Inferring, splicing, and the Stoic analysis of argument*

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IN RECENT YEARS, A NUMBER OF AUTHORS, including the dedicatee of this volume, have advocated the employment of *general elimination rules* in the presentation of harmonious natural deduction rules for logical constants.[1] Motivated by different concerns, in (Milne 2008, Milne 2010) I have given natural deduction formulations of classical propositional and first-order logics employing what in (Milne 2012*b*) I call *general introduction rules*; there I show how general introduction and general elimination rules are in harmony and satisfy a certain inversion principle. Here I want to show how general introduction and general elimination rules narrow, perhaps even close, the gap between Gentzen's two ways of presenting logic: natural deduction and sequent calculus. I shall also describe a surprisingly close connection with the earliest account of propositional logic that we know of, that of the Stoic logicians. We are also led to ask after the significance of Gentzen's *Hauptsatz*. Until the final section, I shall consider only propositional logic.

1 General rules

The distinctive feature of general introduction rules is that they tell us when logically complex assumptions may be discharged; more bluntly, when they are not needed. The introduced connective occurs as main operator in a formula occurring *hypothetically* as a logically complex assumption discharged in the application of the rule; additionally some or all of the component propositions may occur either hypothetically (as assumptions also discharged in the application of the rule) or categorically. In general elimination rules, by contrast, the eliminated connective occurs as main operator in a formula occurring *categorically*, for we are being told what we can draw from a logically complex proposition; some or all of the component propositions may occur either hypothetically (as assumptions discharged in the application of the rule) or categorically. The conclusion of either kind of rule is "general", not determined by the proposition containing the introduced or eliminated connective. The introduction rules for conjunction and disjunction are unexciting rewrites of standard rules; not so for negation and the conditional, whose introduction rules are essentially classical.

*From Catarina Dutilh Novaes and Ole Thomassen Hjortland (eds.), *Insolubles and Consequences: Essays in Honour of Stephen Read*, London: College Publications, 2012, pp. 135-54.

[1]See, *e.g.*, Peter Schroeder-Heister (1984*b*, 1984*a*), Neil Tennant (1992), Stephen Read (2000, 2004, 2010), Sara Negri and Jan von Plato (2001), von Plato (2001), and Nissim Francez and Roy Dyckhoff (2011).

conjunction

$$\frac{\begin{matrix}[\phi \wedge \psi]^m \\ \vdots \\ \chi\end{matrix} \qquad \phi \qquad \psi}{\chi}\, m; \qquad \frac{\phi \wedge \psi \qquad \begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m, \qquad \frac{\phi \wedge \psi \qquad \begin{matrix}[\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m.$$

disjunction

$$\frac{\begin{matrix}[\phi \vee \psi]^m \\ \vdots \\ \chi\end{matrix} \qquad \phi}{\chi}\, m, \qquad \frac{\begin{matrix}[\phi \vee \psi]^m \\ \vdots \\ \chi\end{matrix} \qquad \psi}{\chi}\, m; \qquad \frac{\phi \vee \psi \qquad \begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m.$$

negation

$$\frac{\begin{matrix}[\neg\phi]^m \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m; \qquad \frac{\neg\phi \qquad \phi}{\chi}\,.$$

conditional

$$\frac{\begin{matrix}[\phi \to \psi]^m \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m, \qquad \frac{\begin{matrix}[\phi \to \psi]^m \\ \vdots \\ \chi\end{matrix} \qquad \psi}{\chi}\, m; \qquad \frac{\phi \to \psi \qquad \phi \qquad \begin{matrix}[\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m.$$

These rules give us a formulation of classical propositional logic with the subformula property (*cf.* Milne 2010, §2.7).

The general form of an introduction rule is

$$\frac{\begin{matrix}[\star(\phi_1, \phi_2, \ldots, \phi_n)]^m \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\phi_{i_1}]^m \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\phi_{i_2}]^m \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}\ldots \\ \vdots\;\vdots\;\vdots \\ \ldots\end{matrix} \quad \begin{matrix}[\phi_{i_k}]^m \\ \vdots \\ \chi\end{matrix} \qquad \phi_{j_1} \quad \phi_{j_2} \quad \ldots \quad \phi_{j_l}}{\chi}\, m \quad \star\text{-introduction}$$

where $k + l \leq n$ and $i_p \neq j_q, 1 \leq p \leq k, 1 \leq q \leq l$.

The general form of an elimination rule is

$$\frac{\star(\phi_1, \phi_2, \ldots, \phi_n) \qquad \begin{matrix}[\phi_{r_1}]^m \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\phi_{r_2}]^m \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}\ldots \\ \vdots\;\vdots\;\vdots \\ \ldots\end{matrix} \quad \begin{matrix}[\phi_{r_t}]^m \\ \vdots \\ \chi\end{matrix} \qquad \phi_{s_1} \quad \phi_{s_2} \quad \ldots \quad \phi_{s_u}}{\chi}\, m \quad \star\text{-elimination}$$

where $t + u \leq n$ and $r_p \neq s_q, 1 \leq p \leq t, 1 \leq q \leq u$.

Given the complete set of general introduction rules for a truth-functional connective (of any *arity*), we can read off a complete set of harmonious elimination rules; we can also read off the connective's truth-table. Likewise, given the complete set of general elimination rules for a connective, we can read off a complete set of harmonious introduction rules; we can again read off the connective's truth-table. And given the connective's truth-table, we can read off harmonious general introduction rules and general elimination rules. For more on why this is and ought to be the case, see (Milne 2012*b*).

Derivable argument-forms yield derived inference rules. If $\Sigma \vdash \phi$ is derivable, we can introduce the rule

$$\dfrac{\Sigma \qquad \begin{array}{c}[\phi]^m\\ \vdots\\ \chi\end{array}}{\chi}\,m.$$

The converse holds trivially, for, by an application of the rule, we obtain the derivation

$$\dfrac{\Sigma \qquad [\phi]^1}{\phi}\,1.$$

Corresponding to what is sometimes called the Rule of Assumptions—*e.g.*, in (Lemmon 1965)—we have the "null rule"

$$\dfrac{\phi \qquad \begin{array}{c}[\phi]^m\\ \vdots\\ \chi\end{array}}{\chi}\,m,$$

which has no proper application in natural deduction proofs laid out in tree form for there

$$\phi$$

counts as a proof with conclusion ϕ dependent on ϕ as assumption.

2 Splicing

We form natural deduction proof-trees by treating the conclusion of an application of one rule as the occurrence of a categorical formula in the application of another rule. Formulae standing as assumptions in the application of the first rule continue to stand as assumptions in the application of the second rule if not discharged (as hypothetically occurring side-premisses) in the application of the first. A proof of the argument-form $\Sigma \vdash \psi$ is a proof tree in which only members of Σ remain as undischarged assumptions and ϕ stands as the conclusion, at the root of the proof-tree.

