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Star complements and exceptional graphs ☆

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Abstract

Let G be a finite graph of order n with an eigenvalue μ of multiplicity k. (Thus the μ-eigenspace of a (0, 1)-adjacency matrix of G has dimension k.) A *star complement* for μ in G is an induced subgraph $G - X$ of G such that $|X| = k$ and $G - X$ does not have μ as an eigenvalue. An *exceptional* graph is a connected graph, other than a generalized line graph, whose eigenvalues lie in $[-2, \infty)$. We establish some properties of star complements, and of eigenvectors, of exceptional graphs with least eigenvalue -2.

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1. Introduction

Let G be a finite graph of order n with an eigenvalue μ of multiplicity k. (Thus the corresponding eigenspace of a (0, 1)-adjacency matrix of G has dimension k.) A *star set* for μ in G is a set X of k vertices in G such that the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G (or in [15] a μ*-basic subgraph* of

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G). Star sets and star complements exist for any eigenvalue of any graph, and serve to explain the relation between graph structure and a single eigenvalue μ. When μ is not -1 or 0, they can be used to determine sharp upper bounds for k in arbitrary graphs and in regular graphs [2]; to characterize certain graphs (see for example [12]); and to find all the exceptional graphs (i.e. connected graphs with all eigenvalues at least -2 that are not generalized line graphs) [9]. There are also connections with dominating properties [10, Section 7.6] and independent sets [19]. For a recent survey, see [18] and for basic properties, see [13, Chapter 5]. Here we investigate properties of star sets and star complements related to graphs with least eigenvalue -2, and explain some phenomena observed from earlier computer results. Explicitly, we give a simple computer-free proof that each exceptional graph with least eigenvalue greater than -2 is an induced subgraph of an exceptional graph with least eigenvalue equal to -2; we show how extendability graphs [13, Section 5.1] can be used to investigate the regular exceptional graphs; and we establish a property of eigenvectors of exceptional graphs with -2 as a simple eigenvalue.

The following result [13, Theorem 5.1.7] establishes the fundamental property of star complements: if X is a star set for μ in G, and if H is the star complement $G - X$, then G is determined by μ, H and the H-neighbourhoods of vertices in X.

Theorem 1.1. *Let X be a set of k vertices in the graph G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^{\mathrm{T}} \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^{\mathrm{T}}(\mu I - C)^{-1}B. \tag{1}$$

In this situation, the eigenspace of μ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$, where $\mathbf{x} \in \mathbb{R}^k$.

Recall that μ is a *main* eigenvalue of G if the eigenspace $\mathscr{E}(\mu)$ is not orthogonal to the all-1 vector $\mathbf{j}_n$. In Section 2, we discuss the addition of a vertex to an exceptional star complement for -2 to obtain -2 as a main eigenvalue, and the addition of a star set to obtain -2 as a non-main eigenvalue. In Section 3 we discuss integral eigenvectors of exceptional graphs having -2 as a simple eigenvalue.

2. Eigenvalues of exceptional graphs

We denote the least eigenvalue of a graph G by $\lambda(G)$. Let $\mathscr{H}$ denote the family of 443 exceptional graphs of order 8 with $\lambda(G) > -2$. These graphs were found by Doob and Cvetković [14] in 1979, and are listed as $H001, \ldots, H443$ in [13, Table A2]. The 473 maximal exceptional graphs were found by computer in 1999, as described in [9], and from these calculations we know that each graph in $\mathscr{H}$ arises as a star complement for -2. We begin by verifying this observation theoretically; the desirability of a computer-free proof was noted in [1, p. 17]. Since any exceptional graph G with $\lambda(G) > -2$ is an induced subgraph of a graph in $\mathscr{H}$, one consequence of the result is that no exceptional graph G with $\lambda(G) > -2$ is a maximal exceptional graph.

