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ON EIGENVALUE MULTIPLICITY AND THE GIRTH OF A GRAPH

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In honour of Dragoš Cvetković, on his 70th birthday

Abstract. Suppose that G is a connected graph of order n and girth $g < n$. Let k be the multiplicity of an eigenvalue μ of G. Sharp upper bounds for k are $n - g + 2$ when $\mu \in \{-1, 0\}$, and $n - g$ otherwise. The graphs attaining these bounds are described.

Keywords: Graph, girth, eigenvalue, star complement.

AMS Classification: 05C50

1 Introduction

Let G be a connected graph of order n with an eigenvalue μ of multiplicity k. (Thus the corresponding eigenspace of a $(0, 1)$-adjacency matrix of G has dimension k.) If G has girth g then by interlacing, applied to an induced g-cycle, we have $k \leq n-g+2$ (see [6, Corollary 1.3.12]). When $\mu = -1$ this bound is attained in complete graphs, and when $\mu = 0$ it is attained in most complete bipartite graphs. However, as we show below, the values -1 and 0 are (as usual) exceptional, and $k \leq n - g$ when $\mu \neq -1$ or 0. Two remarks are in order: (i) the inequality $k \leq n - g$ improves the inequality $k \leq n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}}$ implicit in [1, Theorem 2.3] precisely when $g(g+1) > 2n$, (ii) the relation between k and g is tenuous in that large changes in girth may be accompanied by small changes in k. For (ii), note that by adding an appropriate edge to a graph with large girth g, we can reduce the girth to 3, while the multiplicity of any eigenvalue changes by two at most. (Thus it can be advantageous to apply the bound $n-g$ after deleting suitable edges.) We investigate the extremal situation in which $\mu \neq -1$ or 0 and $k = n - g$. In this case, $n \leq \frac{1}{2}g(g+1)$ by [1], and we show that $g \leq 5$ or $k \leq 2$ (or both). Then we can describe all the graphs that arise. Immediate examples of such a graph G are the Petersen graph (with $n = 10$, $g = 5$, $\mu = 1$, $k = 5$) and the graphs obtained from a cycle by adding a pendant edge. In the latter case, $n = g + 1$ while $k = 1$ for any eigenvalue of G. The proof divides naturally into two parts, according as μ is or is not an eigenvalue of the cycle C_g, and the problem reduces to the question of how k pendant edges can be added to a g-cycle to obtain a graph with an eigenvalue of multiplicity k. The notation follows [6], and we make implicit use of the formula [6, Theorem 2.2.3] for the characteristic polynomial of the coalescence of two graphs. In dealing with small graphs ($n \leq 7$) the tables of graph spectra in [2, 3, 4] are helpful.

2 Preliminaries

We assume throughout that $g < n$: this simply excludes the case that G is itself a g-cycle (for which any eigenvalue other than ± 2 has multiplicity $2 = n - g + 2$). We write $c_t(x)$ for the characteristic polynomial of C_t $(t \geq 3)$ and $p_t(x)$ for the characteristic polynomial of the path P_t (of length $t - 1 \geq 0$). Additionally, we define $p_0(x) = 1$. Thus $c_t(2\cos\theta) = 2\cos(t\theta) - 2$, and $p_t(2\cos\theta) = \sin(t+1)\theta/\sin\theta$ when $\sin\theta \neq 0$ (see [3, p.73]). We take H to be an induced g-cycle, say $H = G - X$, and we write $\Delta_H(u)$ for the H-neighbourhood of a vertex $u \in X$. We write $d_H(v, w)$ for the distance in H between vertices v, w of H.

We denote by U_{t+1} the graph obtained from C_t by adding a pendant edge. Note that neither P_t nor U_{t+1}, with characteristic polynomial $xc_t(x) - p_{t-1}(x)$, has a repeated eigenvalue.