I want to consider a different way of manipulating rules. I call it *splicing*. Splicing is carried out by cross-cancelling a hypothetical occurrence of a formula in one rule and a categorical occurrence of the same formula in another rule to obtain a new rule. Splicing is to be thought of as a new primitive operation acting directly on *rules in schematic form*. Employing splicing, we can find (general) introduction and elimination rules for logically complex formulae. For example, exclusive disjunction can be represented as $(\phi \vee \psi) \wedge \neg(\phi \wedge \psi)$. Splicing gives us in a succession of steps:

$$\dfrac{\begin{array}{c}[(\phi \vee \psi) \wedge \neg(\phi \wedge \psi)]^m\\ \vdots\\ \chi\end{array} \qquad \phi \vee \psi \qquad \neg(\phi \wedge \psi)}{\chi}\,m \quad \oplus \quad \dfrac{\begin{array}{c}[\phi \vee \psi]^m\\ \vdots\\ \chi\end{array} \qquad \phi}{\chi}\,m \quad =$$

$$\dfrac{\begin{array}{c}[(\phi \vee \psi) \wedge \neg(\phi \wedge \psi)]^m\\ \vdots\\ \chi\end{array} \qquad \phi \qquad \neg(\phi \wedge \psi)}{\chi}\,m;$$

$$\dfrac{\begin{matrix}[(\phi \lor \psi) \land \neg(\phi \land \psi)]^m\\ \vdots\\ \chi\end{matrix} \qquad \phi \qquad \neg(\phi \land \psi)}{\chi}\,m \quad \oplus \quad \dfrac{\begin{matrix}[\neg(\phi \land \psi)]^m\\ \vdots\\ \chi\end{matrix} \qquad \begin{matrix}[\phi \land \psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m \quad = \quad \dfrac{\begin{matrix}[(\phi \lor \psi) \land \neg(\phi \land \psi)]^m\\ \vdots\\ \chi\end{matrix} \qquad \phi \qquad \begin{matrix}[\phi \land \psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m;$$

$$\dfrac{\begin{matrix}[(\phi \lor \psi) \land \neg(\phi \land \psi)]^m\\ \vdots\\ \chi\end{matrix} \qquad \phi \qquad \begin{matrix}[\phi \land \psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m \quad \oplus \quad \dfrac{\phi \land \psi \qquad \begin{matrix}[\psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m \quad = \quad \dfrac{\begin{matrix}[(\phi \lor \psi) \land \neg(\phi \land \psi)]^m\\ \vdots\\ \chi\end{matrix} \qquad \phi \qquad \begin{matrix}[\psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m.$$

In similar fashion, we obtain the introduction rule

$$\dfrac{\begin{matrix}[(\phi \lor \psi) \land \neg(\phi \land \psi)]^m\\ \vdots\\ \chi\end{matrix} \qquad \begin{matrix}[\phi]^m\\ \vdots\\ \chi\end{matrix} \qquad \psi}{\chi}\,m$$

and the elimination rules

$$\dfrac{(\phi \lor \psi) \land \neg(\phi \land \psi) \qquad \phi \qquad \psi}{\chi} \quad \text{and} \quad \dfrac{(\phi \lor \psi) \land \neg(\phi \land \psi) \qquad \begin{matrix}[\phi]^m\\ \vdots\\ \chi\end{matrix} \qquad \begin{matrix}[\psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m.$$

Had we spliced

$$\dfrac{\begin{matrix}[(\phi \lor \psi) \land \neg(\phi \land \psi)]^m\\ \vdots\\ \chi\end{matrix} \qquad \phi \qquad \begin{matrix}[\phi \land \psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m$$

with the other $\land$-elimination rule, *i.e.*, with

$$\dfrac{\phi \land \psi \qquad \begin{matrix}[\phi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m,$$

we would have obtained this rule

$$\dfrac{\begin{matrix}[(\phi \lor \psi) \land \neg(\phi \land \psi)]^m\\ \vdots\\ \chi\end{matrix} \qquad \phi \qquad \begin{matrix}[\phi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\,m$$

which, being a *weakening* of the null rule, is quite acceptable but utterly uninformative regarding $(\phi \vee \psi) \wedge \neg(\phi \wedge \psi)$. (Quite generally, if we accept a rule, we can accept any weakening obtained by adding formulae whether that be in categorical occurrences or hypothetical.)

The expressive adequacy of, say, $\{\neg, \wedge, \vee\}$ (or, indeed any other expressively adequate set) in classical propositional logic translates into the capacity of splicing to be a technique for generating (sets of) general introduction and general elimination rules for arbitrary n-ary truth-functional connectives. By way of example, the computer programmer's favourite, 'if ϕ then ψ, else χ', which we shall abbreviate as '$\looparrowright(\phi, \psi, \chi)$', is classically equivalent to $(\phi \wedge \psi) \vee (\neg\phi \wedge \chi)$. This is equivalent to the negation of $(\phi \vee \neg\chi) \wedge (\neg\phi \vee \neg\psi)$. Inspection of the results of splicing in the case of exclusive disjunction shows that splicing will yield these rules:

introduction rules

$$\dfrac{\begin{array}{c}[\looparrowright(\phi,\psi,\chi)]^m\\ \vdots\\ v\end{array} \qquad \begin{array}{c}[\phi]^m\\ \vdots\\ v\end{array} \qquad \chi}{v}\, m \quad \text{and} \quad \dfrac{\begin{array}{c}[\looparrowright(\phi,\psi,\chi)]^m\\ \vdots\\ v\end{array} \qquad \phi \qquad \psi}{v}\, m\,;$$

elimination rules

$$\dfrac{\looparrowright(\phi,\psi,\chi) \qquad \phi \qquad \begin{array}{c}[\psi]^m\\ \vdots\\ v\end{array}}{v}\, m \quad \text{and} \quad \dfrac{\looparrowright(\phi,\psi,\chi) \qquad \begin{array}{c}[\phi]^m\\ \vdots\\ v\end{array} \qquad \begin{array}{c}[\chi]^m\\ \vdots\\ v\end{array}}{v}\, m.$$

$\looparrowright(\phi, \psi, \chi)$ is also equivalent to $(\phi \to \psi) \wedge (\neg\phi \to \chi)$. Splicing, starting with this formula, gets us the very same rules.

2.1 Algebra ... or maybe not

With splicing and other operations that we may perform on rules, we have a certain amount of structure.

Splicing is not a function: while in any application of splicing there is only one formula cancelled, a hypothetical occurrence in one rule being crossed off against a categorical occurrence in the other, there may be more than one formula available for cancellation. So, strictly speaking, questions of associativity and commutativity do not arise. Instances of the null rule do act as *identity elements* in that, *when we can splice*, splicing with an instance of the null rule yields the original rule (and there can't be any question about which formula is up for cancellation unless the other rule involved is already an instance of the null rule or a weakening thereof). (Recall that in order to splice two rules, the same formula must occur hypothetically in one rule and categorically in the other, hence the qualification 'when we can splice'.)