Proposition 2.1. *If $H \in \mathscr{H}$ then H has a one-vertex extension H' with $\lambda(H') = -2$.*

Proof. By [13, Theorem 2.3.19], H has an exceptional induced subgraph of order 6, and so there exists a vertex u of H such that $H - u$ is exceptional. In the terminology of [13, Section 3.7], H generates the root system E_8, while $H - u$ generates E_7. If the graph G generates E_k ($k \in \{6, 7, 8\}$)

then its adjacency matrix has the form $Q^{\mathrm{T}}Q - 2I$, where $Q^{\mathrm{T}}Q$ is the Gram matrix of an integral basis for the integral lattice $L(E_k)$ generated by E_k. Now the determinant of such a Gram matrix is a lattice invariant called the *discriminant* of $L(E_k)$, shown in [3, Section 3.10] to be 1 when $k = 8$, 2 when $k = 7$, and 3 when $k = 6$. Thus $P_G(-2) = \det(-Q^{\mathrm{T}}Q) = (-1)^k(9-k)$ (cf. [7, Theorem 3], [13, Lemma 7.5.2]).

Now let H' be the graph obtained from H by attaching a pendant vertex at u. From [6, Theorem 2.11], the characteristic polynomial of H' is given by

$$P_{H'}(x) = xP_H(x) - P_{H-u}(x). \tag{2}$$

In view of the preceding remarks, we have $P_H(-2) = 1$ and $P_{H-u}(-2) = -2$, and so from (2) we obtain $P_{H'}(-2) = 0$. By the Interlacing Theorem [13, Theorem 1.2.21], $\lambda(H') = -2$. $\square$

In the foregoing proof, the appeal to the theory of lattices can be avoided by arguing as follows. If A is the adjacency matrix of H then $A + 2I = Q^{\mathrm{T}}Q$, where each column of the invertible 8×8 matrix Q lies in E_8. The seven columns of Q corresponding to $H - u$ lie in a subsystem E_7 which consists of the vectors in E_8 orthogonal to a fixed vector $\mathbf{b}$ of E_8. Since $E_8^{\perp} = \{\mathbf{0}\}$, $\mathbf{b}$ is not orthogonal to the remaining column $\mathbf{q}$ of Q. Replacing $\mathbf{b}$ with $-\mathbf{b}$ if necessary, we may assume that $\mathbf{b} \cdot \mathbf{q} = 1$. Now let $R = [Q|\mathbf{b}]$. Then $R^{\mathrm{T}}R = A' + 2I$, where A' is the adjacency matrix of H'. Since $R^{\mathrm{T}}R$ has rank 8, -2 is an eigenvalue of H', and by the Interlacing Theorem, $\lambda(H') = -2$ as required.

Given a representation of H' in E_8, it can be shown by the methods of [11, Section 2] that H' has a one-vertex extension for which -2 is an eigenvalue of multiplicity 2.

The graph H' has H as a star complement for -2, but the eigenvalue -2 of H' may or may not be a main eigenvalue. If $\mathbf{v}$ is an eigenvector of H' corresponding to -2 then $R\mathbf{v} = \mathbf{0}$, while -2 is a main eigenvalue if and only if $\mathbf{v} \cdot \mathbf{j}_9 \neq 0$. Let R' be the matrix obtained from R by adding $\mathbf{j}_9^{\mathrm{T}}$ as a ninth row. Then -2 is a non-main eigenvalue if and only if $R'\mathbf{v} = \mathbf{0}$, equivalently $\mathbf{j}_9$ lies in the column space of R^{T}.

For an investigation of main and non-main eigenvalues, we use the notation of Section 1 with $t = n - k$ and $\mathbf{j} = \mathbf{j}_t$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ be the standard orthonormal basis of $\mathbb{R}^k$, and define a bilinear form on $\mathbb{R}^t$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}(\mu I - C)^{-1}\mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^t).$$

By Theorem 1.1, μ is a non-main eigenvalue of G if and only if $\mathbf{j}_n$ is orthogonal to the vectors $\begin{pmatrix} \mathbf{e}_i \\ (\mu I - C)^{-1}B\mathbf{e}_i \end{pmatrix}$ $(i = 1, \ldots, k)$. Since $B\mathbf{e}_i$ is the ith column of B, μ is a non-main eigenvalue of G if and only if $\langle \mathbf{b}, \mathbf{j} \rangle = -1$ for each column $\mathbf{b}$ of B. The computer calculations described in [9] show that each graph H in $\mathscr{H}$ arises as a star complement for -2 in a graph for which -2 is a *main* eigenvalue [5, Theorem 11]. It follows from the foregoing remarks that each such graph H has a one-vertex extension for which -2 is a main eigenvalue; in other words, there exists a column $\mathbf{b}$ such that $\langle \mathbf{b}, \mathbf{b} \rangle = -2$ and $\langle \mathbf{b}, \mathbf{j} \rangle \neq -1$. This fact has not yet been established theoretically.

