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STAR COMPLEMENTS AND CONNECTIVITY IN FINITE GRAPHS

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Abstract

Let G be a finite graph with H as a star complement for an eigenvalue other than 0 or -1. Let $\kappa(G)$, $\delta(G)$ denote respectively the vertex-connectivity and minimum degree of G. We prove that $\kappa(G)$ is controlled by $\delta(G)$ and $\kappa(H)$. In particular, for each $k \in \mathbb{N}$ there exists a smallest non-negative integer $f(k)$ such that $\kappa(G) \geq k$ whenever $\kappa(H) \geq k$ and $\delta(G) \geq f(k)$. We show that $f(1) = 0$, $f(2) = 2$, $f(3) = 3$, $f(4) = 5$ and $f(5) = 7$.

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1 Introduction

Let G be a finite simple graph of order n with μ as an eigenvalue of multiplicity k. (Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix A of G has dimension k.) A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. We use the notation of [7], where the fundamental properties of star sets and star complements are established in Chapter 5.

It is well known that if $\mu \neq -1$ or 0 and $n > 4$ then $|X| \leq \binom{n-k}{2}$ [1]; in particular, there are only finitely many graphs with a prescribed star complement H for some eigenvalue other than 0 or -1. Certain graphs can be characterized by a star complement: for surveys, see [9] and [11]. More generally, it is of interest to investigate properties of H that are reflected in G: connectedness is one such property, as noted in [8, Section 2]. Here we discuss k-connectedness for $k > 1$. In Section 2 we show that for each $k \in \mathbb{N}$ there exists a non-negative integer $F(k)$ with the following property: if $\mu \notin \{-1, 0\}$, H is k-connected and G has least degree $\delta(G) \geq F(k)$ then G is k-connected. It is straightforward to show that if $f(k)$ is the smallest non-negative integer with this property, then $f(1) = 0$ and $f(2) = 2$. In Sections 3 and 4 we show that $f(3) = 3$, $f(4) = 5$, $f(5) = 7$ and $8 \leq f(6) \leq 20$.

We take $V(G) = \{1, \ldots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent. For $S \subseteq V(G)$, we write G_S for the subgraph induced by S, and $\Delta_S(u)$ for the S-neighbourhood $\{v \in S : v \sim u\}$. For the subgraph H of G we write $\Delta_H(u)$ for $\Delta_{V(H)}(u)$. Let P be the matrix of the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$ with respect to the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

We shall require the following properties of star sets and star complements; the first follows from [7, Proposition 5.1.1].

Lemma 1.1 *The subset S of $V(G)$ lies in a star set for μ if and only if the vectors $P\mathbf{e}_i$ $(i \in S)$ are linearly independent.*

Since P is a polynomial in A [7, Equation 1.5] we have $\mu P\mathbf{e}_i = AP\mathbf{e}_i = PA\mathbf{e}_i$ $(i = 1, \ldots, n)$, whence:

Lemma 1.2 $\mu P\mathbf{e_i} = \sum_{j \sim i} P\mathbf{e}_j$ $(i = 1, \ldots, n)$.

As a consequence of Lemmas 1.1 and 1.2 we have:

Lemma 1.3 [7, Proposition 5.1.4] *Let X be a star set for μ in G, and let $H = G - X$. If $\mu \notin \{-1, 0\}$, then $V(H)$ is a location-dominating set in G, that is, the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$ are non-empty and distinct.*

By interlacing [7, Corollary 1.3.12] we have:

Lemma 1.4 *If S is a star set for μ in G and if U is a proper subset of S then $S \setminus U$ is a star set for μ in $G - U$.*

The next result strengthens [1, Theorem 2.3], which says that if G has H as a star complement of order t, for an eigenvalue $\mu \notin \{-1, 0\}$, then either (a) G has order at most $\binom{t+1}{2}$, or (b) $\mu = 1$ and $G = K_2$ or $2K_2$.