Counting every rule as a weakening of itself, there is an uninteresting partial ordering given by the relation *rule X is a weakening of rule Y*.

The negation rules play a special role: splicing an introduction rule with the instance of the negation elimination rule containing the introduced formula yields an elimination rule for the introduced formula's negation. Conversely, splicing an elimination rule with the instance of the negation introduction rule containing the eliminated formula yields an introduction rule for the eliminated formula's negation.

Duality Conjunction and disjunction are duals. There's an obvious duality between their respective introduction and elimination rules. We can spell it out in a simple instruction for dualising a complete set of (harmonious) general introduction and elimination rules for a connective:

> To obtain the (dual) rules for the dual connective, in each rule turn all hypothetical occurrences into categorical occurrences and all categorical occurrences into hypothetical.

Accordingly, negation is self-dual, as we would expect. The dual of $\looparrowright(\phi, \psi, \chi)$ is then readily seen to be $\looparrowright(\phi, \chi, \psi)$, a fact that can be checked rather more laboriously by writing out a truth table for $\looparrowright(\phi, \psi, \chi)$, turning it upside down and swapping '0's ('T's) for '1's ('F's) and *vice versa*.

"The levelling of local peaks"[2] By inspection, we see that in the examples at hand, splicing an introduction rule and an elimination rule for the same connective and cancelling the formula containing the introduced/eliminated connective produces either an instance of the null rule (in the case of negation) or a weakening of the null rule. In fact, for general introduction and elimination rules harmonious in the sense of (Milne 2012*b*) this holds not only for the examples at hand but is always the case.

2.2 *bullet*

Steve Read has introduced a one-place connective which, symbolised by '•', is often nowadays called '*bullet*': *bullet* is interderivable with its negation. In his (2000), Steve gave it these (impure) introduction and elimination rules:

$$\frac{\neg\bullet}{\bullet}\ \bullet\text{-introduction} \qquad \text{and} \qquad \frac{\bullet \qquad \begin{matrix}[\neg\bullet]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\ \bullet\text{-elimination (Read 2000, p. 141)}$$

The introduction rule doesn't have a general conclusion but we can easily rectify that:

$$\frac{\neg\bullet \qquad \begin{matrix}[\bullet]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\ \bullet\text{-introduction}$$

—and now we see that, according to the recipe for dualising given above, *bullet* is self-dual.[3]

Splicing with the negation rules, we obtain these (pure) general introduction and elimination rules—

[2] The term is, of course, Dummett's.

[3] In the context of certain three-valued logics, such as Graham Priest's *Logic of Paradox* (Priest 1979) or my *Logic of Conditional Assertions* (Milne 2004), *bullet* would act as a harmless, propositional constant/0-ary connective having

$$\dfrac{\begin{matrix}[\bullet]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m \ \bullet\text{-introduction} \qquad \text{and} \qquad \dfrac{\bullet}{\chi}\ \bullet\text{-elimination}$$

(*cf.* Read 2010, p. 571).[4]

—which gives *bullet* the introduction rule of the *verum* constant and the elimination rule of the *falsum* constant! And upon splicing these rules for *bullet* we get neither an instance nor a weakening of an instance of the null rule, rather we get the explicitly contradictory rule

$$\frac{}{\chi}$$

which makes me think there's something more than a little fishy about *bullet* in the (classical) context of general introduction and elimination rules.[5]

3 Soundness, completeness, and sequent calculus

Since rules are general in the sense that the conclusion is in effect the same (arbitrary) formula, and the indexing of discharges has no significance for splicing, we simplify our notation. Given a rule, we write '$\Sigma \Rightarrow \Delta$', where Σ is the set of formulae occurring categorically and Δ is the set of formulae occurring hypothetically (and discharged in the application of the rule); either set may be empty. The null rule gives us all instances of $\phi \Rightarrow \phi$.[6]

We say that the rule $\Sigma \Rightarrow \Delta$ is *classically sound* if there is no assignment of truth-values to atoms under which all the formulae in Σ are *true* and all the formulae in Δ are *false*. We confine attention to formulae built up employing $\wedge$, $\vee$, $\rightarrow$ and $\neg$.

Theorem 1 (Soundness Theorem)**.** If the rule $\Sigma \Rightarrow \Delta$ is obtained by splicing from instances of the null rule and the rules for negation, conjunction, disjunction, and the conditional or is a weakening of any such rule then it is classically sound.

Proof. The rules for negation, conjunction, disjunction, and the conditional are all classically sound. Weakening preserves classical soundness, as does splicing which in this setting is just Gentzen's *Schnitt*. □

Theorem 2 (Completeness Theorem)**.** If the rule $\Sigma \Rightarrow \Delta$ is classically sound then it is obtainable by splicing and/or weakening from instances of the null rule and the rules for negation, conjunction, disjunction, and the conditional.

the intermediate truth-value. With application to the first, Steve says of *bullet* that 'it constitutes a kind of proof-conditional Liar sentence' (2000, p. 142); with application to the second, *bullet* just is the constant N used, following (Hailperin 1996), in defining de Finetti's binary *conditioning* connective(Milne 2004, §§1.4-1.5, 9.1). In these contexts the negation is, of course, not governed by both *ex falso quodlibet* and the Rule of Dilemma, *i.e.*, not governed both by general introduction and by general elimination rules.

[4]Strictly, because he is being very careful regarding structural rules, Steve has two occurrences of • "above the line" in the elimination rule and would no doubt want the analogue in the introduction rule (which is mine, not his). But splicing, being based on the standard use of natural deduction rules, does not care about such niceties.

[5]For more on *bullet* in the context of general introduction and elimination rules, see (Milne 2012*b*, §4.5).

[6]Note that $\star(\phi_1, \phi_2, \ldots, \phi_n)$ and $*(\phi_1, \phi_2, \ldots, \phi_n)$ are dual only where $\Sigma \Rightarrow \star(\phi_1, \phi_2, \ldots, \phi_n), \Delta$ is an introduction rule for $\star$ if, and only if, $\Delta, *(\phi_1, \phi_2, \ldots, \phi_n) \Rightarrow \Sigma$ is an elimination rule for $*$ and likewise, *mutatis mutandis*, for $\star$-elimination and $*$-introduction rules.

Proof sketch. The idea of the proof is that we can replicate the operational rules of Gentzen's calculus *LK* for $\neg$, $\wedge$, $\vee$, and $\to$. We can also replicate the effects of Gentzen's structural rules. Proof is then by recursion on the structure of proofs of classically sound sequents in *LK*.

negation Suppose we have derived $\Sigma, \phi \Rightarrow \Delta$. By splicing with the negation introduction rule, we obtain $\Sigma \Rightarrow \neg\phi, \Delta$.

