C^1$.

Theorem 1. The characteristic equation $K(\lambda) = 1$ for any stationary solution $p_1(a)$ is stable if and only if $R'(P_1) < 0$.

Proof. Under the assumptions for the vital rates above the characteristic equation reduces to the following

$$\int_0^M e^{-\lambda a} \pi(a)\beta(a,P_1)da + \int_0^M e^{-\lambda a}\pi(a)da \int_0^M \beta'_P(a,P_1)p_1(0)\pi(a)da = 1$$

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and

$$\int_0^M \beta_P'(a, P_1) p_1(0) \pi(a) da = \int_0^M f'(P_1) p_1(0) b(a) \pi(a) da.$$

For any stationary solution

$$\int_0^M \beta(a, P_1) \pi(a) da = \int_0^M f(P_1) b(a) \pi(a) da = 1,$$

and with

$$p_1(0) = \frac{P_1}{\int_0^M \pi(a) da}$$

we get

$$K(\lambda) = 1 = \int_0^M e^{-\lambda a} \pi(a) b(a) f(P_1) da + \int_0^M e^{-\lambda a} \pi(a) da \frac{P_1}{\int_0^M \pi(a) da} \frac{f'(P_1)}{f(P_1)}.$$

Now suppose that $f'(P_1) > 0$ holds. Then it is easy to see that for the characteristic equation $K(\lambda)$

$$lim_{\lambda \to -\infty} K(\lambda) = +\infty \quad lim_{\lambda \to +\infty} K(\lambda) = 0$$

and $K(\lambda)$ is a strictly monotone decreasing function of λ, so there exists exactly one real λ for which $K(\lambda) = 1$ holds. Now we are going to show that $\lambda \geq 0$ holds. Contrary, suppose that $\lambda < 0$. Then for every $a \in [0, M]$ we have $e^{-\lambda a} \geq 1$. Then

$$\int_0^M e^{-\lambda a} \pi(a) b(a) f(P_1) da > \int_0^M \pi(a) b(a) f(P_1) da = 1,$$

and

$$\int_0^M e^{-\lambda a} P_1 \frac{\pi(a)}{\int_0^M \pi(a) da} \frac{f'(P_1)}{f(P_1)} da > 0.$$

This shows that $K(\lambda) > 1$ for every $\lambda < 0$.

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On the other hand if $f'(P_1) < 0$ then suppose that there exists a solution $\lambda = x + iy$ with $x \geq 0$. Then the characteristic equation assumes the form

$$1 = Re(K(\lambda)) = \int_0^M e^{-xa} cos(ya)\pi(a)b(a)f(P_1)da$$

$$+ \int_0^M e^{-xa} cos(ya)\pi(a)da \frac{P_1}{\int_0^M \pi(a)da} \frac{f'(P_1)}{f(P_1)}$$

and $Im(K(\lambda)) = 0$.

If $x \geq 0$ then $e^{-xa} \leq 1$ and $|cos(ya)| \leq 1$, so that

$$Re(K(\lambda)) \leq \int_0^M \pi(a)b(a)f(P_1)da + P_1 \frac{f'(P_1)}{f(P_1)}da \leq 1 + P_1 \frac{f'(P_1)}{f(P_1)} < 1,$$

a contradiction.

Finally observe that $R'(P_1) < 0 \iff f'(P_1) < 0$ because $R(P) = \int_0^M b(a)f(P)\pi(a)da.$ ∎

This result is in accordance with the example of Gurtin-MacCamy (see [1],[3]), because now $m'(P) \equiv 0$ and $-\beta'(P_1) > 0 \iff f'(P_1) < 0$.

3 μ does depend on P

Now suppose the following vital rates, both of them depending on P

$$\beta(a, P) = b(a)f(P), \quad \mu(a, P).$$

The characteristic equation is

$$K(\lambda) = A_{11}(\lambda)A_{22}(\lambda) - A_{12}(\lambda)A_{21}(\lambda) + A_{12}(\lambda) + A_{21}(\lambda) = 1,$$

where now

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$$A_{11}(\lambda) = \int_0^M e^{-\lambda a} \pi(a, P_1) da,$$

$$A_{12}(\lambda) = -p_1(0) \int_0^M e^{-\lambda a} \pi(a, P_1) \int_0^a \mu_P'(s, P_1) e^{\lambda s} ds da$$

$$A_{21}(\lambda) = \int_0^M e^{-\lambda a} \beta(a, P_1) \pi(a, P_1) da$$

$$A_{22}(\lambda) = p_1(0) \left(\int_0^M \beta_P'(a, P_1) \pi(a, P_1) da - \int_0^M e^{-\lambda a} \beta(a, P_1) \int_0^a \mu_P'(s, P_1) e^{\lambda s} ds \pi(a, P_1) da \right).$$

Theorem 2. The stationary solution $p_1(a)$ corresponding to the population quantity P_1 is asymptotically stable if $f'(P_1) < 0$ and $\mu_P'(., P_1) > 0$.