The *extendability graph* $\Gamma(H; \mu)$ [13, p. 121] has as vertices the (0, 1)-vectors $\mathbf{b} \in \mathbb{R}^t$ such that $\langle \mathbf{b}, \mathbf{b} \rangle = \mu$, with an edge between $\mathbf{b}$ and $\mathbf{b}'$ if and only if $\langle \mathbf{b}, \mathbf{b}' \rangle \in \{-1, 0\}$. A clique on $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ in $\Gamma(H; \mu)$ determines a graph G with H as a star complement for μ: in the notation of Theorem 1.1, $H = G - X$ where $X = \{1, 2, \ldots, k\}$, $B = [\mathbf{b}_1|\mathbf{b}_2|\cdots|\mathbf{b}_k]$, and vertices i, j of X are adjacent if and only if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = -1$. We may define the *non-main extendability graph* $\Gamma^*(H; \mu)$ as the subgraph of $\Gamma(H; \mu)$ induced by those (0, 1)-vectors $\mathbf{b}$ for which $\langle \mathbf{b}, \mathbf{j} \rangle = -1$.

Proposition 2.2. *Let $H \in \mathscr{H}$ and suppose that the cone $K_1 \nabla H$ has -2 as an eigenvalue. Then $\overline{\Gamma^*(H;-2)}$ has a perfect matching, say $\mathbf{b}_1\mathbf{c}_1, \ldots, \mathbf{b}_m\mathbf{c}_m$, with $\mathbf{b}_i + \mathbf{c}_i = \mathbf{j}$ $(i = 1, \ldots, m)$. Moreover the following hold.*

(i) *If $\Gamma^*(H;-2)$ has a clique of order m then $\Gamma^*(H;-2)$ is a cocktail-party graph $CP(2m) = \overline{mK_2}$. In this situation every maximal clique has order m, there are 2^m maximal cliques, and the 2^m corresponding graphs with H as a star complement for -2 are switching-equivalent.*
(ii) *$m \leqslant 20$ and if G has H as a star complement for -2 as a non-main eigenvalue then G is switching-equivalent to an induced subgraph of $L(K_8)$.*

Proof. Since $K_1 \nabla H$ has -2 as an eigenvalue, we have $\langle \mathbf{j}, \mathbf{j} \rangle = -2$. Hence, for any vertex $\mathbf{b}$ of $\Gamma^*(H;-2)$ we have $\langle \mathbf{j}-\mathbf{b}, \mathbf{j}-\mathbf{b} \rangle = -2$, so that $\mathbf{j}-\mathbf{b}$ is a vertex of $\Gamma(H;-2)$. In addition we have $\langle \mathbf{j}-\mathbf{b}, \mathbf{j} \rangle = -1$, and so $\mathbf{j}-\mathbf{b}$ is a vertex of $\Gamma^*(H;-2)$. Since $\langle \mathbf{j}-\mathbf{b}, \mathbf{b} \rangle = 1$, it follows that $\mathbf{b}$ and $\mathbf{j}-\mathbf{b}$ are non-adjacent vertices of $\Gamma^*(H;-2)$, and hence that $\overline{\Gamma^*(H;-2)}$ has a perfect matching with the property claimed.

For (i), note first that a clique of order m in $\Gamma^*(H;-2)$ has precisely one vertex from each pair $\{\mathbf{b}_i, \mathbf{c}_i\}$; without loss of generality, $\mathbf{b}_1, \ldots, \mathbf{b}_m$ induce a clique K. Secondly, note that the map $\mathbf{b} \mapsto \mathbf{j}-\mathbf{b}$ is an isomorphism from the graph on $\mathbf{b}_1, \ldots, \mathbf{b}_m$ to the graph on $\mathbf{c}_1, \ldots, \mathbf{c}_m$, and so $\mathbf{c}_1, \ldots, \mathbf{c}_m$ also induce a clique. Thirdly, if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ then $\langle \mathbf{b}_i, \mathbf{c}_j \rangle = -1$, while if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = -1$ then $\langle \mathbf{b}_i, \mathbf{c}_j \rangle = 0$. Thus $\Gamma^*(H;-2) \cong CP(2m)$. Hence there are 2^m cliques of order m, each obtained from K by replacing $\mathbf{b}_i$ with $\mathbf{c}_i$ for all i in some subset of $\{1, \ldots, m\}$. As noted in [13, Section 5.5], the corresponding graphs with H as a star complement for -2 are switching-equivalent.