Proposition 1.5 *Let G be a graph with X as a star set for μ, and let $H = G - X$. Let $s = |\cup_{i\in X} \Delta_H(i)|$.*
(i) *If $|X| > s$ then G_X has μ as an eigenvalue of multiplicity at least $|X| - s$.*
(ii) *If $\mu \notin \{-1, 0\}$ then $|X| \leq \binom{s+1}{2}$.*

Proof. Since $s \leq \binom{s+1}{2}$, the second assertion is immediate when $|X| \leq s$. Accordingly we assume throughout the proof that $|X| > s$. We show first that μ is an eigenvalue of G_X. Let $S = \cup_{i\in X}\Delta_H(i)$ and $X = \{1, 2, \ldots, k\}$. By Lemma 1.2, the vectors $\mu P\mathbf{e}_i - \Sigma\{P\mathbf{e}_j : j \in \Delta_X(i)\}$ $(i \in X)$ lie in the subspace $\langle P\mathbf{e}_h : h \in S\rangle$, and so there exist $\alpha_1, \alpha_2, \ldots \alpha_k$, not all zero, such that

$$\sum_{i=1}^{k} \alpha_i(\mu P\mathbf{e}_i - \Sigma\{P\mathbf{e}_j : j \in \Delta_X(i)\}) = \mathbf{0}.$$

Since the vectors $P\mathbf{e}_i$ $(i \in X)$ are linearly independent, it follows that $(\mu I - A_X)\mathbf{a} = \mathbf{0}$, where $\mathbf{a} = (\alpha_1, \alpha_2, \ldots \alpha_k)^\top$ and A_X is the adjacency matrix of G_X.

Let Y be a star set for μ in G_X and consider the graph $G - Y$. If $|X| - |Y| > s$ then the above argument shows that μ is an eigenvalue of $G_{X\setminus Y}$. This is a contradiction because $G_{X\setminus Y}$ is a star complement for μ in G_X. Hence $|Y| \geq |X| - s$, and we have proved the first assertion.

Since G_X has a star complement $G_{X\setminus Y}$ of order at most s, we have $|X| \leq \binom{s+1}{2}$ whenever $\mu \notin \{-1, 0\}$. (Note that by Lemma 1.3, we have $G_X \neq K_2$, while $s = 3$ when $G_X = 2K_2$.) □

2 Controlling connectivity

Let G be a graph with a k-connected star complement H for an eigenvalue other than -1 or 0. In effect the following result establishes a quadratic upper bound for $\delta(G)$ in the case that G is not k-connected. The vertex-connectivity of G is denoted by $\kappa(G)$, and we refer to a separating set of size $\kappa(G)$ as a *minimum* separating set.

Theorem 2.1 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0. If $\kappa(H) \geq k \geq 1$ and $\delta(G) \geq \frac{1}{2}(k-1)(k+2)$ then $\kappa(G) \geq k$.*

Proof. The proof is by induction on k; the result holds for $k = 1$ since $V(H)$ is a dominating set in G. Assume that $k > 1$ and that the result holds for $k-1$. Suppose by way of contradiction that $\kappa(G) < k$; then $\kappa(G) = k-1$ by the induction hypothesis.

Let S be a cutset in G of size $k-1$. If S contains a vertex v outside H then (by Lemma 1.4) $G - v$ has H as a star complement, while $\delta(G - v) \geq \frac{1}{2}(k-1)(k+2) - 1 > \frac{1}{2}(k-2)(k+1)$. By the induction hypothesis, we have $\kappa(G - v) \geq k - 1$. This is a contradiction because $G - v$ has $S \setminus \{v\}$ as a separating set of size $k - 2$.

Thus $S \subseteq V(H)$. Since $H - S$ is connected, $H - S$ lies in some component C of $G - S$. Let $R = V(G - S) \setminus V(C)$. Note that if $j \in R$ then $\Delta_H(j) \subseteq S$, and so $|\Delta_R(j)| \geq \frac{1}{2}(k-1)(k+2) - (k-1) = \frac{1}{2}k(k-1)$. Hence $|R| > \binom{k}{2}$.

