Rowlinson P (2012) Regular star complements in strongly regular graphs, *Linear Algebra and Its Applications*, 436 (5), pp. 1482-1488.

This is the peer reviewed version of this article

REGULAR STAR COMPLEMENTS IN STRONGLY REGULAR GRAPHS

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Abstract

We prove that, aside from the complete multipartite graphs and graphs of Steiner type, there are only finitely many connected strongly regular graphs with a regular star complement of prescribed degree $s \in \mathbb{N}$. We investigate the possible parameters when $s \leq 5$.

AMS Classification: 05C50

Keywords: strongly regular graph, eigenvalue, star complement.

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1 Introduction

Let G be a finite simple graph of order n with μ as an eigenvalue of multiplicity k. (Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix of G has dimension k.) A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. The fundamental properties of star sets and star complements are established in [8, Chapter 5]. A survey of star complements in regular graphs may be found in [18], along with a description of the regular graphs with a star or windmill as a star complement. The cubic graphs with a regular star complement are determined in [15], and the regular graphs with a 1-regular star complement are determined in [17]. As the following examples show, it can happen that a strongly regular graph has a regular star complement. We use the notation of [8].

Examples 1.1 (i) The Petersen graph has $3K_2$ as a 1-regular star complement for the eigenvalue -2.
(ii) The Petersen graph has C_5 as a 2-regular star complement for the eigenvalue 1.
(iii) The Gewirtz graph [10] has the Sylvester graph [2, p.223] as a 5-regular star complement for -4. (The Gewirtz graph has spectrum $10, 2^{(35)}, -4^{(20)}$, and the Sylvester graph has spectrum $5, 2^{(16)}, -1^{(10)}, -3^{(9)}$.)
(iv) The complete multipartite graph $\overline{(s+1)K_u}$ $(u \in I\!N)$ has K_{s+1} as an s-regular star complement for the eigenvalue 0.
(v) The line graph $L(K_u)$ $(u > 4)$ has a union of disjoint odd cycles, of order u, as a 2-regular star complement for the eigenvalue -2.

We say that a strongly regular graph is of *Steiner type* $S(2, \tilde{k}, \tilde{v})$ if its parameters n, r, e, f coincide with those of the block graph of a Steiner system $S(2, \tilde{k}, \tilde{v})$, that is (see [11, Section 9]),

$$n = \frac{\tilde{v}(\tilde{v}-1)}{\tilde{k}(\tilde{k}-1)}, \qquad r = \tilde{k}\frac{\tilde{v}-\tilde{k}}{\tilde{k}-1}, \qquad e = (\tilde{k}-1)^2 + \frac{\tilde{v}-1}{\tilde{k}-1} - 2, \qquad f = \tilde{k}^2. \quad (1)$$

Recall that strongly regular graphs have the same parameters if and only if they are cospectral [8, Section 3.6]. For example, the Chang graphs [8, Example 1.2.6] are of Steiner type because they are cospectral with $L(K_8)$, while $L(K_q)$ is the block graph of the unique design $S(2, 2, q)$. We show in Section 2 that, aside from the complete multipartite graphs and graphs of Steiner type, there are only finitely many connected strongly regular graphs with a regular star complement of prescribed degree $s \in I\!N$. Note that complete graphs are excluded from our considerations, and so the case $s = 0$ does not arise (see Proposition 1.6). In Section 3, we investigate the cases $s = 1, 2, 3, 4, 5$. The results are of potential interest in relation to the construction of strongly regular graphs from star complements (cf. Examples 1.3). For instance, the existence of a strongly regular graph with parameters $(85, 14, 3, 2)$ remains open, but the parameters are consistent with the presence of a 4-regular graph of order 35 as a star complement for -3.

Here we first recall the required properties of star complements. For $X \subseteq V(G)$, we write G_X for the subgraph of G induced by X, and '$u \sim v$' to mean that vertices u and v are adjacent.

Theorem 1.2 [8, Theorem 5.1.7] *Let X be a set of k vertices in the graph G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \tag{2}$$

In this situation, $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in \mathbb{R}^k)$.

Writing $H = G - X$, we see that the columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u) = \{v \in V(H) : u \sim v\}$ $(u \in X)$. Thus G is determined by μ, a star complement H for μ, and the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$.