For (ii), note that we may add to G a vertex v adjacent to every vertex of G to obtain a cone $K_1 \nabla G$ which has H as a star complement for -2. Now we argue as in [13, Proposition 6.2.1]: by choosing a suitable representation of $K_1 \nabla G$ in the root system E_8, we see that G is switching-equivalent to an induced subgraph of $L(K_8)$. In particular, v can have at most 28 neighbours and so $m \leqslant 20$. □

Example 2.3. Let H consist of disjoint cycles of lengths 3 and 5 together with a bridge between them, i.e. H is the exceptional graph $H010$. It is straightforward to verify that $K_1 \nabla H$ has -2 as an eigenvalue. Moreover, H is an induced subgraph of a Chang graph G; and since G is regular of order 28, $\Gamma^*(H;-2)$ has a clique of order 20 (necessarily maximal by Proposition 2.2(ii)). By Proposition 2.2(i), $\Gamma^*(H;-2) \cong CP(40)$ and any maximal graph having H as a star complement for -2 as a non-main eigenvalue is switching-equivalent to G, hence to $L(K_8)$.

The arguments of Example 2.3 apply whenever (i) $\langle \mathbf{j}, \mathbf{j} \rangle = -2$, and (ii) H is a star complement for -2 in a graph of order 28 with -2 as a non-main eigenvalue. M. Lepović (private communication) has verified by computer that exactly 198 of the 443 graphs $H \in \mathscr{H}$ satisfy condition (i), i.e. are such that the cone $K_1 \nabla H$ has -2 as an eigenvalue. The computer investigations reported in [8] show that all but one ($H434$) are star complements for -2 in some $K_1 \nabla G$, where G is a Chang graph. Moreover 172 of the remaining 197 graphs have maximal degree less than 7, hence are induced subgraphs of a Chang graph G. Thus for each of these 172 graphs H we have $\Gamma^*(H;-2) \cong CP(40)$. The same holds when H is the graph $H440$ (with maximal degree 7): in this case, there are many non-isomorphic graphs among the corresponding 2^{20} graphs of order 28 that have -2 as a non-main eigenvalue. (In [5, Section 6] it was asserted wrongly that only one such graph of order 28 exists; however Theorem 11 of [5] remains valid.)

In the next example, we construct $\Gamma^*(H; -2)$ explicitly when H is $H443$, another graph in $\mathscr{H}$ with maximal degree 7. In this case we do not have prior knowledge of a graph of order 28 with -2 as a non-main eigenvalue. The calculations show that there is no regular graph with $H443$ as a star complement for -2.

Example 2.4. Let H be the complement of $K_{1,2} \dot\cup 5K_1$, i.e. H is the exceptional graph $H443$ which features as a versatile star complement for -2 in [13, Section 6.3]. A vertex u in X is said to be of type abc if its H-neighbourhood $\Delta_H(u)$ consists of a, b, c vertices of degree 5, 6, 7 respectively. Let C be the adjacency matrix of H, with vertices labelled so that their degrees are in non-decreasing order. To use Theorem 1.1 with $\mu = -2$, note that

$$(2I_8 + C)^{-1} = \begin{pmatrix} 8 & 5\mathbf{j}_2^{\mathrm{T}} & -3\mathbf{j}_5^{\mathrm{T}} \\ 5\mathbf{j}_2 & 3J_{2,2} + I_2 & -2J_{2,5} \\ -3\mathbf{j}_5 & -2J_{5,2} & J_{5,5} + I_5 \end{pmatrix},$$

where I_m denotes the $m \times m$ identity matrix and $J_{m,n}$ denotes the all-1 matrix of size $m \times n$. If we now equate diagonal entries in Eq. (1), we obtain

$$2 = a + b + c + 7a^2 + 10ab - 6ac - 4bc + 3b^2 + c^2. \tag{3}$$

If u, v are distinct vertices of types $a_1b_1c_1$, $a_2b_2c_2$ then, equating off-diagonal entries in Eq. (1), we have

$$\begin{aligned} a_{uv} = {} & |\Delta_H(u) \cap \Delta_H(v)| + 7a_1a_2 + 3b_1b_2 + c_1c_2 + 5(a_1b_2 + a_2b_1) \\ & - 3(a_1c_2 + a_2c_1) - 2(b_1c_2 + b_2c_1). \end{aligned} \tag{4}$$