However, $|\cup_{j\in R}\Delta_H(j)| \leq k-1$, and so by Proposition 1.5(ii) we have $|R| \leq \binom{k}{2}$, a contradiction. □

Corollary 2.2 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0. Then $\kappa(G)$ is controlled by $\kappa(H)$ and $\delta(G)$.*

Proof. Let $\kappa = \kappa(H)$, $\delta = \delta(G)$, and $F(k) = \frac{1}{2}(k-1)(k+2)$ $(k \in \mathbb{N})$. Define $g(\kappa, \delta)$ as the largest $k \leq \kappa$ such that $F(k) \leq \delta$. Then $\kappa(G) \geq g(\kappa, \delta)$ by the Theorem. Since also $\kappa(G) \leq \delta$, the result follows. □

In view of Theorem 2.1, we may define $f(k)$ $(k \in \mathbb{N})$ as the least non-negative integer such that $\kappa(G) \geq k$ whenever $\kappa(H) \geq k$ and $\delta(G) \geq f(k)$. Thus $f(k) \leq F(k)$ for all $(k \in \mathbb{N})$. The following example shows that $f(k) \geq k$ for all $k > 1$.

Example 2.3 For $k \geq 2$, let G_k be the graph obtained from a $(k+1)$-clique H_k by adding a vertex of degree $k-1$, and let μ be the largest eigenvalue of G_k. Since G_k is connected, we have $\mu > k$ and so H_k is a star complement for μ. Now $\kappa(G_k) = k-1 = \delta(G_k)$, while $\kappa(H_k) = k$. Hence $f(k) \geq k$. □

Since $F(2) = 2$, we have $f(2) = 2$, an observation which follows also from [8, Proposition 2.1(ii)]. We investigate $f(k)$ $(k = 3, 4, 5, 6)$ in the next two sections; there we shall require the following result, which is proved by refining an argument in the proof of Theorem 2.1.

Proposition 2.4 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0. If $\kappa(H) \geq k > 1$, $\delta(G) > f(k-1)$ and $\kappa(G) < k$ then every minimum separating set for G lies in $V(H)$.*

Proof. Let S be a minimum separating set for G, and suppose by way of contradiction that S contains a vertex v outside H. Then H is a star complement for μ in $G - v$. Since $\kappa(H) \geq k-1$ and $\delta(G-v) \geq f(k-1)$, we have $\kappa(G-v) \geq k-1$. This is a contradiction because $G - v$ has $S \setminus \{v\}$ as a separating set of size at most $k-2$. □

3 The cases $\kappa(H) = 3, 4$

In this section we determine $f(3)$ and $f(4)$. We make use of the following observation.

Lemma 3.1 *Let G be a graph with X as a star set for an eigenvalue other than -1 or 0, and let $H = G - X$. Then X does not contain vertices $1, 2, 3$ such that $\Delta_H(1)$ is the disjoint union of $\Delta_H(2)$ and $\Delta_H(3)$.*

Proof. Suppose by way of contradiction that $\Delta_H(1) = \Delta_H(2) \dot\cup \Delta_H(3)$. By Lemma 1.4, we may take $X = \{1, 2, 3\}$. Let $|V(G)| = n$ and let P be the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$, so that

$$\Sigma\{P\mathbf{e}_i : i \in \Delta_H(1)\} = \Sigma\{P\mathbf{e}_j : j \in \Delta_H(2)\} + \Sigma\{P\mathbf{e}_k : k \in \Delta_H(3)\}.$$

By Lemma 1.2, we have

$$\mu P\mathbf{e}_1 - \Sigma\{P\mathbf{e}_i : i \in \Delta_X(1)\} = \Sigma\{P\mathbf{e}_i : i \in \Delta_H(1)\} =$$

$$\mu P\mathbf{e}_2 - \Sigma\{P\mathbf{e}_j : j \in \Delta_X(2)\} + \mu P\mathbf{e}_3 - \Sigma\{P\mathbf{e}_k : k \in \Delta_X(3)\}. \quad (1)$$