Examples 1.3 (i) The Petersen graph can be constructed from a 5-cycle as a star complement H for 1 by adding 5 vertices whose H-neighbourhoods are the singleton subsets of $V(H)$. It follows from (2) that if u, v are added, with neighbours $u', v' \in V(H)$, then $u \sim v$ if and only if $u' \not\sim v'$ [8, Example 5.2.3] .
(ii) For odd $n \geq 5$, the line graph $L(K_n)$ can be constructed from an n-cycle as a star complement H for -2 by adding $\frac{1}{2}n(n-3)$ vertices whose H-neighbourhoods have the form $\{u_1, u_2, u_3, u_4\}$ with $u_1 \sim u_2$ and $u_3 \sim u_4$. It follows from (2) that if u, v are added, then $u \sim v$ if and only if $\Delta_H(u)$, $\Delta_H(v)$ intersect in two adjacent vertices of H (cf. [1, Theorem 2.4]).

If G is r-regular and $\mu \neq r$ then the all-1 vector $\mathbf{j}_n$ is orthogonal to $\mathcal{E}(\mu)$; in other words, μ is a non-main eigenvalue (see [16], for example). From the description of $\mathcal{E}(\mu)$ in Theorem 1.1, we have the following result, where we write $\mathbf{j}$ for $\mathbf{j}_{n-k}$.

Proposition 1.4 [7, Proposition 0.3] *With the notation above, μ is a non-main eigenvalue if and only if*

$$\mathbf{b}_u^\top(\mu I - C)^{-1}\mathbf{j} = -1 \text{ for all } u \in X. \tag{3}$$

Proposition 1.5 *Let G be an r-regular graph with an s-regular subgraph $H = G - X$ as a star complement for the eigenvalue $\mu \neq r$. If μ has multiplicity k then $|\Delta_H(u)| = s - \mu$ for all $u \in X$ and*

$$k(r - \mu) = n(r - s). \tag{4}$$

Proof. By Proposition 1.4, we have $-1 = \mathbf{b}_u^\top(\mu - s)^{-1}\mathbf{j}$, whence $\mathbf{b}_u^\top\mathbf{j} = s - \mu$ for each $u \in X$. Counting edges between X and its complement $\bar{X}$, we see that $k(s - \mu) = (n - k)(r - s)$, equivalently $k(r - \mu) = n(r - s)$. □

It follows that, in the situation of Proposition 1.5, μ is an integer, while X and $\bar{X}$ form an equitable bipartition of $V(G)$; equivalently, X and $\bar{X}$ are regular sets in the sense of [5, 13]. The following observation disposes of the case $s = 0$.

Proposition 1.6 *If G is an r-regular graph ($r > 0$) with $\overline{K_t}$ as a star complement for the eigenvalue μ then either*
(a) $\mu = -1$ *and* $G = tK_{r+1}$, *or*
(b) $\mu = 1$ *and* $G = tK_2$.

Proof. Let X be a star set for μ, with $H = G - X = K_t$. Suppose first that $\mu \neq r$. Then from Equation (3) we have $\mathbf{b}_u^\top \mathbf{j} = -\mu$ for each $u \in X$. On the other hand, Equation (2) yields $\mathbf{b}_u^\top \mathbf{b}_u = \mu^2$, and so $\mu^2 = -\mu$. Since μ is not an eigenvalue of $\overline{K_t}$, we have $\mu = -1$; moreover, each neighbourhood $\Delta_H(u)$ ($u \in X$) is a singleton. For distinct vertices u, v in X, we see from Equation (2) that $u \sim v$ if and only if $\mathbf{b}_u^\top \mathbf{b}_v = 1$, equivalently $\Delta_H(u) = \Delta_H(v)$. It follows that each component of G is complete, and we have case (a).