Lemma 2.1 *If X contains a vertex u such that $|\Delta_H(u)| > 1$ then $g \leq 4$ and $H + u$ is one of the graphs shown in Fig. 1.*

Proof. Let v, w be distinct vertices in $\Delta_H(u)$. Since $d_H(v, w) \leq \frac{1}{2}g$, G has a cycle of length at most $\frac{1}{2}g + 2$, and so $g \leq 4$. If $g = 4$ then $H + u$ is the graph shown in Fig. 1(a), and if $g = 3$ then we have the two possibilities shown in Figs. 1(b)(c). □

Lemma 2.2 *If u, v are adjacent vertices of X such that $|\Delta_H(u)| = |\Delta_H(v)| = 1$ then $g \leq 6$ and $H + u + v$ is one of the graphs shown in Fig. 2.*

Proof. Let $\Delta_H(u) = \{u'\}$, $\Delta_H(v) = \{v'\}$. If $u' = v'$ then $g = 3$ and $H + u + v$ is the graph shown in Fig. 2(f). If $u' \neq v'$ then $0 < d_H(u', v') \leq \frac{1}{2}g$, and so G has a cycle of length at most $\frac{1}{2}g + 3$. Hence $g \leq 6$ in this case, and Figs. 2(a), 2(b), 2(c)(d), 2(e) show the possibilities for $H + u + v$ when $g = 6, 5, 4, 3$ respectively. □

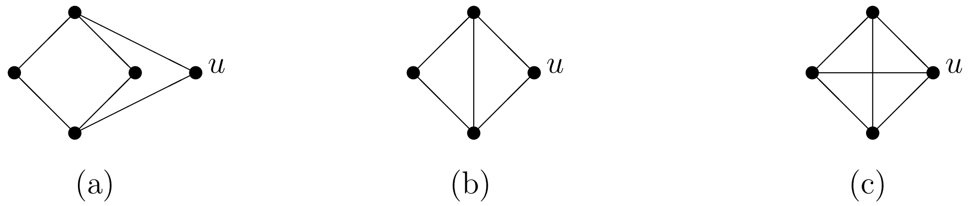


Figure 1: The graphs from Lemma 2.1.

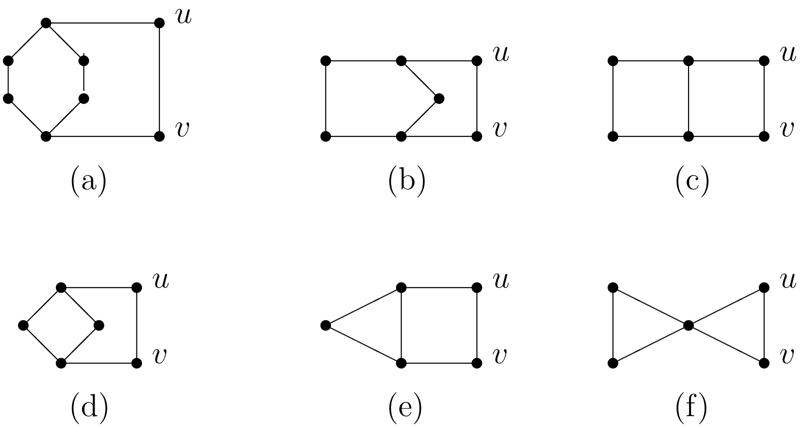


Figure 2: The graphs from Lemma 2.2.

Proposition 2.3 *Let μ be an eigenvalue of multiplicity k in a connected graph G of order n and girth $g < n$. Then $k = n - g + 2$ if and only if either*
(a) $g = 3$, $G = K_n$ $(n > 3)$, $\mu = -1$ or
(b) $g = 4$, $G = K_{r,s}$ $(n = r + s > 4,\ r > 1,\ s > 1)$, $\mu = 0$.

Proof. Suppose that $k = n - g + 2$, and let u be a vertex of X such that $H + u$ is connected. By interlacing, μ is a double eigenvalue of H, and the addition to H of any k' vertices in X increases the multiplicity of μ by k'. Since μ has multiplicity 3 in $H + u$, u has at least two neighbours in H, and so $g \leq 4$ by Lemma 2.1. If $g = 3$ then $k = n - 1$ and (a) holds. If $g = 4$ then $H + u = K_{2,3}$ (Fig. 1(a)) and $\mu = 0$. In this case, the spectrum of G has the form $-\lambda, 0^{(n-2)}, \lambda$ and so (b) holds (see [6, Theorem 3.2.4]). Conversely, $k = n - g + 2$ in cases (a) and (b). □

Proposition 2.4 *Let μ be an eigenvalue of multiplicity k in a connected graph G of order n and girth $g < n$. If $\mu \neq -1$ or 0 then $k \leq n - g$.*