Suppose, next, that we have derived $\Sigma \Rightarrow \phi, \Delta$. By splicing with the negation elimination rule, we obtain $\Sigma, \neg\phi \Rightarrow \Delta$. Thus we replicate the left and right introduction rules for negation in Gentzen's *LK*.

conditional Suppose we have derived $\Sigma, \phi \Rightarrow \psi, \Delta$. By splicing with one of the conditional introduction rules we obtain $\Sigma \Rightarrow \phi \to \psi, \psi, \Delta$; by splicing with the other conditional introduction rule and recalling that Σ and Δ stand for sets, we obtain $\Sigma \Rightarrow \phi \to \psi, \Delta$.

Suppose, next, that we have derived $\Sigma_1 \Rightarrow \phi, \Delta_1$ and $\Sigma_2, \psi \Rightarrow \Delta_2$. By splicing the first with the elimination rule for the conditional, we get $\Sigma_1, \phi \to \psi \Rightarrow \psi, \Delta_1$; by splicing this with $\Sigma_2, \psi \Rightarrow \Delta_2$ we obtain, just as we should, $\Sigma_1, \Sigma_2, \phi \to \psi \Rightarrow \Delta_1, \Delta_2$.

conjunction and disjunction These are treated similarly.

Structural rules: contraction and permutation are hidden from view in the use of sets, rather than sequences; in the present context Gentzen's Weakening (Thinning, Augmentation) just is the weakening of a rule by adding side-formulae either categorically or hypothetically (left or right); splicing is Cut.

We now appeal to the known completeness of *LK*. □

What starts out as, seemingly, only a notational convenience turns out to be not just a genuine sequent calculus but, in effect, Gentzen's *LK*.

Of course, the system is not exactly Gentzen's *LK*. But now, notice that the uses made of splicing/cut to show derivability of Gentzen's rules satisfy this constraint: in every application

$$\frac{\Sigma_1, \phi \Rightarrow \Delta_1 \qquad \Sigma_2 \Rightarrow \phi, \Delta_2}{\Sigma_1, \Sigma_2 \Rightarrow \Delta_1, \Delta_2}$$

of splicing/cut, ϕ is a subformula of at least one formula in $\Sigma_1 \cup \Sigma_2 \cup \Delta_1 \cup \Delta_2$.

This is the rule of *Analytic Cut* of Raymond Smullyan's (1968*a*), where a system is developed in order to show that 'The real importance of cut-free proofs is not the elimination of cuts per se, but rather that such proofs obey the subformula principle' (Smullyan 1968*a*, p. 560).[7]

4 The Stoic analysis of argument

The Stoic account of propositional logic proceeds by presenting a handful of basic argument-forms and by reducing more complex arguments to these by a using a fixed number of techniques. Viewed coarsely, the approach is similar to Gentzen's sequent calculus account: there

[7] In serendipitous conformity with Smullyan's dictum, I showed that the $\{\wedge, \vee, \to, \neg, \exists\}$-fragment of the system for classical logic in (Milne 2010) has the subformula property using model-theoretic means.

are basic argument forms taken as valid and a set of rules for generating valid argument-forms from valid argument forms. In detail there are philosophically significant differences and a difference of direction: reduction of the more complex to less rather than the other way around. Perhaps a better approximation to Stoic intentions would be achieved by inverting the proof-trees of a sequent calculus so that one works downwards from the complex forms towards the basic argument-forms,[8] although the difference of direction may not be of great moment when giving a formal account of Stoic logic.

The basic argument forms, the *indemonstrables*, are, in effect, direct elimination rules. (By 'direct' I mean that all formulae save one occur categorically.) Stoic logicians provide elimination rules for conditionals and (exclusive) disjunctions and for negations of conjunctions.

According to Benson Mates,

> The Stoics maintained that their system of propositional logic was complete in the sense that every valid argument could be reduced to a series of arguments of five basic types. Even the method of reduction was not left vague, but was exactly characterized by four meta-rules, of which we possess two, and possibly three. Whether or not the Stoic system was actually complete could be decided only with the help of the missing rules. (Mates 1961, p. 4)

The claim to completeness is, *if meant in the modern sense*, well wide of the mark.[9] But we can give a formulation of classical propositional logic Stoic in spirit—at least in some respects.

Splicing a rule with instances of the negation elimination rule, hypothetical occurrences of formulae are replaced by categorical occurrences of their negations. Likewise, splicing a rule with instances of the negation introduction rule, categorical occurrences of formulae are replaced by hypothetical occurrences of their negations. We obtain these rules for the conditional, conjunction and exclusive disjunction:

$\phi \rightarrow \psi, \phi \Rightarrow \psi, \quad \phi \rightarrow \psi, \neg\psi \Rightarrow \neg\phi, \quad \neg(\phi \rightarrow \psi) \Rightarrow \phi, \quad \neg(\phi \rightarrow \psi) \Rightarrow \neg\psi;$

$\phi \wedge \psi \Rightarrow \phi, \quad \phi \wedge \psi \Rightarrow \psi, \quad \neg(\phi \wedge \psi), \phi \Rightarrow \neg\psi, \quad \neg(\phi \wedge \psi), \psi \Rightarrow \neg\phi;$

$\phi + \psi, \phi \Rightarrow \neg\psi, \quad \phi + \psi, \psi \Rightarrow \neg\phi, \quad \phi + \psi, \neg\phi \Rightarrow \psi, \quad \phi + \psi, \neg\psi \Rightarrow \phi,$

$\neg(\phi + \psi), \phi \Rightarrow \psi, \quad \neg(\phi + \psi), \psi \Rightarrow \phi, \quad \neg(\phi + \psi), \neg\phi \Rightarrow \neg\psi, \quad \neg(\phi + \psi), \neg\psi \Rightarrow \neg\phi.$

Here the rules for the conditional, the negated conjunction, and the (exclusive) disjunction are exactly the Stoic indemonstrables (at least if we accept informal ancient accounts—see, *e.g.*, (Mueller 1978, pp. 10-11)). The rest are present in order to spell out the truth-functional accounts of these connectives (accounts by-and-large accepted by Stoic logicians only in the cases of negation and conjunction, necessitarian readings being favoured for the conditional and exclusive disjunction).

Splicing $\neg\neg\phi, \neg\phi \Rightarrow$ and $\Rightarrow \neg\phi, \phi$, we get the double negation elimination argument-form/rule $\neg\neg\phi \Rightarrow \phi$; similarly, we may obtain $\phi \Rightarrow \neg\neg\phi$. We need some rule to get complex consequences, not just atomic formulae and their negations, double negations, triple negations, *etc*. For this we need some indirect rule, such as the Stoic's *first thema*:

[8] *Cf.* (Corcoran 1974). See also (Frede 1974, pp. 186-190).