If $\mu = r$ let $v \in X$, and let C be the component C of G containing v. Then $C - v$ is a star complement for μ in C, necessarily a star $K_{1,r}$. Since G is r-regular and $r > 0$, it follows that $r = 1$, $\mu = 1$, and we have case (b). □

2 Arithmetic

Let G be a connected strongly regular graph with parameters n, r, e, f ($2 \leq r \leq n - 2$). In particular (see [4, 3] for example),

$$(n - r - 1)f = r(r - e - 1) \tag{5}$$

and

$$2(r + 1) \leq n + f. \tag{6}$$

Suppose that G has an s-regular star complement $H = G - X$ for the eigenvalue μ, where μ has multiplicity k. Note that $k \neq 1$ (since G is not complete) and so $\mu \neq r$. Hence μ is a non-main eigenvalue. Since μ is an integer, G is not a 5-cycle, and so $r \geq 3$. Moreover (see [4, Chapter 2]),

$$\mu = \tfrac{1}{2}(e - f + \Delta), \quad k = \tfrac{1}{2}\{n - 1 + \tfrac{(n-1)(f-e)-2r}{\Delta}\}, \tag{7}$$

where

$$\Delta^2 = (e - f)^2 + 4(r - f). \tag{8}$$

We do not specify a sign for Δ. Substituting for μ, k and n in Equation (4), we obtain:

$$(2s - r)\Delta = r(e - f + 2). \tag{9}$$

Taking squares and using Equation (8) again, we obtain

$$r^2(r - e - 1) = s(r - s)\Delta^2. \tag{10}$$

Now let m be the greatest common divisor of r and s, say $r = pm$ and $s = qm$, where p and q are coprime. Then $p^2(r - e - 1) = q(p - q)\Delta^2$, whence p^2 divides Δ^2, say $\Delta^2 = a^2p^2$, where $a > 0$.

Lemma 2.1 *If $f < r$ then $a \leq m$, with strict inequality when $s > 2$.*

Proof. Since $0 < f < r$, we have $\Delta^2 \leq (r-1)^2 + 4(r-1) < (r+1)^2$, whence $a^2p^2 \leq r^2$ and $a \leq m$. If $a = m$ then Equation (10) becomes $r - e - 1 = s(r - s)$, whence $e = (1 - s)(r - 1 - s)$. In this situation, if $s > 1$ then $s = r - 1$, $e = 0$ and Equation (8) becomes $(r - f)(r + f - 4) = 0$. From this it follows that $r + f = 4$, and hence that $r = 3, f = 1, s = 2$ (cf. Example 1.1(ii)). □

We are now in a position to prove our finiteness result:

Theorem 2.2 *For each $s \in \mathbb{N}$, there exists a finite family $\mathcal{R}_s$ of strongly regular graphs with the following property. If G is a connected strongly regular graph with an s-regular star complement for the eigenvalue μ then exactly one of the following holds:*
(a) $\mu = 0$ *and* $G = \overline{(s+1)K_q}$ $(q \in \mathbb{N})$,
(b) $\mu = -1 - v$, $s = v(v+1)$ *and G is of Steiner type* $S(2, v+1, vw+1)$ $(v, w \in \mathbb{N})$,
(c) $G \in \mathcal{R}_s$.

Proof. If $f = r$ then G is a complete multipartite graph (with parts of size $n - r$), say $G = \overline{(s+1)K_q}$ $(q \in \mathbb{N})$, with spectrum $qs, 0^{((q-1)(s+1))}$, $-s^{(q)}$. If $r \neq 2s$ then by Equations (7) and (9), there is a unique solution for μ, necessarily $\mu = 0$. If $r = 2s$ then $q = 2$, and to verify (a) we must eliminate the possibility $\mu = -2$. In this case, a star complement for -2 has order $s + 2$, and so is a cocktail party graph (of order at least 4): this is a contradiction because such a graph has -2 as an eigenvalue. Thus (a) holds when $f = r$, and we now assume that $f < r$.

We write $\alpha = \frac{a}{m}$, so that Equation (10) becomes

$$e = s^2\alpha^2 - 1 - r(s\alpha^2 - 1). \tag{11}$$

Substituting for e and Δ^2 in Equation (8), and solving the resulting quadratic in f, we obtain:

$$f = s^2\alpha^2 + 1 - r(s\alpha^2 - 1) \pm \alpha(r - 2s). \tag{12}$$

We have

$$\frac{r(r-e-1)}{f} = \frac{rs(r-s)\alpha^2}{s^2\alpha^2 + 1 - r(s\alpha^2 - 1) \pm \alpha(r - 2s)},$$

and this is an integer by Equation (5). It is expressible as the quotient $(Jp^2 + Kp)/(Lp + M)$, where

$$J = qma^2, \quad K = -q^2ma^2, \quad L = m - qa^2 \pm a, \quad M = q^2a^2 + 1 \mp 2aq.$$

Now $L^2(Jp^2 + Kp) = J(Lp + M)^2 + (Lp + M)(LK - 2JM) + M(JM - KL)$ and so $Lp + M$ divides $M(JM - KL)$. This enables us to bound $Lp + M$ when $M(JM - KL) \neq 0$.