The 10 solutions of Eq. (3) are given in [13, Table 1, p. 148] along with information from Eq. (4) sufficient to construct $\Gamma(H; -2)$. Since $(a, b, c) = (1, 2, 5)$ is one solution of (3), $\mathbf{j}$ arises as a vertex and its neighbours include the vertices of $\Gamma^*(H; -2)$. The neighbours $\mathbf{b}$ of $\mathbf{j}$ for which $\langle \mathbf{b}, \mathbf{j} \rangle = -1$ can be identified from Eq. (4). They correspond to ten vertices in X of type 011, ten of type 114, ten of type 023 and ten of type 102. (Note that if $\mathbf{b}$ is of type 011 or 023 then $\mathbf{j} - \mathbf{b}$ is of type 114 or 102, respectively.) We deduce that again $\Gamma^*(H; -2) \cong CP(40)$, and so we obtain 2^{20} maximal graphs with H as a star complement for -2 as a non-main eigenvalue. M. Lepović (private communication) has shown by computer that 356 non-isomorphic graphs arise in this way. We note in passing that the cones over H and any of these graphs of order 28 not only have -2 as a main eigenvalue but also have K_8 (a subgraph of $K_1 \nabla H$) as a star complement for -2.

None of the graphs of order 28 here is regular; indeed we can show as follows that there is no regular graph G with H as a star complement for -2. For suppose that $H = G - X$ where G is r-regular and the star set X consists of e, f, g, h vertices of type 102, 114, 023, 011 respectively. Let $e + g = p$, $f + h = q$ and note that $0 \leqslant p \leqslant 10$, $0 \leqslant q \leqslant 10$. Counting edges between X and vertices in H of degree 5, 6, 7 in turn, we have:

$$r = 5 + e + f, \quad 2r = 12 + f + 2g + h, \quad 5r = 35 + 2e + 4f + 3g + h.$$

We can now write $e = p - g$, $h = q - f$ and solve these equations for r, f, g in terms of p and q. We find that $f = \frac{1}{2}(q - 14)$, a contradiction.

We know that every exceptional graph G with least eigenvalue -2 has an exceptional star complement H for -2 [13, Theorem 5.31]. We shall see that if G is regular and H has order 8 then H satisfies the hypotheses of Proposition 2.2.

Lemma 2.5. *Let G be an r-regular graph of order n, and let μ be an eigenvalue of G other than r. If C is an adjacency matrix of any star complement for μ, then*

$$\mathbf{j}^{\mathrm{T}}(\mu I - C)^{-1}\mathbf{j} = \frac{n}{\mu - r}.$$

Proof. In the notation of Theorem 1.1, let $S = (B|C - \mu I)$. Then Eq. (1) may be written

$$\mu I - A = S^{\mathrm{T}}(\mu I - C)^{-1}S.$$

Now $S\mathbf{j}_n = (r-\mu)\mathbf{j}$ and so $\mathbf{j}_n^{\mathrm{T}}(\mu I - A)\mathbf{j}_n = (r-\mu)^2\mathbf{j}^{\mathrm{T}}(\mu I - C)^{-1}\mathbf{j}$. The result follows since $\mathbf{j}_n^{\mathrm{T}}(\mu I - A)\mathbf{j}_n = \mu n - rn$. □

Now we apply Lemma 2.5 in the case that G is an exceptional regular graph, $\mu = -2$, and C is the adjacency matrix of an exceptional star complement H for -2. Then H has order $k \in \{6, 7, 8\}$ and $A + 2I = M^{\mathrm{T}}M$ for some $k \times n$ matrix M of rank k: the columns of M are a representation of G in the root system E_t (cf. [13, Chapter 3]). Let $\mathbf{u} = \frac{1}{r+2}M\mathbf{j}$, so that $\mathbf{u}^{\mathrm{T}}\mathbf{u} = \frac{n}{r+2}$. From the proof of [13, Theorem 4.1.5] we know that $\mathbf{u}^{\mathrm{T}}\mathbf{u} \in \left\{2, \frac{4}{3}\right\}$ if $t = 6$, $\mathbf{u}^{\mathrm{T}}\mathbf{u} \in \left\{2, \frac{3}{2}\right\}$ if $t = 7$ and $\mathbf{u}^{\mathrm{T}}\mathbf{u} = 2$ if $t = 8$. Thus necessarily $n = 2(r+2)$ when $k = 8$; in this case we have $\langle \mathbf{j}, \mathbf{j}\rangle = -2$ by Lemma 2.5, and so H satisfies the hypotheses of Proposition 2.2. Accordingly, we have the following result.