[9] See (Hitchcock 2005) on what might have been intended. See (Mueller 1979) and (Milne 1995, Milne 2012*a*) for more on the completeness of Stoic logic.

$$\text{if } \Sigma, \phi \Rightarrow \psi \text{ then } \Sigma, \neg\psi \Rightarrow \neg\phi.$$

This gives us a complete set of rules for a pseudo-Stoic, natural deduction account of classical propositional logic (with exclusive disjunction as primitive), confined to argument-forms with at least one premiss. (And by splicing with the negation rules we can get back from these rules to general introduction and elimination rules.)[10]

I have just said that the *rules* for the conditional, the negated conjunction, and the (exclusive) disjunction are *exactly* the Stoic indemonstrables but a while back I said that the indemonstrables are *argument-forms*. What we see is that there is really no difference between taking (single conclusion) general introduction and elimination rules for natural deduction and combining rules by splicing and taking argument-forms and generating new ones by Cut. We have two ways of presenting the same underlying structure, albeit that the starting point may seem quite different.

4.1 Tableaux

The "Stoic" argument-forms/rules all have a complex formula occurring in the antecedent/occurring categorically and a single formula in the succeedent/occurring hypothetically. Using the negation rules we can move all bar the complex formula across to the succeedent, adding or deleting negations as we go. When we do so, what we have are, in effect, the rules for Smullyan's (1968*b*) analytic tableaux. We can mimic the elaboration of tableaux (trees) by splicing/cutting—we have to take a little care in setting this up, for we have to mimic both what Smullyan calls [non-branching] α rules and what he calls β [branching] rules.[11]

The negation rules let us turn categorical occurrences of negated formulae into hypothetical occurrences of the unnegated original and *vice versa*. Consequently, we can mimic tableaux directly with putative rules, splicing on the main formula, not side formulae. The argument-form $\Sigma \vdash \phi$ is valid if, and only if, every permitted sequence of splices on (complex) main formulae, starting from

[10] In the interests of historical accuracy, I should say that (*a*) the Stoics had no analogue of the null rule, considering arguments with fewer than two premisses not to be syllogisms; (*b*) they had no analogue of Weakening, declaring arguments invalid in virtue of redundancy of premisses (although exactly what that amounts to is unclear), and (*c*) they had a rule, the third *thema*, whose effect is very much like Gentzen's Cut rule. Stoic logicians seem to have recognised the equivalence in meaning of a proposition and its double negation although how this was accommodated formally is unclear—one possibility is along the lines of the account of the contrary in (Read 1988, pp. 178-182) (see (Mueller 1979, p. 204)). But what is important here is the style of their analysis: basic argument-forms and a set of procedures for the transformation of argument-forms. Regarding their treatment of negation we should note, too, a couple of remarks of Gentzen's:

- For *negation* ($\neg$) the situation is not quite as simple; here there are several distinct forms of inference and these cannot be divided clearly into $\neg$-introductions and $\neg$-eliminations. (Gentzen 1936, ¶4.56)
- The following must be said about the *rules of inference* for *negation*; as already mentioned at 4.56, the choice of elementary forms of inference is here *more arbitrary* than in the case of the other logical constants. (Gentzen 1936, ¶5.26)

[11] It is no co-incidence that our rules/argument-forms for the pseudo-Stoic formulation of classical logic are close to the rules of the unsigned version of the tableaux system *KE* in (D'Agostino & Mondadori 1994).

$$\dfrac{\Sigma \qquad \begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m.$$

terminates in an instance of the null rule or a weakening thereof.[12]

4.2 Negation and two kinds of disjunctions

The Stoics have disjunctive syllogism as one of their five indemonstrables, albeit their disjunction is exclusive. The late Michael Dummett called disjunctive syllogism (with inclusive disjunction) 'a fundamental form of argument' (1991, p. 293) . Treating it as basic goes hand in hand with a suggestion Dummett made on another occasion:

> We must first recall that a non-classical negation can be readily introduced in terms of justifiability alone. The utterance "ϕ or ψ" may now be thought of as expressing a conditional claim to be able to justify the claim "ϕ", given a justification of "Not ψ", or, conversely, to justify the claim "ψ", given a justification of "Not ϕ": this is, in effect, to take the logical law *modus tollendo ponens* as giving the basic meaning of disjunction.

He went on to say:

> The interpretation of "or" proposed above in effect equates "ϕ or ψ" with "If not ϕ, then ψ, and, if not ψ, then ϕ", understood intuitionistically, a rendering of course weaker than the ordinary intuitionistic interpretation of "ϕ or ψ". (Dummett 1990, pp. 7-8, with a change of notation)[13]

Rather than settle for one or other disjunction, Steve has urged that there are two disjunctions in common usage, one governed by Gentzen's $\vee$-introduction, another governed by disjunctive syllogism—see (Read 1988, pp. 31-34) and (Read 1994, pp. 60, 160, 163). By distinguishing the two, the "Lewis proof"[14] that anything follows from a contradiction, which infers "ϕ or ψ" from "ϕ" and then "ψ" from "ϕ or ψ" and "$\neg\phi$", is seen to trade on an ambiguity. Given the strongly classical nature of general introduction and elimination rules, it is unsurprising that this route to blocking the proof is unavailable in the present setting. It is interesting to see why not, though.

[12]Given the links between natural deduction with general introduction and elimination rules and Smullyan's analytic cut sequent calculus, on the one hand, and between the latter and Carlo Cellucci's (2000) analytic cut trees, on the other, the way of mimicking trees by splicing rules just suggested should be in some sense equivalent to Cellucci's tableaux system; spelling out the equivalence is work for another occasion.

[13]That "$\phi \vee \psi$" says nothing more nor less than "if not ϕ, ψ, and if not ψ, ϕ" is by no means a new idea. We find this statement in Mill's *System of Logic*:

> As has been well remarked by Archbishop Whately and others, the disjunctive form is resolvable into the conditional; every disjunctive proposition being equivalent to two or more conditional ones. "Either A is B or C is D," means, "if A is not B, C is D; and if C is not D, A is B." (Mill 1843, Book 1, Ch. IV 'Of propositions', sec. 3, p. 110); (1846, p. 56)

[14]William of Soissons's proof, as Steve doubtless prefers me to say—see (Martin 1986).

Our sequent calculus started out as a convenient means for the representation of general introduction and general elimination rules. We can read it again as a system of rules for multiple conclusion natural deduction somewhat along the lines of (Boričić 1985) in that proofs are in tree form. But splicing with the negation rules allows us to turn any multiple conclusion rule into a plurality of impure, single conclusion rules—and to recover pure multiple conclusion rules from a family of impure, single conclusion rules. Thus, *in the present setting*, there is *essentially no difference* between Gentzen's $\vee$-elimination rule and the left and right forms of *disjunctive syllogism*,

$$\frac{\neg\phi \qquad \phi \vee \psi}{\psi} \quad \text{and} \quad \frac{\neg\psi \qquad \phi \vee \psi}{\phi}\,.$$

Splicing $(\neg\phi \rightarrow \psi) \wedge (\neg\psi \rightarrow \phi)$ to obtain general introduction rules, one arrives naturally at these *three* rules—

$$\frac{\begin{array}{c}(\neg\phi \rightarrow \psi) \wedge (\neg\psi \rightarrow \phi)\\ \vdots \\ \chi\end{array} \qquad \phi}{\chi}\;, \qquad \frac{\begin{array}{c}(\neg\phi \rightarrow \psi) \wedge (\neg\psi \rightarrow \phi)\\ \vdots \\ \chi\end{array} \qquad \psi}{\chi} \quad \text{and}$$

$$\frac{\begin{array}{c}(\neg\phi \rightarrow \psi) \wedge (\neg\psi \rightarrow \phi)\\ \vdots \\ \chi\end{array} \qquad \phi \qquad \psi}{\chi}$$

—but the third, being a weakening of each of the previous two, is redundant.