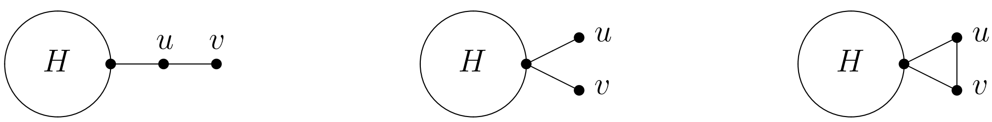
Proof. In view of Proposition 2.3 it suffices to exclude the case $k = n - g + 1$. Suppose that $k = n - g + 1$ and let u be a vertex of X such that $H + u$ is connected. Since μ is a multiple eigenvalue of $H + u$, u has at least two neighbours in H. By Lemma 2.1, $g \leq 4$. If $g = 3$ then $\mu = 2$, but the graphs in Figs. 1(b)(c) do not have 2 as an eigenvalue. If $g = 4$ then $\mu = \pm 2$, but $K_{2,3}$ (Fig. 1(a)) does not have 2 or -2 as an eigenvalue. $\square$

To investigate the graphs with $k = n - g$ when $\mu \neq -1$ or 0, we distinguish two cases (I) and (II) according as μ is or is not an eigenvalue of C_g.

3 Case I

In this section we assume that $k = n - g > 0$, $\mu \neq -1$ or 0, and μ is an eigenvalue of the induced g-cycle $H = G - X$. If $|\Delta_H(u)| > 1$ for some $u \in X$ then by Lemma 2.1 either $g = 4$ and $\mu = \pm 2$ or $g = 3$ and $\mu = 2$. In either case, we have a contradiction to the fact that (by interlacing) μ is an eigenvalue of $H + u$. If X contains a vertex with no neighbour in H then X contains vertices u, v such that $H + u + v$ has the form shown in Fig. 3(a). Now $H + u$ has no repeated roots, and so by interlacing, the addition to $H + u$ of each successive vertex in X increases the multiplicity of μ by 1. Hence $H + u + v$ has μ as a double eigenvalue. But $H + u + v$ has characteristic polynomial $c_g(x)(x^2 - 1) - xp_{g-1}(x)$, and this is not divisible by $(x - \mu)^2$. We conclude that each vertex u of X has a unique neighbour u' in H.

The vertices u' ($u \in X$) are distinct, for otherwise X contains vertices u, v such that $H + u + v$ has the form shown in Fig. 3(b) or 3(c). In the former case, $H + u + v$ has characteristic polynomial $x^2 c_g(x) - 2xp_{g-1}(x)$, which is not divisible by $(x - \mu)^2$. In the latter case, $g = 3$ and $H + u + v$ is the graph shown in Fig. 2(f), for which -1 is the only repeated eigenvalue.



(a) (b) (c)

Figure 3: Configurations for Case I.

If X is not independent then we may apply Lemma 2.2 to adjacent vertices u, v of X. Of the graphs in Fig. 2, only (a) and (b) have a double eigenvalue $\mu \notin \{-1, 0\}$, and $\mu = 1$ in both cases. Since 1 is not an eigenvalue of C_5, we have $g = 6$, with $H + u + v$ the graph in Fig. 2(a). Since $g = 6$ and $w' \neq u', v'$, there is just one way to add a vertex w to $H + u + v$, and we find that 1 is not a triple eigenvalue of $H + u + v + w$. Thus only one graph arises when the edges uu' ($u \in X$) are not independent.

It remains to consider the case in which G consists of the g-cycle H and k independent pendant edges. When $k > 1$ we consider a graph $H + u + v$, and let r, s be the lengths of the two u'-v'

paths in H. Then $r+s=g$ and $H+u+v$ has characteristic polynomial

$$x^2 c_g(x) - 2x p_{g-1}(x) + p_{r-1}(x) p_{s-1}(x).$$

Since μ is an eigenvalue of both H and $H+u$, we have $\mu = 2\cos\alpha$ where $\alpha = \frac{2\pi h}{g}$ for some integer h, $0 < h < \frac{1}{2}g$. Without loss of generality, $x-\mu$ divides $p_{r-1}(x)$, equivalently $\sin r\alpha = 0$. Then $\sin s\alpha = 0$, equivalently $x-\mu$ divides $p_{s-1}(x)$. Since also $(x-\mu)^2$ divides $c_g(x)$, we deduce that $(x-\mu)^2$ divides $p_{g-1}(x)$, a contradiction. We summarize our results as follows.