We examine the various possibilities for G_X. To within a transposition of the vertices 2 and 3, there are six cases to consider, namely those in which the edge-set of G_X is one of $\{12, 13, 23\}$, $\{12, 23\}$, $\{12, 13\}$, $\{12\}$, $\{23\}$, $\emptyset$. Equation (1) becomes respectively:

$$\begin{gathered}
\mu P\mathbf{e}_1 - P\mathbf{e}_2 - P\mathbf{e}_3 = \mu P\mathbf{e}_2 - P\mathbf{e}_1 - P\mathbf{e}_3 + \mu P\mathbf{e}_3 - P\mathbf{e}_1 - P\mathbf{e}_2, \\
\mu P\mathbf{e}_1 - P\mathbf{e}_2 = \mu P\mathbf{e}_2 - P\mathbf{e}_1 - P\mathbf{e}_3 + \mu P\mathbf{e}_3 - P\mathbf{e}_2, \\
\mu P\mathbf{e}_1 - P\mathbf{e}_2 - P\mathbf{e}_3 = \mu P\mathbf{e}_2 - P\mathbf{e}_1 + \mu P\mathbf{e}_3 - P\mathbf{e}_1, \\
\mu P\mathbf{e}_1 - P\mathbf{e}_2 = \mu P\mathbf{e}_2 - P\mathbf{e}_1 + \mu P\mathbf{e}_3, \\
\mu P\mathbf{e}_1 = \mu P\mathbf{e}_2 - P\mathbf{e}_3 + \mu P\mathbf{e}_3 - P\mathbf{e}_2, \\
\mu P\mathbf{e}_1 = \mu P\mathbf{e}_2 + \mu P\mathbf{e}_3.
\end{gathered}$$

By Lemma 1.1, the vectors $P\mathbf{e}_1, P\mathbf{e}_2, P\mathbf{e}_3$ are linearly independent, and this leads to a contradiction in all cases. $\square$

Proposition 3.2 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0. If H is 3-connected and $\delta(G) \geq 3$ then G is 3-connected.*

Proof. Suppose by way of contradiction that $\kappa(G) < 3$. Since $f(2) = 2$, we have $\kappa(G) = 2$. Let $H = G - X$ and let S be a minimum separating set in G, say $S = \{v, w\}$. Then $S \subseteq V(H)$ by Proposition 2.4. As in the proof of Theroem 2.1, let R be the set of vertices in X that lie outside the component of $G - S$ containing $H - S$. Then $\Delta_H(u) \subseteq S$ for all $u \in R$. Since $\delta(G) \geq 3$, while the neighbourhoods $\Delta_H(u)$ $(u \in R)$ are non-empty and distinct, R consists of 3 pairwise adjacent vertices whose H-neighbourhoods are $\{v\}, \{w\}$ and $\{v, w\}$. Now Lemma 3.1 provides a contradiction. $\square$

It follows that $f(3) = 3$. We begin our investigation of $f(4)$ with an example which demonstrates that $f(4) > 4$. (This example was found experimentally using the computer package GRAPH [4].)

Example 3.3 (Here non-integer eigenvalues are given to 4 decimal places.) Let H be the graph obtained from an 8-clique by deleting an edge, and let u, v, w be pairwise adjacent vertices in H. Let G be the graph obtained from H by adding three pairwise adjacent vertices with H-neighbourhoods $\{u, v\}, \{u, w\}, \{v, w\}$. The spectrum of G is 7.1017, 2.3416, $0^{(3)}, -1^{(2)}$, $-1.4433, -2^{(3)}$, and the spectrum of H is 6.7720, $0, -1^{(5)}, -1.7720$. Thus G has H as a star complement for -2. Now $\kappa(H) = 6$ and $\delta(G) = 4$, while $\kappa(G) = 3$. It follows that $f(4) \geq 5$. $\square$

Proposition 3.4 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0. If H is 4-connected and $\delta(G) \geq 5$ then G is 4-connected.*

Proof. Suppose by way of contradiction that $\kappa(G) < 4$. Since $f(3) = 3$, we have $\kappa(G) = 3$. Let $H = G - X$ and let S be a minimum separating set in G, say $S = \{u, v, w\}$. Then $S \subseteq V(H)$ by Proposition 2.4. Defining R as before, we have $|R| \leq 7$ by Lemma 1.3. On the other hand, $|R| \geq 4$ because $\delta(G) \geq 5$. Moreover, if $|R| = 4$ then R induces a clique and the H-neighbourhoods of the vertices in R are the four subsets of size 2 or 3.