In the present setting one cannot coherently tell Steve's story of there being two disjunctions, for when spelt out in terms of general introduction and elimination rules, the supposedly distinct connectives have the same introduction and elimination rules. In itself, this in no way serves to impugn Steve's account of how to by-pass the Lewis proof; it merely serves to underline the classical nature of the setting provided by (my account of) general introduction and general elimination rules.

5 Gentzen's *Hauptsatz*

According to Dummett, it was Gerhard Gentzen 'who, by inventing both natural deduction and the sequent calculus, first taught us how logic should be formalised' (Dummett 1991, p. 251). In view of what we have—all too briefly—seen of the Stoic approach to argument, such a claim may seem to slight the achievement of Stoic logicians. However, on an earlier occasion Dummett had been more specific in the credit he gives Gentzen. There he said,

> It can be said of Gentzen that it was he who first showed how proof theory should be done. By replacing the old axiomatic formalizations of logic by sequent calculi, and, in particular, by the cut-free systems, he not only corrected our conceptual perspective on logic, but also restored the balance of power as between proof-theoretic methods and algebraic methods. (Dummett 1981, p. 434)

In the light of this clarification, perhaps Dummett is not unfair to the Stoics. Their system was nothing like a *cut-free* sequent-calculus. But what is the significance of cut-free systems?

Gentzen sums up the *Hauptsatz* like this:

> HAUPTSATZ Every *LJ*- or *LK*-derivation can be transformed into an *LJ*- or *LK*-derivation with the same endsequent and in which the inference figure called a 'cut' does not occur.

and notes this consequence

> COROLLARY OF THE HAUPTSATZ (SUBFORMULA PROPERTY) In an *LJ*- or *LK*-derivation without cuts, all occurring formulae are subformulae of the formulae that occur in the endsequent. (Gentzen 1934-35, ¶¶2.5 & 2.513).

A page or so before the statement of the Hauptsatz, Gentzen had said,

> In general, we could *simplify* the calculi in various respects if we attached no importance to the *Hauptsatz*. To indicate this briefly: the inference figures [right $\wedge$-introduction, left $\vee$-introduction, left $\wedge$-introduction, right $\vee$-introduction, left $\forall$-introduction, right $\exists$-introduction, right $\neg$-introduction, left $\neg$-introduction, and left $\rightarrow$-introduction] in the calculus *LK* could be replaced by basic sequents according to the following schemata:
>
> $\phi, \psi \Rightarrow \phi \,\&\, \psi \quad \phi \vee \psi \Rightarrow \phi, \psi \quad \phi \wedge \psi \Rightarrow \phi \quad \phi \wedge \psi \Rightarrow \psi$
> $\phi \Rightarrow \phi \vee \psi \quad \psi \Rightarrow \phi \vee \psi \quad \forall x \phi x \Rightarrow \phi a \quad \phi a \Rightarrow \exists x \phi x$
> $\Rightarrow \phi, \neg\phi$ (law of excluded middle)
> $\neg\phi, \phi \Rightarrow$ (law of contradiction)
>
> These basic sequents and our inference figures may easily be shown to be equivalent. (Gentzen 1934-35, ¶2.2, with modernisation of nomenclature and changes of notation).

The propositional basic argument-forms are *exactly* the rewrites of the general introduction and general elimination rules for conjunction, disjunction, and the conditional of §1 when put in the sequent format of §3. (The sketch of a proof of the completeness theorem above goes some way towards filling in the steps in proving one half of the easily shown equivalence.)

Let us remind ourselves

- that we obtained our sequent calculus by a straight-forward rewriting of the rules of a natural deduction calculus for propositional logic that has the subformula property;
- that the rules for Gentzen's cut-free sequent calculus are derived rules of our sequent calculus;
- that there is an easy back-and-fore switch we can do between our sequent calculus and the "Stoic" form in which all basic argument-forms have a single conclusion;

- ○ that we can reread the "Stoic" formulation as a set of impure, single conclusion natural deduction elimination rules (keeping Dilemma as the sole indirect rule).

(There's one further fact that bears mention: the most natural way to transcribe general introduction and elimination rules into a sequent system yields a sequent calculus in which all operational rules are left and right *elimination* rules (see Milne 2012*b*, §6)!)

What is common to all these systems is that, if Cut is used, its uses may be constrained—in deriving Gentzen's operational rules, we cut only side formulae/minor premisses in rules governing connectives—, and, quite generally, although derivations may not contain only subformulae of premisses and conclusion of the derived argument-form, what formulae may appear can be restricted, say, perhaps, to subformulae of premisses and conclusion of the derived argument-form *and their negations*. Do we then "attach no importance to the *Hauptsatz*"? Hardly—the *Hauptsatz* provides an elegant (and vivid) way to make the broader point about *constraints* on cuts/formulae. This is made clearer when we consider the first-order case.[15]

6 The move to first order

We can readily cast the standard $\exists$-introduction rule of natural deduction in general introduction form; the standard (Gentzen) $\exists$-elimination rule is already in general elimination form. When added to the natural deduction formulation of classical propositional logic from which we started out, we have a formulation of the $\{\neg, \wedge, \vee, \rightarrow, \exists\}$-fragment of classical first-order logic *with the subformula property*.[16]

When we add the (general introduction and general elimination rewrites) of the standard rules for the universal quantifier, we no longer have the subformula property. We *do* have an optimal strengthening of the subformula results obtained, as consequences of normalization theorems, by Dag Prawitz (1965, §§3.1, 3.2, 4.1, and 4.2), for the $\vee$- and $\exists$-free fragment of classical first-order logic, and by Gunnnar Stålmarck (1991) for full classical first-order logic. We have:

> When $\Sigma \vdash \phi$ there is a proof in which, over and above subformulae of members of $\Sigma \cup \{\phi\}$, only $\neg\phi$ and negations of instances of universal generalizations that are themselves subformulae of members of $\Sigma \cup \{\phi\}$ occur. But $\neg\phi$ need not occur if ϕ itself has negation dominant, and, likewise, the negation $\neg\xi(v/c)$ of an instance of a subformula $\forall v\xi$ need not occur if ξ has negation dominant. Those non-subformulae that do occur need occur only as assumptions discharged in applications of Dilemma. If a non-subformula that is the negation $\neg\xi(v/c)$ of an instance of a subformula $\forall v\xi$ does so occur, it occurs in an application of Dilemma whose

[15] There's a parallel that would bear closer scrutiny than I have had time to give it. Inspired, on the one hand, by Smullyan's remark, quoted above, regarding the subformula principle and, on the other, by Smullyan's own demonstration of the equivalence of standard tableaux and cut-free sequent calculi (Smullyan 1968*b*), a number of authors have advocated non-standard tableaux systems which, among other things, allow for more efficient proofs—*e.g.* Boolos (1984), D'Agostino & Mondadori (1994), Cellucci (2000).