Theorem 3.1 *Let μ be an eigenvalue of multiplicity k in a connected graph G of order n and girth $g < n$. Suppose that $\mu \neq -1$ or 0, and that μ is an eigenvalue of C_g. Then $k = n-g$ if and only if either*
(a) *$k=2$, $g=6$, $\mu=1$ and G is the graph in Fig. 2(a), or*
(b) *$k=1$, $\mu = \cos\frac{2\pi h}{g}$ $(h = 1, 2, \ldots, \lfloor\frac{1}{2}(g-1)\rfloor)$ and $G = U_{g+1}$.*

4 Case II

In this section we assume that $k = n-g > 0$, $\mu \neq -1$ or 0, and μ is not an eigenvalue of the induced g-cycle $H = G - X$. Thus H is a star complement for μ and the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$ are distinct and non-empty [6, Proposition 5.1.4]. Moreover, G has μ-eigenvectors $\mathbf{x}_u = (x_{ui})$ $(u \in X)$ such that $x_{uv} = \delta_{uv}$ $(u, v \in X)$ [5, Theorem 7.2.6]. By interlacing, the addition to H of k' vertices of X results in a graph with μ as an eigenvalue of multiplicity k'.

If X contains a vertex u such that $|\Delta_H(u)| > 1$ then $g \leq 4$ by Lemma 2.1. If $g=4$ then $H+u = K_{2,3}$ (Fig. 1(a)) and $\mu = \pm\sqrt{6}$, while no extension $H+u+v$ has μ as an eigenvalue. (The five possibilities yield four different graphs, those numbered 52, 74, 90, 91 in [4].) If $g=3$ then $H+u$ is as shown in Fig. 1(b) or (c). In the latter case, $\mu = 3$ while no extension of K_4 by a single vertex has 3 as an eigenvalue, and so $G = K_4$. In the former case, $\mu^2 - \mu - 4 = 0$ and no extension $H+u+v$ can have μ as an eigenvalue of multiplicity two. To see this, let μ^* be the algebraic conjugate of μ and let λ be the largest eigenvalue of $H+u+v$; then $H+u+v$ has an eigenvalue $-2\mu - 2\mu^* - \lambda$ with absolute value greater than λ, a contradiction. Thus $k=1$ when X contains a vertex u such that $|\Delta_H(u)| > 1$, and Fig. 1 shows the three possibilities for G.

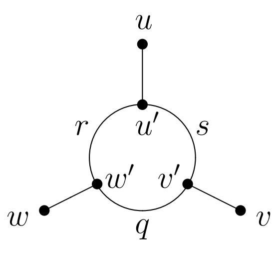


Figure 4: A configuration for Case II.

Now suppose that $|\Delta_H(u)| = 1$ for all $u \in X$. If X contains adjacent vertices u, v then by Lemma 2.2, $H+u+v$ is one of the graphs shown in Fig. 2. Of these, the first is excluded because the double eigenvalue 1 is an eigenvalue of H, and the last four are excluded because none has a double eigenvalue $\mu \notin \{-1, 0\}$. Thus $H+u+v$ is the graph shown in Fig. 2(b), and then $\mu = 1$,

$g = 5$. The graphs with C_5 as a star complement for 1 are determined in [6, Example 5.2.3], and those with girth 5 are induced subgraphs of the Petersen graph.

It remains to consider the case in which G consists of the g-cycle H and k independent pendant edges uu' ($u \in X$). We show that $k \leq 2$. Suppose by way of contradiction that μ is a triple eigenvalue of $H + u + v + w$, where u', v', w' are separated by q, r, s edges of H as shown in Fig. 4. Thus $q + r + s = g$. Note that $g > 3$ (by [4] for example). Since μ is a double eigenvalue of each of $H + v + w$, $H + u + w$, $H + u + v$, we know that $(x - \mu)^2$ divides each of

$$x^2 c_g(x) - 2xp_{g-1}(x) + p_{q-1}(x)p_{r+s-1}(x), \tag{1}$$

$$x^2 c_g(x) - 2xp_{g-1}(x) + p_{r-1}(x)p_{s+q-1}(x), \tag{2}$$

$$x^2 c_g(x) - 2xp_{g-1}(x) + p_{s-1}(x)p_{q+r-1}(x). \tag{3}$$

On subtracting (2) from (1), we see that $(x - \mu)^2$ divides $f(x)$, where

$$f(x) = p_{q-1}(x)p_{r+s-1}(x) - p_{r-1}(x)p_{s+q-1}(x).$$

With some trigonometric manipulation when $x = 2\cos\theta$, we find that

$$f(x) = \begin{cases} p_{s-1}(x)p_{q-r-1}(x) & \text{if } q > r, \\ -p_{s-1}(x)p_{r-q-1}(x) & \text{if } q < r. \end{cases}$$