Theorem 2.6. *Let G be an exceptional regular graph with least eigenvalue -2, having H as an exceptional star complement for -2. If H has order 8, then*

(i) *H is one of 198 graphs in $\mathscr{H}$,*
(ii) *both $K_1 \nabla H$ and $K_1 \nabla G$ have -2 as a main eigenvalue,*
(iii) *G is switching-equivalent to an induced subgraph of $L(K_8)$.*

Example 2.7. Consider a cone $K_1 \nabla G$ such that some star complement H for -2 in G is also a star complement for -2 in $K_1 \nabla G$; for example, 430 of the 432 maximal exceptional graphs of order 29 have this property (see [13, Section 6.1]. By [10, Eq. (4.3.7)], -2 is a non-main eigenvalue of G, a fact we can also establish as follows. The extendability graph $\Gamma(H; -2)$ has $\mathbf{j}$ as a vertex such that $\langle \mathbf{b}, \mathbf{j}\rangle = -1$ for all other vertices $\mathbf{b}$; thus deletion of the vertex of the cone leaves G as a graph in which -2 is a non-main eigenvalue. Moreover (cf. [13, Section 5.5]) all vertices of G outside H are amenable to switching, and any switching yields another graph G' with -2 as a non-main eigenvalue. If $K_1 \nabla G'$ is a maximal exceptional graph, then G' is a maximal graph with H as a star complement for -2 as a non-main eigenvalue.

Further remarks and examples may be found in [5, Section 6].

3. Eigenvectors of exceptional graphs

In this section we discuss exceptional graphs with -2 as a simple eigenvalue. Such graphs have a star complement for -2 of order 6, 7 or 8, and the eigenvectors corresponding to -2 are all scalar multiples of an eigenvector $\mathbf{v}$ whose entries are integers. If $\mathbf{v}$ is chosen with minimal norm, then $\mathbf{v}$ is called a *minimal integral eigenvector*, and its *height* is the maximum modulus of its coordinates. We establish theoretically a property of heights noted from computer results given in [5].

First we give a short proof of the following theorem, established in [17] in a chemical context, and generalized in [16].

Theorem 3.1. *Let λ be a simple eigenvalue of the graph G. Then there exists an eigenvector $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ corresponding to λ such that*

$$x_j^2 = |P_{G-j}(\lambda)| \quad (j = 1, \ldots, n).$$

Proof. Let $\mu_1, \ldots, \mu_m$ be the distinct eigenvalues of G, so that [10, Section 4.2]

$$P_{G-j}(x) = P_G(x) \sum_{i=1}^{m} \frac{\|P_i \mathbf{e}_j\|^2}{x - \mu_i} \quad (j = 1, \ldots, n),$$

where P_i is the orthogonal projection of $\mathbb{R}^n$ onto $\mathscr{E}(\mu_i)$ and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Now suppose that $\lambda = \mu_h$, so that

$$\|P_h \mathbf{e}_j\|^2 = \frac{P_{G-j}(\lambda)}{P'_G(\lambda)} \quad (j = 1, \ldots, n).$$

On the other hand, $\|P_h \mathbf{e}_j\|^2 = \mathbf{e}_j^{\mathrm{T}} P_h \mathbf{e}_j$, the (j, j)-entry of P_h; and if $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{T}}$ is a unit eigenvector which spans $\mathscr{E}(\lambda)$ then $P_h = \mathbf{u}\mathbf{u}^{\mathrm{T}}$, with (j, j)-entry u_j^2. The result follows by defining

$$x_j = \sqrt{|P'_G(\lambda)|}\, u_j \quad (j = 1, \ldots, n). \qquad \square$$

Following [7], we define the *discriminant* d_G of a graph G with $\lambda(G) \geqslant -2$ as $(-1)^n P_G(-2)$; and for $k = 6, 7, 8$, we define $\mathscr{H}_k$ as the set of exceptional graphs on k vertices with $\lambda(G) > -2$. (Thus $\mathscr{H}_8 = \mathscr{H}$.) As we saw in the proof of Proposition 2.1, if G belongs to $\mathscr{H}_k$ then $d_G = 9 - k$.