[16] As noted above, (Milne 2010) contains a model-theoretic proof; Tor Sandqvist has an unpublished constructive proof.

immediate conclusion is $\xi(v/c)$, and, likewise, if $\neg\phi$ does occur as an assumption discharged in an application of Dilemma, it is an application whose immediate conclusion is ϕ. (*cf.* Milne 2010, ¶3.5.3, Theorem 8)

One might, as I did in (Milne 2010), grub around trying to loosen up the rules for the universal quantifier in order to restore the subformula property. But it is more in line with present concerns to note that, classically, $\forall$ is dual to $\exists$ and use this fact to obtain the standard introduction and elimination rules for $\forall$. As a first step in this direction, we notice with some alarm that the sequent rewriting in the manner of §3 of the (standard and general) elimination rule for $\exists$ gives us

$$\exists x\phi x \Rightarrow \phi a,$$

(where x replaces *all* occurrences of a in ϕa) which is far from classically sound (a fact which no doubt accounts for the absence of a basic argument-form replacing left $\exists$-introduction—and likewise right $\forall$-introduction—in Gentzen's list, quoted above).

The proof-theoretic semantics tradition has its roots in the work of Gentzen, especially his (1934-35), and Prawitz's (1965), which largely followed Gentzen's style of natural deduction. In an appendix, Prawitz (1965, Appendix C, §3) discusses "rules for existential instantiation" which take their inspiration from Quine's (1950*a*) and (1950*b*).[17] Read as a rule, the sequent above is exactly a rule of existential instantiation and, as such, must be subject to constraints. In terms of splicing, I suggest that in splicing

$$\dfrac{\exists x\phi x \qquad \begin{matrix}[\phi a]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\,m$$

the formula cancelled *must* be ϕa, and in any such splicing a does not occur in any other formula featuring either hypothetically or categorically in the instance of the rule with which it is spliced. (Since the conclusion is treated as "general", *for the purposes of splicing* we do not need to say anything specific about occurrences of a in the conclusion.) Now, suppose we have

$$\Sigma, \phi a \Rightarrow \Delta,$$

where a does not occur in any formula in Σ, nor in any formula in Δ. Then we may splice with $\exists x\phi x \Rightarrow \phi a$ to obtain

$$\Sigma, \exists x\phi x \Rightarrow \Delta.$$

We have replicated Gentzen's *LK* rule for left $\exists$-introduction, with exactly Gentzen's restriction—see (Gentzen 1934-35, ¶1.22).

Dualising—*i.e.*, splicing $\neg\exists x\neg\phi x$ with negation rules—to obtain an introduction rule for $\forall$, we end up with

$$\phi a \Rightarrow \forall x\phi x,$$

[17] See (Anellis 1991) and (Pelletier 1999) for details of the decade-and-a-half long travail to perfect this approach.

where x replaces all occurrences of a in ϕa and when spliced ϕa *must* be the formula cancelled; in any such splicing a does not occur in any other formula featuring either hypothetically or categorically in the instance of the rule with which it is spliced.

Now, suppose we have

$$\Sigma \Rightarrow \phi a, \Delta,$$

where a does not occur in any formula in Σ, nor in any formula in Δ. Then we may splice with $\phi a \Rightarrow \forall x \phi x$ to obtain

$$\Sigma \Rightarrow \forall x \phi x, \Delta.$$

We have replicated Gentzen's *LK* rule for right $\forall$-introduction with exactly Gentzen's restriction—see again (Gentzen 1934-35, ¶1.22).

6.1 ε-terms

$$\exists x \phi x \Rightarrow \phi a,$$

is not sound when a is read as a standard name. We can make it sound by giving a a special reading *in context*: a stands for some one of the items satisfying ϕx, if there are any, else it is any item (in the domain of discourse). The interpretation of a is tied to the predicate ϕx and its interpretation. To mark that dependence, let us subscript a accordingly.

Working from the (classical) equivalence of $\forall x \phi x$ and $\neg \exists x \neg \phi x$, we can use splicing in order to obtain an introduction rule for $\forall x \phi x$. What we end up with is

$$\phi a_{\neg \phi x} \Rightarrow \forall x \phi x.$$

By doing the minimum to retain classical soundness at first-order, we end up with the ε-calculus rules for the existential and universal quantifiers—see (Hazen 1987).

7 Conclusion

Smullyan (1968*b*) showed how proofs with analytic tableaux and cut-free derivations in Gentzen's sequent calculus *LK* match up; Prawitz (1965) showed how normalised natural deduction proofs employing Gentzen's rules are a close but not perfect match.

If we are prepared to side with Smullyan, apparently against Gentzen, on the true significance of the *Hauptsatz*, there is merit in investigating other approaches. In particular, that the formulation of classical propositional logic employing (what I have called) general introduction and elimination rules should match Smullyan's analytic cut sequent calculus is, if only from a purely technical/practical point of view, a mark in their favour. That, as I think they are, general introduction and elimination rules are philosophically well motivated—a claim I have done little to substantiate here but address at length in (Milne 2012*b*)—is an added bonus.

In philosophy, analogies are sometimes helpful. In more formal inquiries, what is often more important is seeing that one is describing the very same phenomenon in different ways. That the first step in the direction of the confluence of ideas sketched above should have been taken by the earliest investigators of propositional logic is a remarkable coincidence.

References

Anellis, I. H. (1991). 'Forty years of "unnatural" natural deduction and quantification: A history of first-order systems of natural deduction, from Gentzen to Copi'. *Modern Logic*, **2**: 113–152. See also (Anellis 1992).

——— (1992). 'Leblanc correction'. *Modern Logic*, **3**: 98. Correction to (Anellis 1991).

Boolos, G. (1984). 'Don't eliminate cut'. *Journal of Philosophical Logic*, **13**: 373–378.

Boričić, B. R. (1985). 'On sequence-conclusion natural deduction systems'. *Journal of Philosophical Logic*, **14**: 359–377.

Cellucci, C. (2000). 'Analytic cut trees'. *Logic Journal of the IGPL*, **8**: 733–750.

Corcoran, J. (1974). 'Remarks on Stoic deduction'. In *Ancient Logic and Its Modern Interpretations: Proceedings of the Buffalo Symposium on Modernist Interpretations of Ancient Logic, 21 and 22 April, 1972*, edited by J. Corcoran, *Synthese Historical Library*, vol. 9. Reidel, Dordrecht, pp. 169–181.