In this case, let $R = \{1, 2, 3, 4\}$, with $\Delta_H(1) = \{u, v, w\}$, $\Delta_H(2) = \{v, w\}$, $\Delta_H(3) = \{u, w\}$ and $\Delta_H(4) = \{u, v\}$. By Lemma 1.4, we may take $X = R$. Defining P as before, we have from Lemma 1.2:

$$2(\mu P\mathbf{e}_1 - \Sigma\{P\mathbf{e}_i : i \in \Delta_X(1)\}) = 2\Sigma\{P\mathbf{e}_i : i \in S\} =$$

$$\sum_{j=2}^{4}(\mu P\mathbf{e}_j - \Sigma\{P\mathbf{e}_k : k \in \Delta_X(j)\}).$$

It follows that $(2\mu+3)P\mathbf{e}_1 = \mu(P\mathbf{e}_2+P\mathbf{e}_3+P\mathbf{e}_4)$, and hence that $P\mathbf{e}_1, P\mathbf{e}_2, P\mathbf{e}_3, P\mathbf{e}_4$ are linearly dependent, contradicting Lemma 1.1.

We deduce that $|R| \in \{5, 6, 7\}$. Now the neighbourhoods $\Delta_H(i)$ $(i \in R)$ must include two disjoint subsets whose union is a third such neighbourhood, and Lemma 3.1 provides a final contradiction. □

We deduce that $f(4) = 5$.

4 The cases $\kappa(H) = 5, 6$

We begin with two examples which show that $f(5) \geq 7$ and $f(6) \geq 8$. Here we make use of the following result from [10, Section 3] (see also [6, Example 5.1.14] and [7, Example 5.2.16]).

Proposition 4.1 *Let G be a finite graph with a proper induced subgraph $G - X = H \cong K_8$ such that*
(i) $|\Delta_H(u)| = 3$ *for all* $u \in X$,
(ii) *for distinct vertices u, v of X, $|\Delta_H(u) \cap \Delta_H(v)| = 2$ when $u \sim v$, and $|\Delta_H(u) \cap \Delta_H(v)| = 1$ when $u \not\sim v$.*
Then H is a star complement in G for the eigenvalue -2.

Example 4.2 Let G be the graph obtained from an 8-clique by adding four pairwise adjacent vertices whose H-neighbourhoods are the four 3-subsets of a 4-set in $V(H)$. By Proposition 4.1, G has H as a star complement for -2. Now $\kappa(H) = 7$, $\delta(G) = 6$ and $\kappa(G) = 4$. Hence $f(5) > 6$. □

Example 4.3 Here G is constructed from an 8-clique H by first adding a set X of ten vertices whose H-neighbourhoods are the ten 3-subsets of a 5-set in $V(H)$; secondly, we add an edge between vertices u and v of X whenever $|\Delta_H(u) \cap \Delta_H(v)| = 2$. (It follows that X induces the line graph $L(K_5)$.) By Proposition 4.1, G has H as a star complement for -2. We have $\kappa(H) = 7$, $\delta(G) = 7 = \delta(H)$ and $\kappa(G) = 5$. Hence $f(6) > 7$. □

Since $F(5) = 14$ and $F(6) = 20$, we deduce that $7 \leq f(5) \leq 14$ and $8 \leq f(6) \leq 20$. We conclude by showing that $f(5) = 7$.

Proposition 4.4 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0. If H is 5-connected and $\delta(G) \geq 7$ then G is 5-connected.*

Proof. Suppose by way of contradiction that $\kappa(G) < 5$. Since $f(4) = 5$, we have $\kappa(G) = 4$. Let $H = G - X$ and let S be a minimum separating set in G. Then $S \subseteq V(H)$ by Proposition 2.4. Defining R as before, we have $|R| \geq 5$ because $\delta(G) \geq 7$. Moreover, if $|R| = 5$ then R induces a clique and the H-neighbourhoods of the vertices in R are the five subsets of size 3 or 4. If $R = \{1, 2, 3, 4, 5\}$, with $\Delta_H(1) = S$ then (arguing as before) we find that $(3\mu + 4)P\mathbf{e}_1 = \mu(P\mathbf{e}_2 + P\mathbf{e}_3 + P\mathbf{e}_4 + P\mathbf{e}_5)$, contradicting Lemma 1.1.