Thus if $q \neq r$ then $x - \mu$ divides $p_{s-1}(x)$. From (3) we see that $x - \mu$ divides $xc_g(x) - 2p_{g-1}(x)$. Since $x - \mu$ divides $xc_g(x) - p_{g-1}(x)$, we deduce that $c_g(\mu) = 0$, contrary to hypothesis. Therefore, $q = r$, and similarly $r = s$. Thus the vertices of H may be labelled $1, 2, \ldots, 3r$, with $u' = 3r$, $v' = r$ and $w' = 2r$. Now $H + u + v + w$ has a μ-eigenvector $\mathbf{x} = (x_i)$ with $x_u = 1$, $x_v = 0$ and $x_w = 0$. Then $x_{3r} = \mu$ ($\neq 0$). Let $x_{r-1} = c$. Then $c \neq 0$, for otherwise the eigenvalue equations for μ force $\mathbf{x} = \mathbf{0}$. There exist polynomials $f_0, f_1, f_2, \ldots, f_{2r}$ such that $x_{r+i} = cf_i(\mu)$ ($i = 0, 1, \ldots, 2r$). (Applying the eigenvalue equations along the path $r, r+1, \ldots 3r$, we find that $f_0(\mu) = 0$, $f_1(\mu) = -1$ and $f_i(\mu) = \mu f_{i-1}(\mu) - f_{i-2}(\mu)$ ($i > 1$).) Now $f_r(\mu) = 0$ and we let m be the least positive integer i such that $f_i(\mu) = 0$. Note that $m > 1$, and let $c' = f_{m-1}(\mu)$ ($\neq 0$). Then $x_{r+m+i} = c'f_i(\mu)$ ($i = 0, 1, \ldots, m-1$) and we see that $x_{r+j} = 0$ if and only if m divides j. Thus m divides r and $x_{3r} = 0$, a contradiction.

We summarize our results as follows.

Theorem 4.1 *Let μ be an eigenvalue of multiplicity k in a connected graph G of order n and girth $g < n$. Suppose that $\mu \neq -1$ or 0 and μ is not an eigenvalue of C_g. If $k = n - g$ then one of the following holds:*

(a) $k = 1$, $\mu = 3$ *and* $G = K_4$,
(b) $k = 1$, $\mu = \frac{1}{2}(1 \pm \sqrt{17})$ *and G is obtained from K_4 by deleting an edge,*
(c) $k = 1$, $\mu = \pm\sqrt{6}$ *and* $G = K_{2,3}$,
(d) $3 \leq k \leq 5$, $\mu = 1$ *and G is an induced subgraph of the Petersen graph,*
(e) $k = 1$, $G = U_{g+1}$ *and* $\mu \neq \cos\frac{2\pi h}{g}$ ($h = 1, 2, \ldots, \lfloor\frac{1}{2}(g-1)\rfloor$),
(f) $k = 2$ *and G consists of a g-cycle and two independent pendant edges.*

5 Case II revisited

It remains to investigate the graphs that arise in case (f) of Theorem 4.1. For positive integers r, s, we write $C_{r,s}$ for the graph $H + u + v$ consisting of a g-cycle H and pendant edges uu', vv' with r, s the lengths of the two u'-v' paths in H.