Remark. We are to prove that under the conditions above the characteristic equation cannot have roots with positive or zero real part, and we refer to [4] Th.I.5.1, where it is proven that if there exist a root with positive real part then there exists a real positive root, too.

Proof. We are going to prove that under the conditions for the vital rates for any real $\lambda \geq 0$ we have $K(\lambda) < 1$.

Observe that $A_{21}(\lambda) \leq 1 \iff \lambda \geq 0$, so it is enough to prove

$$A_{11}(\lambda) A_{22}(\lambda) - A_{12}(\lambda) A_{21}(\lambda) + A_{12}(\lambda) < 0$$

or

$$-A_{11}(\lambda) A_{22}(\lambda) - A_{12}(\lambda) > -A_{12}(\lambda) A_{21}(\lambda). \tag{3.1}$$

Simplifying the first term of $A_{22}(\lambda)$ we get:

$$A_{22}(\lambda) = p_1(0) \left(\frac{f'(P_1)}{f(P_1)} - \int_0^M e^{-\lambda a} b(a) f(P_1) \int_0^a e^{\lambda s} \mu_P'(s, P_1) ds \pi(a, P_1) da \right).$$

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Now we are going to substitute the $A_{ij}(\lambda)$ into the inequality (3.1):

$$-\frac{f'(P_1)}{f(P_1)}p_1(0)\int_0^M e^{-\lambda a}\pi(a,P_1)da+$$

$$+p_1(0)\int_0^M e^{-\lambda a}\pi(a,P_1)da(\int_0^M e^{-\lambda a}b(a)f(P_1)\pi(a,P_1)\int_0^a e^{\lambda s}\mu'_P(s,P_1)dsda)+$$

$$+p_1(0)\int_0^M e^{-\lambda a}\pi(a,P_1)\int_0^a e^{\lambda s}\mu'_P(s,P_1)dsda >$$

$$> p_1(0)(\int_0^M e^{-\lambda a}\pi(a,P_1)\int_0^a e^{\lambda s}\mu'_p(s,P_1)dsda)(\int_0^M e^{-\lambda a}\beta(a,P_1)\pi(a,P_1)da).$$
$$(3.2)$$

Omitting the first term of the left-hand side and dividing by $p_1(0)$ we get

$$(\int_0^M e^{-\lambda a}\pi(a,P_1)da)(\int_0^M e^{-\lambda a}\beta(a,P_1)\pi(a,P_1)\int_0^a e^{\lambda s}\mu'_P(s,P_1)dsda)+$$

$$+\int_0^M e^{-\lambda a}\pi(a,P_1)\int_0^a e^{\lambda s}\mu'_P(s,P_1)dsda >$$

$$> (\int_0^M e^{-\lambda a}\pi(a,P_1)\int_0^a e^{\lambda s}\mu'_P(s,P_1)dsda)(\int_0^M e^{-\lambda a}\beta(a,P_1)\pi(a,P_1)da)\quad(3.3)$$

which implies (3.2).

Subtracting the second term of the left-hand side we get

$$(\int_0^M e^{-\lambda a}\pi(a,P_1)da)(\int_0^M e^{-\lambda a}\beta(a,P_1)\pi(a,P_1)\int_0^a e^{\lambda s}\mu'_P(s,P_1)dsda) >$$

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$$> (\int_0^M e^{-\lambda a}\pi(a,P_1)\int_0^a e^{\lambda s}\mu'_P(s,P_1)dsda)(\int_0^M e^{-\lambda a}\beta(a,P_1)\pi(a,P_1)da - 1).$$

$$(3.4)$$

Finally observe that for $\lambda \geq 0$ we have $e^{-\lambda a} \leq 1$ so that

$$(\int_0^M e^{-\lambda a}\beta(a,P_1)\pi(a,P_1)da - 1) \leq 0.$$

The first factor of the right-hand side of (3.4) is positive because $\mu'(.,P_1) > 0$, thus the right-hand side is negativ while the left-hand side is positive. This proves (3.3) wich implies (3.2). ∎

Remark. Under the assumptions on the vital rates in the theorem $R'(P_1) < 0$ holds.

$$R(P(t)) = \int_0^M \beta(a,P(t))\pi(a,P(t))da$$

$$R'(P_1) = \int_0^M \beta'_P(a,P_1)\pi(a,P_1) + \beta(a,P_1)\pi'_P(a,P_1)da$$

$$\pi'_P(a,P_1) = -\int_0^a \mu'_P(s,P_1)ds e^{-\int_0^a \mu(s,P_1)ds} = -\pi(a,P_1)\int_0^a \mu'_p(s,P_1)ds$$

$$R'(P_1) = \int_0^M \beta'_P(a,P_1)\pi(a,P_1)da - \int_0^M \beta(a,P_1)\pi(a,P_1)\int_0^a \mu'_P(s,P_1)dsda$$

If $\beta'_p(.,P_1) < 0,\ \mu'_P(.,P_1) > 0$ then $R'(P_1) < 0$.

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