Let $\mathscr{H}_k^*$ be the set of graphs which have -2 as a simple eigenvalue and a graph in $\mathscr{H}_k$ as a star complement for -2. (It is noted in [5] that $|\mathscr{H}_6^*| = 51$, $|\mathscr{H}_7^*| = 512$ and $|\mathscr{H}_8^*| = 4206$.)

Corollary 3.2. *If G belongs to $\mathscr{H}_k^*$, then G has an integral minimal eigenvector corresponding to -2 with a coordinate equal to* 1.

Proof. By Theorem 3.1, the simple eigenvalue -2 of G has an eigenvector $\mathbf{x}$ such that $|x_i| = \sqrt{d_{G-i}} (i = 1, \ldots, n)$. For some i the subgraph $G - i$ is an exceptional star complement for -2, so that (replacing $\mathbf{x}$ with $-\mathbf{x}$ if necessary) we have $x_i = \sqrt{9-k}$. Hence $\mathbf{x} = \sqrt{9-k}\mathbf{x}'$, where $\mathbf{x}' = (x'_1, \ldots, x'_{k+1})^{\mathrm{T}}$, and each x'_j is rational. Now $\sqrt{9-k} \in \{1, \sqrt{2}, \sqrt{3}\}$, while each x_j^2 is an integer, and so each x'_j is an integer. Since also $x'_i = 1$, $\mathbf{x}'$ is an eigenvector satisfying the conclusions of the Corollary. $\square$

Corollary 3.2 confirms an empirical observation from computer calculations of eigenvectors reported in [4]. We saw in the proof that $x_j = \sqrt{9-k}x'_j$ where x'_j is an integer $(j = 1, \ldots, k+1)$, and so we can also deduce the following result from Theorem 3.1.

Corollary 3.3. *If G belongs to $\mathscr{H}_k^*$, then each $|P_{G-j}(-2)|$ $(j = 1, \ldots, k+1)$ is of the form $(9-k)s^2$ where s is an integer.*

Proposition 3.4. *If G belongs to $\mathscr{H}_k^*$ then the height of an integral minimal eigenvector is less than or equal to* 3, 4, 6 *for $k =$* 6, 7, 8 *respectively.*

Proof. If G has an induced subgraph K isomorphic to $K_{1,4}$ then for each vertex j outside K, $G - j$ has -2 as an eigenvalue (by interlacing) and so the jth entry of an integral minimal eigenvector $\mathbf{x} = (x_1, \ldots, x_{k+1})^{\mathrm{T}}$ is zero. Since 1 and -2, or -1 and 2, are the components of an integral minimal eigenvector of $K_{1,4}$, it follows that $\mathbf{x}$ has coordinates 0, 1, -2 or 0, -1, 2. Accordingly suppose that G has no induced $K_{1,4}$. Then each $G - i$ has at most three components. When $x_i \neq 0$ we find an upper bound for d_{G-i} as the product of the discriminants of the possible components. The possible values of the discriminant of a component of order t here are: $9 - t$ ($t = 6, 7, 8$) for an exceptional graph, 4 for the line graphs of odd unicyclic graphs or line graphs of trees with one petal, and $t + 1$ for the line graphs of trees (cf. [7, Theorem 3]). By considering all distributions of the vertices of $G - i$ among at most three components we find easily that d_{G-i} is at most 27, 36 or 48 depending on whether the order of G is 7, 8 or 9. (For example, 48 is the product of discriminants of line graphs of orders 2, 3 and 3.) In the notation of Corollary 3.3, we have $3s^2 \leqslant 27$ when $k = 6$, $2s^2 \leqslant 36$ when $k = 7$, and $s^2 \leqslant 48$ when $k = 8$. Thus the height of $\mathbf{x}$ is at most 3, 4 or 6 respectively. □

The bounds in Proposition 3.4 are attained by the exceptional Smith graphs (cf. [13, Section 3.4]). See [5, Table 1] for additional data (obtained by computer) on the heights of eigenvectors.

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