Lemma 5.1 *No graph $C_{r,r}$ $(r > 1)$ has C_{2r} as a star complement for an eigenvalue $\mu \neq 0$.*

Proof. We use the notation above. If H is a star complement for μ then $C_{r,r}$ has a μ-eigenvector $\mathbf{x}$ with $x_u = 1, x_v = 0$. Then $x_{v'} = 0$, and if we apply the eigenvalue equations along each v'-u' path in H, we find that $x_{u'} = -x_{u'}$. It follows that $\mu = 0$, contrary to assumption. □

Lemma 5.2 *The graph $C_{1,g-1}$ has C_g as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$ if and only if $\mu = 1$ and $g \equiv -1 \bmod 6$.*

Proof. In this case, $H + u + v$ has characteristic polynomial

$$x(xc_g(x) - p_{g-1}(x)) - p_g(x). \tag{4}$$

Suppose that H is a star complement for μ. Since μ is an eigenvalue of $H + u$ and $H + u + v$, it follows that $p_g(\mu) = 0$. Hence $\mu = 2\cos\alpha$ where $\alpha = \frac{a\pi}{g+1}$ for some integer a, $1 \leq a \leq g$. If a is odd and we evaluate (4) at $2\cos\alpha$, we find that $(2\cos\alpha + 1)^2 = 0$, whence $\mu = -1$, contrary to hypothesis. Hence a is even, and then we have $(2\cos\alpha - 1)^2 = 0$. Thus $\mu = 1$, $\alpha = \frac{\pi}{3}$, $g + 1 = 3a$ and $g \equiv -1 \bmod 6$.

Suppose that the vertices of H are labelled $1, 2, \ldots, g$ in sequence, with $u' = 1$, $v' = 2$. Let $\mathbf{x}$ be a 1-eigenvector of $H + u + v$ with $x_u = 1$, $x_v = 0$. Then $x_1 = 1$, $x_2 = 0$ and hence $x_g = 0$. Now the sequence $x_1, x_2, x_3, \ldots, x_{g-1}, x_g, x_1$ consists of recurrent subsequences $1, 0, -1, -1, 0, 1$. Conversely, if $g \equiv -1 \bmod 6$ then 1 is not an eigenvalue of H and we can construct two linearly independent 1-eigenvectors using these subsequences; hence H is a star complement for μ. □

Proposition 5.3 *Let $G = C_{r,s}$, where $r > 1, s > 1$ and $r \neq s$. Suppose that $\mu \notin \{-1, 0\}$, and that μ is not an eigenvalue of C_{r+s}. Then μ is a double eigenvalue of G if and only if*

(*) $\mu = 2\cos\alpha$, $\alpha = \frac{h\pi}{r-s}$ *(h an odd integer) and* $\tan s\alpha = 2\sin 2\alpha$.

[Note that $\tan r\alpha = \tan s\alpha$ when $\alpha = \frac{h\pi}{r-s}$.]

Proof. We may assume that $r > s$. Suppose that μ is a double eigenvalue of $G = H + u + v$. If we delete in turn the neighbours of u' in H, we see that $x - \mu$ divides each of

$$xp_{r+s}(x) - p_{s+1}(x)p_{r-2}(x), \quad xp_{r+s}(x) - p_{r+1}(x)p_{s-2}(x),$$

Hence $x - \mu$ divides

$$p_{r+1}(x)p_{s-2}(x) - p_{s+1}(x)p_{r-2}(x),$$

which is equal to $(x^2 - 1)p_{r-s-1}(x)$. Thus $\mu = 1$ or $p_{r-s-1}(\mu) = 0$ (or both).

If $\mu = 1$ then we consider a 1-eigenvector $\mathbf{x}$ of $H + u + v$ with $x_{u'} = 1$ and $x_{v'} = 0$. Applying the eigenvalue equations along both u'-v' paths in H, we find that $r \equiv s \equiv 1 \bmod 3$ and $r \not\equiv s \bmod 6$. Hence $r - s$ is an odd multiple of 3, and (*) holds with $\alpha = \frac{\pi}{3}$.

If $p_{r-s-1}(\mu) = 0$ then $\mu = 2\cos\alpha$ where $\alpha = \frac{h\pi}{r-s}$ for some integer h strictly between 0 and $r - s$. (Thus $\sin\alpha \neq 0$.) Now G has characteristic polynomial

$$f_{r,s}(x) = x^2 c_{r+s}(x) - 2xp_{r+s-1}(x) + p_{r-1}(x)p_{s-1}(x)$$

and so $x-\mu$ divides
$$f_{r,s}(x) - 2x(xc_{r+s}(x) - p_{r+s-1}(x)). \tag{5}$$
If $x = 2\cos\alpha$ the expression (5) becomes
$$\frac{1}{\sin^2\alpha} \times \{16\sin^2\alpha\cos^2\alpha\sin^2\frac{r+s}{2}\alpha + (-1)^h\sin^2 s\alpha\}.$$
Since $\cos\alpha \neq 0$, it follows that if h is even then $\sin\frac{r+s}{2}\alpha = 0$ and hence $\sin(r+s)\alpha = 0$. But then $x-\mu$ divides $p_{r+s-1}(x)$ and hence also $c_{r+s}(x)$, a contradiction. Therefore h is odd.