D'Agostino, M. & M. Mondadori (1994). 'The taming of the cut. Classical refutations with analytic cut'. *Journal of Logic and Computation*, **4**: 285–319.

Dummett, M. A. E. (1981). *Frege: Philosophy of Language*. Duckworth, London, second edn. First edition, 1973.

——— (1990). 'The source of the concept of truth'. In *Meaning and Method: Essays in Honor of Hilary Putnam*, edited by G. Boolos. Cambridge University Press, Cambridge, pp. 1–15.

——— (1991). *The Logical Basis of Metaphysics*. Duckworth, London.

Francez, N. & R. Dyckhoff (2011). 'A note on harmony'. *Journal of Philosophical Logic*. DOI 10.1007/s10992-011-9208-0.

Frede, M. (1974). *Die Stoische Logik*, *Abhandlungen der Akademie der Wissenschaften in Göttingen. Philologisch-Historische Klasse. Dritte Folge.*, vol. 88. Vandenhoeck & Ruprecht, Göttingen,.

Gentzen, G. (1934-35). 'Untersuchungen über das logische Schließen'. *Mathematische Zeitschrift*, **39**: 176–210, 405–31. English translation by Manfred Szabo *as* 'Investigations into Logical Deduction', *American Philosophical Quarterly*, **1**, 1964, 288-306, **2**, 1965, 204-218; *also in* M.E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, 1969, pp. 68-131.

——— (1936). 'Die Widerspruchfreiheit der reinen Zahlentheorie'. *Mathematische Annalen*, **112**: 493–565. English translation by Manfred Szabo *as* The Consistency of Elementary Number Theory *in* M.E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, 1969, pp. 132-201.

Hailperin, T. (1996). *Sentential Probability Logic: Origins, Development, Current Status, and Technical Applications*. Lehigh University Press, Bethlehem, PA.

Hazen, A. (1987). 'Natural deduction and Hilbert's ε-operator'. *Journal of Philosophical Logic*, **16**: 411–421.

Hitchcock, D. (2005). 'The peculiarities of Stoic logic'. In *Mistakes of Reason: Essays in Honour of John Woods*, edited by K. A. Peacock & A. D. Irvine, chap. 13. Toronto Studies in Philosophy, University of Toronto Press, Toronto, pp. 224–242.

Lemmon, E. (1965). *Beginning Logic*. Nelson, London. Second edition, Chapman Hall, London, 1987.

Martin, C. J. (1986). 'William's machine'. *Journal of Philosophy*, **83**: 564–572.

Mates, B. (1961). *Stoic Logic*. University of California Press, Berkeley and Los Angeles, second edn. First edition, 1953.

Mill, J. S. (1843). *A System of Logic, Ratiocinative and Inductive, being a connected view of the principles of evidence, and the methods of scientific investigation*, vol. I. John W. Parker, London, first edn. Published in two volumes.

——— (1846). *A System of Logic, Ratiocinative and Inductive, being a connected view of the principles of evidence, and the methods of scientific investigation*. Harper & Brothers, New York, first edn. Published in a single volume.

Milne, P. (1995). 'On the completeness of non-Philonian Stoic logic'. *History and Philosophy of Logic*, **16**: 39–64.

——— (2004). 'Algebras of intervals and a logic of conditional assertions'. *Journal of Philosophical Logic*, **33**: 497–548.

——— (2008). 'A formulation of first-order classical logic in natural deduction with the subformula property'. In *The Logica Yearbook 2007*, edited by M. Peliš. Filosofia, Institue of Philosophy, Czech Academy of Sciences, Prague, pp. 97–110.

——— (2010). 'Subformula and separation properties in natural deduction via small Kripke models'. *Review of Symbolic Logic*, **3**: 175–227.

——— (2012*a*). 'Completeness theorems for stoic logic'. Submitted for publication.

——— (2012*b*). 'Inversion principles and introduction rules'. To appear in a Festschrift for Dag Prawitz edited by Heinrich Wansing.

Mueller, I. (1978). 'An introduction to Stoic logic'. In *The Stoics*, edited by J. M. Rist. University of California Press, Berkeley and Los Angeles, pp. 1–26.

——— (1979). 'The completeness of Stoic logic'. *Notre Dame Journal of Formal Logic*, **20**: 201–215.

Negri, S. & J. von Plato (2001). *Structural Proof Theory*. Cambridge University Press, Cambridge. With an appendix by Arne Ranta.

Pelletier, F. J. (1999). 'A brief history of natural deduction'. *History and Philosophy of Logic*, **20**: 1–31.

von Plato, J. (2001). 'Natural deduction with general elimination rules'. *Archive for Mathematical Logic*, **40**: 541–567.

Prawitz, D. (1965). *Natural Deduction: A Proof-Theoretical Study*. Almqvist & Wiksell, Stockholm. Reprinted with new preface, Mineola, NY: Dover Publications, 2006.

Priest, G. (1979). 'The logic of paradox'. *Journal of Philosophical Logic*, **8**: 219–241.

Quine, W. V. (1950*a*). 'On natural deduction'. *Journal of Symbolic Logic*, **15**: 93–102.

Quine, W. V. O. (1950*b*). *Methods of Logic*. Holt, New York. Third edition, Routledge and Kegan Paul, London, 1980.

Read, S. (1988). *Relevant Logic*. Basil Blackwell, Oxford.

——— (1994). *Thinking About Logic*. Oxford University Press, Oxford.

——— (2000). 'Harmony and autonomy in classical logic'. *Journal of Philosophical Logic*, **29**: 123–154.

——— (2004). 'Identity and harmony'. *Analysis*, **64**: 113–119.

——— (2010). 'General-elimination harmony and the meaning of the logical constants'. *Journal of Philosophical Logic*, **39**: 557–576.

Schroeder-Heister, P. (1984*a*). 'A natural extension of natural deduction'. *Journal of Symbolic Logic*, **49**: 1284–1300.

——— (1984*b*). 'Generalized rules for quantifiers and the completeness of the intuitionistic operators &, $\vee$, $\rightarrow$, $\curlywedge$, $\forall$, $\exists$'. In *Computation and Proof Theory. Proceedings of the Logic Colloquium held in Aachen, July 18-23, 1983, Part II*, edited by M. Richter, E. Börger, W. Oberschelp, B. Schinzel, & W. Thomas, *Lecture Notes in Mathematics*, vol. 1104. Springer, Berlin/Heidelberg/New York/Tokyo, pp. 399–426.

Smullyan, R. M. (1968*a*). 'Analytic cut'. *Journal of Symbolic Logic*, **33**: 560–564.

——— (1968*b*). *First-Order Logic*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 43. Springer, New York. Reprinted with new preface, New York: Dover Publications, 1995.

Stålmarck, G. (1991). 'Normalization theorems for full first order classical natural deduction'. *Journal of Symbolic Logic*, **56**: 129–149.

Tennant, N. (1992). *Autologic*. Edinburgh University Press, Edinburgh.