Consider the first choice of sign, and note that the argument in this case embraces the case $r = 2s$. We have $M \neq 0$ for otherwise $qa = 1$ and then $a = 1, s = m, \alpha = \frac{1}{s}$, whence $f = r$, contrary to assumption. Also $JM - KL = qma^2(mq - qa + 1)$, and this is non-zero by Lemma 2.1. Consequently $M(JM - KL) \neq 0$. From Lemma 2.1 we see also that, for given s, there are only finitely many possibilities m, a, q. If $L = 0$ then $m = a(qa - 1)$ and Equation (11) yields

$$s^2\alpha^2 - e - 1 = r(s\alpha^2 - 1) = \frac{ra}{m}, \tag{13}$$

whence $r < s^2a/m$. Then there are only finitely many possibilities for r, and (since $n \leq r^2 + 1$) only finitely many possibilities for n. On the other hand, if $L \neq 0$ then the relation $|Lp + M| \leq |M(JM - KL)|$ shows that p (and hence r and hence n) is bounded in terms of m, a, q (and hence in terms of s).

Now consider the second choice of sign, with $r \neq 2s$. Here $M = (qa+1)^2 \neq 0$, and $JM - KL = qma^2(1 + qa + qm) \neq 0$. Thus $M(JM - KL) \neq 0$, and if $L \neq 0$ then n is bounded as before. If $L = 0$ then $m = a(qa + 1)$ and so $r = pa(qa+1)$. From Equations (11) and (12), we have $e = q^2a^2 + pa - 1$, $f = (qa + 1)^2$. It follows that $n = (pa + 1)(pqa^2 + qa + 1)/(qa + 1)$. Comparing these parameters with those in (1), we see that G is of Steiner type $S(2, v+1, vw+1)$, where $v = qa$ and $w = pa+1$. In this situation, Equation (9) yields $\Delta = r(e - f + 2)/(2s - r) = -qa$ and so $\mu = \frac{1}{2}(e - f + \Delta) = -1 - qa = -1 - v$. Finally, $s = qa(qa + 1) = v(v + 1)$, and so we have case (b) of the Theorem. □

Note that $\mathcal{R}_1$ contains the Petersen graph, and in view of [17, Proposition 3.1], $\mathcal{R}_s$ contains the graph $\overline{L(K_{s+3})}$ whenever $s > 1$: if H is a 2-regular star complement for -2 in $L(K_{s+3})$ then $\overline{H}$ is an s-regular star complement for 1 in $\overline{L(K_{s+3})}$. Example 1.1(iii) shows that $\mathcal{R}_5$ contains the Gewirtz graph, and that a strongly regular graph with a regular star complement is not necessarily of the form $L(K_q)$ or $\overline{L(K_q)}$.

3 Regular star complements of small degree

In this section we investigate the parameters of G that arise when $s \leq 5$. We retain the notation of Section 2, and exclude the complete multipartite graphs by taking $f < r$. The possibilities for the parameters of a graph in $\mathcal{R}_s$ ($s \leq 5$) are listed in the accompanying table; information on the existence and uniqueness of the corresponding graphs may be found at `http://www.win.tue.nl/~aeb/graphs/srg/srgtab.html`, courtesy of A. E. Brouwer. There are at least 32649 strongly regular graphs whose parameters appear in the table, and the graphs themselves are not investigated here.

s	1	2	3	4	4	4	4	4	5	5	5	5
μ	-2	1	1	1	1	-3	-3	-3	1	-3	-3	-4
n	10	10	15	10	21	26	45	85	28	36	126	56
r	3	3	6	6	10	10	12	14	15	15	25	10
e	0	0	1	3	3	3	3	3	6	6	8	0
f	1	1	3	4	6	4	3	2	10	6	4	2

With minor variations, the parameters are found as follows. For prescribed s, a, m we use Equation (10) to find e in terms of r, and Equation (8) to determine the (one or two) possibilities for f before imposing the condition that f divides $r(r-e-1)$. When $r \neq 2s$ we find Δ from Equation (9), μ from Equation (7). We give just an outline of the calculations, together with a description of the graphs involved where appropriate.