Next suppose that $|R| = 6$. Applying Proposition 1.5(i) with $s \leq 4$ we see that μ is a multiple eigenvalue of G_R. Now each vertex of G_R has degree at least 3, with at most one vertex of degree equal to 3, and so each vertex of the complement $\overline{G_R}$ has degree at most 2, with at most one vertex of degree equal to 2. Thus G_R is one of the graphs numbered 1,2,3,4,8,9 in [5]. Only the last of these has a multiple eigenvalue other than -1 or 0, and it follows that G_R is the skeleton of an octahedron (with $\mu = -2$). Hence $|\Delta_H(j)| \geq 3$ for all $j \in R$. This contradicts Lemma 1.3 because S has only five distinct subsets of size at least 3.

Finally, suppose that $|R| \geq 7$. Let Q be a 7-subset of R and consider the subgraph G' of G induced by $V(H) \dot\cup Q$. By Proposition 1.5(i), G_Q $(= G'_Q)$ has μ as an eigenvalue of multiplicity at least 3. Hence $G_Q = 3K_2 \dot\cup K_1$, with $\mu = 1$. (This follows from an inspection of the spectra of (i) the connected graphs of order 7 [2, pp.176-232] and (ii) the connected graphs of order at most 6 [3, 5]. Note that in case (i), if μ is not an integer then μ has an algebraic conjugate $\mu^* \neq \mu$ with the same multiplicity. In this situation, $\mu + \mu^*$ is an integer and the largest eigenvalue λ_1 of G is $-3(\mu + \mu^*)$; since G_Q is not complete, necessarily $\lambda_1 = 3$.)

Let $Q = \{1, 2, 3, 4, 5, 6, 7\}$, with $1 \sim 2$, $3 \sim 4$, $5 \sim 6$, and let $W = \langle P\mathbf{e}_h : h \in S \rangle$, where now P represents the orthogonal projection onto $\mathcal{E}(\mu)$ associated with G'. The subspace W has dimension 4 because it contains the vectors $P\mathbf{e}_1 - P\mathbf{e}_2$, $P\mathbf{e}_3 - P\mathbf{e}_4$, $P\mathbf{e}_5 - P\mathbf{e}_6$ and $P\mathbf{e}_7$. Hence the vectors $P\mathbf{e}_h$ $(h \in S)$ are linearly independent. Now

$$P\mathbf{e}_1 - P\mathbf{e}_2 = \Sigma\{P\mathbf{e}_i : i \in \Delta_S(1)\},\ P\mathbf{e}_2 - P\mathbf{e}_1 = \Sigma\{P\mathbf{e}_i : i \in \Delta_S(2)\}.$$

It follows that

$$\Sigma\{P\mathbf{e}_h : h \in \Delta_S(1) \setminus \Delta_S(2)\} + \Sigma\{P\mathbf{e}_h : h \in \Delta_S(2) \setminus \Delta_S(1)\} + 2\Sigma\{P\mathbf{e}_h : h \in \Delta_S(1) \cap \Delta_S(2)\} = \mathbf{0},$$

a final contradiction. □

Combining the various results from Sections 2, 3 and 4, we have:

Theorem 4.5 *Let G be a graph with H as a star complement for the eigenvalue μ. For each $k \in \mathbb{N}$ there exists a smallest non-negative integer $f(k)$ with the property*

$$\mu \notin \{-1, 0\},\ \kappa(H) \geq k,\ \delta(G) \geq f(k) \Rightarrow \kappa(G) \geq k.$$

We have $k \leq f(k) \leq \frac{1}{2}(k-1)(k+2)$ for all $k \geq 2$; moreover, $f(1) = 0$, $f(2) = 2$, $f(3) = 3$, $f(4) = 5$, $f(5) = 7$ and $f(6) \geq 8$. (In particular, if $\mu \notin \{-1, 0\}$ and H is connected, then G is connected, while if $\mu \notin \{-1, 0\}$ and H is 2-connected then either G is 2-connected or G has a pendant vertex.)

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