Again if $x = 2\cos\alpha$ then
$$xc_{r+s}(x) - p_{r+s-1}(x) = \frac{2\cos s\alpha}{\sin\alpha}\{\sin s\alpha - 2\sin 2\alpha\cos s\alpha\}.$$
Now $\cos s\alpha \neq 0$ for otherwise $\sin(r+s)\alpha = -\sin 2s\alpha = 0$, leading to a contradiction as before. Hence $\sin s\alpha - 2\sin 2\alpha\cos s\alpha = 0$, and condition (*) holds.

Conversely, if (*) holds then μ is a double eigenvalue of $C_{r,s}$ because it is a root of both $f_{r,s}(x)$ and $f'_{r,s}(x)$. To see this, note that
$$f_{r,s}(2\cos\theta)\sin^2\theta = (2\sin 2\theta\cos r\theta - \sin r\theta)(2\sin 2\theta\cos s\theta - \sin s\theta) - 2\sin^2 2\theta(1+\cos(r-s)\theta).$$
□

For $n > 10$, we may summarize our results as follows. Note that the graphs in Lemma 5.2 satisfy condition (*) with $\alpha = \frac{\pi}{3}$, $r = g-1, s = 1, h = a-1$.

Theorem 5.4 *let G be a connected graph of order $n > 10$ and girth $g < n$. Suppose that G has an eigenvalue μ of multiplicity k.*

(1) *If $\mu \in \{-1, 0\}$ then $k \leq n-g+2$ with equality if and only if either*
  (a) *$k = n-1$, $g = 3$, $G = K_n$, $\mu = -1$ or*
  (b) *$k = n-2$, $g = 4$, $G = K_{r,s}$ $(n = r+s,\ r > 1,\ s > 1)$, $\mu = 0$.*

(2) *If $\mu \notin \{-1, 0\}$ then $k \leq n-g$ with equality if and only if either*
  (a) *$k = 1$, $G = U_{g+1}$ and μ is an eigenvalue of U_{g+1} other than -1 or 0, or*
  (b) *$k = 2$, $G = C_{r,s}$ $(r+s = g,\ r \neq s)$, μ satisfies (*) and μ is not an eigenvalue of C_g.*

We conclude with some examples of graphs that arise in case (2)(b) of Theorem 5.4 If $r \equiv 8 \bmod 12$ and $s \equiv 2 \bmod 12$ then (*) is satisfied with $\alpha \in \{\frac{\pi}{6}, \frac{5\pi}{6}\}$, and we have $\mu = \pm\sqrt{3}$. If $r \equiv 15 \bmod 24$ and $s \equiv 3 \bmod 24$ then (*) is satisfied with $\alpha \in \{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}\}$, and we have $\mu = \pm\sqrt{2\pm\sqrt{3}}$. If $r \equiv 4 \bmod 6$ and $s \equiv 1 \bmod 6$ then (*) is satisfied with $\alpha = \frac{\pi}{3}$ and $\mu = 1$ (as in Lemma 5.2).

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References

[1] Bell F. K., Rowlinson P., On the multiplicities of graph eigenvalues, *Bull. London Math. Soc.* **35** (2003), 401-408.

[2] Cvetković D., Doob M., Gutman I., Torgašev A., *Recent Results in the Theory of Graph Spectra*, North-Holland (Amsterdam), 1988.

[3] Cvetković D., Doob M., Sachs H., *Spectra of Graphs*, 3rd edn., Johann Ambrosius Barth (Heidelberg), 1995.

[4] Cvetković D., Petrić M., A table of connected graphs on six vertices, *Discrete Math.* **50** (1984), 37-49.

[5] Cvetković D., Rowlinson P., Simić S., *Eigenspaces of Graphs*, Cambridge University Press (Cambridge), 1997.

[6] Cvetković D., Rowlinson P., Simić S., *An Introduction to the Theory of Graph Spectra*, Cambridge University Press (Cambridge), 2010.