The case $s = 1$. Here $r^2(r-e-1) = (r-1)\Delta^2$, whence $\Delta^2 = r^2$, $e = 0$ and Equation (8) becomes $(r-f)(r+f-4) = 0$. Thus $r + f = 4$ and so $(n, r, e, f) = (10, 3, 0, 1)$. From Equation (9), we have $\Delta = -3$ and so $\mu = -2$. In this case, G is the Petersen graph, arising as in Example 1.1(i). This result is just a special case of [17, Theorem 3.2].

The case $s = 2$. Here $r^2(r-e-1) = 2(r-2)\Delta^2$. If r is odd then $r^2 = \Delta^2$ and $r + e = 3$. Hence $r = 3$, $e = 0$ and we find in turn that $f = 1$, $n = 10$, $\Delta = 3$, $\mu = 1$. Thus G is the Petersen graph, arising as in Example 1.1(ii).

If r is even then $r^2 \neq \Delta^2$ (by the argument above) and so $r^2 = 4\Delta^2$ by Lemma 2.1. It follows that $r = 2e$, and then Equation (8) becomes $(f-4)(f-2e) = 0$. Hence $f = 4$, $e \neq 2$ and $n = \frac{1}{2}(e+1)(e+2)$. From Equation (9), we have $\Delta = -e$, and so $\mu = -2$. The parameters of G are those of $L(K_u)$, where $u = e+2$, and so G is cospectral with $L(K_u)$. Thus either $G = L(K_u)$ or $u = 8$ and G is a Chang graph (see [7, Chapter 4]). All three Chang graphs arise in case (b) of Theorem 2.2 because each has $C_3 \dot\cup C_5$ as a star complement for -2.

The case $s = 3$. Here $r^2(r-e-1) = 3(r-3)\Delta^2$, and by Lemma 2.1, either $\Delta^2 = \frac{1}{9}r^2$ or $\Delta^2 = \frac{4}{9}r^2$.

If $\Delta^2 = \frac{1}{9}r^2$ then $e = \frac{2}{3}r$, $f = \frac{1}{3}r + 4$ and

$$n - r - 1 = \frac{r(r-e-1)}{f} = \frac{r(r-3)}{r+12}.$$

It follows that $r + 12$ divides 180. Since $f \neq r > 3$ and r is divisible by 3, we have $r \in \{18, 24, 33, 48, 78, 168\}$. The cases $r = 24, 48, 168$ do not arise because they lead to non-integer values of k in (7). The cases $r = 18, 33, 78$ are ruled out by the 'absolute bound': $n \leq \frac{1}{2}k'(k'+3)$, where k' is the multiplicity of either multiple eigenvalue (see [19, Section 6] or [8, Theorem 3.6.7]).

If $\Delta^2 = \frac{4}{9}r^2$ then $\frac{1}{3}r = 3 - e$. Here the possibilities for (n, r, e, f) are $(28, 9, 0, 4)$ and $(15, 6, 1, 3)$, with associated spectra $28, 1^{(21)}, -5^{(6)}$ and $5, 1^{(9)}, -3^{(5)}$ respectively. The absolute bound is violated in the first case. In the second case, by considering $\overline{G}$ we see that $G = \overline{L(K_6)}$, an example noted in Section 2. Here $\mu = 1$ because a star complement H for μ has even order; thus H is a 3-regular graph of order 6. Since 1 is an eigenvalue of $\overline{C_6}$, necessarily $H = \overline{2K_3}$.

The case $s = 4$. Here $r^2(r - e - 1) = 4(r - 4)\Delta^2$, and by Lemma 2.1, $\Delta^2 \in \{\frac{1}{16}r^2, \frac{1}{4}r^2, \frac{9}{16}r^2\}$.

If $\Delta^2 = \frac{1}{16}r^2$ then $e = \frac{3}{4}r$ and Equation (8) yields $f = \frac{1}{2} + 4$. Writing $r = 4p$, we see that

$$\frac{r(r-e-1)}{f} = \frac{2p(p-1)}{p+2},$$

whence $p + 2$ divides 12. Since $f < r$, we deduce that either $(n, r, e, f) = (21, 16, 12, 12)$ or $(n, r, e, f) = (56, 40, 30, 24)$. In both cases, condition (6) is violated.

If $\Delta^2 = \frac{1}{4}r^2$ then $e = 3$ and $f \in \{\frac{1}{2}r + 1, 9 - \frac{1}{2}r\}$. If $f = \frac{1}{2}r + 1$ then $(n, r, e, f) \in \{(10, 6, 3, 4), (21, 10, 3, 6), (56, 22, 3, 12)\}$. Here the third possibility is ruled out by the absolute bound. If $(n, r, e, f) = (10, 6, 3, 4)$ then $G = L(K_5)$, $\mu = 1$ and $H = \overline{3K_2}$, an example complementary to Example 1.1(i). If $(n, r, e, f) = (21, 10, 3, 6)$ then $G = \overline{L(K_7)}$. Secondly, suppose that $f = 9 - \frac{1}{2}r$. Then $(n, r, e, f) \in \{(17, 8, 3, 5), (26, 10, 3, 4), (45, 12, 3, 3), (85, 14, 3, 2), (209, 16, 3, 1)\}$. The first possibility is excluded by the requirement that $f \mid r(r - e - 1)$, and the last by the condition: if $f = 1$ then $r \geq (e+1)(e+5)$ [9, Theorem 4.2]. The graphs with $(n, r, e, f) = (26, 10, 3, 4)$ are complements of graphs of Steiner type $S(2, 3, 13)$, and there are 10 of them (see [14]). There are 78 graphs with $(n, r, e, f) = (45, 12, 3, 3)$ [6]. The existence of a strongly regular graph with parameters $(85, 14, 3, 2)$ remains an open question.

If $\Delta^2 = \frac{9}{16}r^2$ then $5r = 32 - 4e$, impossible since $r \geq 5$.

The case s=5. Here $r^2(r - e - 1) = 5(r - 5)\Delta^2$, and by Lemma 2.1, $r = 5p$ for some integer $p > 1$. Moreover, $\Delta^2 \in \{\frac{1}{25}r^2, \frac{4}{25}r^2, \frac{9}{25}r^2, \frac{16}{25}r^2\}$.

If $\Delta^2 = \frac{1}{25}r^2$ then $e = \frac{4}{5}r$ and $f = \frac{3}{5}r + 4$. Since $r(r - e - 1)/f = 5p(p - 1)/3p + 4$ and $f < r$, we have $p \in \{8, 22\}$. Then $(n, r, e, f) = (51, 40, 32, 28)$ or $(144, 110, 88, 70)$, and in both cases condition (6) is violated.

If $\Delta^2 = \frac{4}{25}r^2$ then $r = 5(e - 3)$ and $f \in \{3e-8, 12-e\}$. We find that $(n, r, e, f) = (28, 15, 6, 10)$ or $(144, 65, 16, 40)$. In the first case, $\mu = 1$ and $\overline{G}$ is either $L(K_8)$ or a Chang graph. The second case is ruled out by the absolute bound. When $f = 12 - e$ we find that $r = 5p$, where $1 < p < 9$ and $9 - p$ divides $5p(4p - 3)$. Hence $p \in \{3, 5\}$ and $(n, r, e, f) = (36, 15, 6, 6)$ or $(126, 25, 8, 4)$. In both cases, $\mu = -3$. There are 32548 strongly regular graphs with parameters $(36, 15, 6, 60)$ [12], while a strongly regular graph with parameters $(126, 25, 8, 4)$ is described in [3].

If $\Delta^2 = \frac{9}{25}r^2$ then $r = 5p$, $e = 8 - 4p$ and necessarily $r = 10, e = 0$. Then $(n, r, e, f) = (56, 10, 0, 2)$ and G is the Gewirtz graph [10]. Here $\mu = -4$ because a 5-regular star complement has even order (cf. Example 1.1(iii)).

If $\Delta^2 = \frac{16}{25}r^2$ then $r = 5p$ and $e = -11p + 15$, impossible since $p > 1$.

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