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Graphs for which the least eigenvalue is minimal, II☆

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Dedicated to Michael Doob on his 65th birthday.

Abstract

We continue our investigation of graphs G for which the least eigenvalue $\lambda(G)$ is minimal among the connected graphs of prescribed order and size. We provide structural details of the bipartite graphs that arise, and study the behaviour of $\lambda(G)$ as the size increases while the order remains constant. The non-bipartite graphs that arise were investigated in a previous paper [F.K. Bell, D. Cvetković, P. Rowlinson, S.K. Simić, Graphs for which the least eigenvalue is minimal, I, Linear Algebra Appl. (2008), doi:10.1016/j.laa.2008.02.032]; here we distinguish the cases of bipartite and non-bipartite graphs in terms of size.

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1. Introduction

Let $G = (V_G, E_G)$ be a simple graph, with vertex set V_G and edge set E_G. Its *order* is $|V_G|$, denoted by n, and its *size* is $|E_G|$, denoted by m. We write $u \sim v$ to indicate that vertices u and v are adjacent, and we write A_G for the (0, 1)-adjacency matrix of G. The characteristic polynomial $\det(xI - A_G)$ is denoted by $\phi_G(x)$. The zeros of $\phi_G(x)$ are called the *eigenvalues of* G; recall that they are real since A_G is symmetric. We write $\lambda(G)$ for the least eigenvalue of G, $\rho(G)$ for the largest eigenvalue (the *index*) of G, and $\lambda_i(G)$ for the ith largest eigenvalue of G $(i = 1, 2, \ldots, n)$. The degree of a vertex v is denoted by $\deg(v)$.

In a previous paper [1] we investigated the graphs G for which $\lambda(G)$ is minimal among the connected graphs of prescribed order and size. We showed that if G is not complete then $\lambda(G)$ is a simple eigenvalue and G is either bipartite or a join of two graphs of a simple form. In this paper, we provide structural details of the bipartite graphs that arise, and study the behaviour of $\lambda(G)$ as the size increases while the order remains constant.

The main structural result in [1] is Theorem 3.7 which reads:

Theorem 1.1. *Let G be a connected graph whose least eigenvalue is minimal among the connected graphs of order n and size m $\left(0 < m < \binom{n}{2}\right)$. Then G is either*

(i) *a bipartite graph, or*
(ii) *a join of two nested split graphs (not both totally disconnected).*

A graph G is called a *nested split graph* if its vertices can be ordered so that $jq \in E_G$ implies $ip \in E_G$ whenever $i \leqslant j$ and $p \leqslant q$. The nested split graphs are the graphs without $2K_2$, P_4 or C_4 as an induced subgraph (cf. [5]); they are precisely the graphs with a stepwise adjacency matrix (see [4, Section 3.3]). For subsequent reference we provide further details from [1] of the graphs that arise in case (ii) of Theorem 1.1. Here, let $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ be an eigenvector corresponding to $\lambda(G)$, and let $V^- = \{u \in V_G : x_u < 0\}$, $V^0 = \{u \in V_G : x_u = 0\}$, $V^+ = \{u \in V_G : x_u > 0\}$. Let H^-, H^+ be the subgraphs of G induced by V^-, V^+, respectively. By [1, Proposition 3.5], if H^-, H^+ are not both totally disconnected then every vertex in V^- is adjacent to every vertex in V^+. Otherwise, $V_0 \neq \emptyset$ (since G is non-bipartite), and each vertex v in $V^- \cup V^+$ has a neighbour outside V_0 (by consideration of the corresponding eigenvalue equation $\lambda(G)x_v = \sum_{u \sim v} x_u$). Recall also that each vertex in V^0 is adjacent to all other vertices [1, Lemma 3.1]. Accordingly we can deduce the following:

Proposition 1.2. *In case* (ii) *of Theorem* 1.1, *G has an edge $e = vw$ such that $x_v x_w \geqslant 0$, $x_v \neq 0$ and $G - e$ is connected.*

For a bipartite graph G, we have $\lambda(G) = -\rho(G)$, and so in Section 2 we determine the structure of connected bipartite graphs with maximal index for prescribed n and m. Here, $m \leqslant \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, with equality if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. In Section 3, we investigate how the minimal least eigenvalue of bipartite graphs varies with m when n is fixed, while in Section 4 we use these results to study the same question for all connected graphs; in particular, we are in a position to distinguish cases (i) and (ii) of Theorem 1.1 when m varies.

2. The structure of extremal bipartite graphs

Before we state our main result in this section we need a definition.

Let G be a bipartite graph with colour classes U and V. We say that G is a *double nested graph* if there exist partitions $U = U_1 \dot\cup U_2 \dot\cup \cdots \dot\cup U_h$ and $V = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_h$, such that the neighbourhood of each vertex in U_1 is $V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_h$, the neighbourhood of each vertex in U_2 is $V_1 \dot\cup \cdots \dot\cup V_{h-1}$, and so on. If $|U_i| = m_i$ $(i = 1, 2, \ldots, h)$ and $|V_i| = n_i$ $(i = 1, 2, \ldots, h)$ then G is denoted by $D(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$.

Theorem 2.1. *If G is a graph for which $\lambda(G)$ is minimal (equivalently, $\rho(G)$ is maximal) among all connected bipartite graphs of order n and size m, then G is a double nested graph.*

Thus double nested graphs play the same role among bipartite graphs (with respect to the index) as nested split graphs among non-bipartite graphs. The proof of Theorem 2.1 is based on the following lemmas, the first of which is taken from [6]. Recall that the index ρ of a connected graph G is a simple eigenvalue, and that there exists a unique unit eigenvector corresponding to ρ having only positive entries; this eigenvector is called the *Perron eigenvector* of G.

Lemma 2.2. *Let G' be the graph obtained from a connected graph G by rotating the edge $r_i s$ around r_i to the non-edge position $r_i t$ for each $i \in \{1, \ldots, k\}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the Perron eigenvector of G. If $x_t \geqslant x_s$ then $\rho(G') > \rho(G)$.*

The next lemma will be very helpful when we encounter a bridge in a graph whose index is assumed to be maximal. Given two rooted graphs $P(=P_u)$ and $Q(=Q_v)$ with u and v as roots, let G be the graph obtained from the disjoint union $P \dot\cup Q$ by adding the edge uv. Let G' be the graph obtained from the coalescence of P_u and Q_v by attaching a pendant edge at the vertex identified with u and v.

Lemma 2.3. *With the above notation, if P and Q are two non-trivial connected graphs then $\rho(G) < \rho(G')$.*

Proof. Let $(x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the Perron eigenvector of G. Without loss of generality, we may suppose that $x_u \leqslant x_v$. Let Δ be the neigbourhood of u in P; since P is non-trivial, $\Delta \neq \emptyset$. Now G' is obtained from G by replacing the edges uw $(w \in \Delta)$ by the edges vw $(w \in \Delta)$, and so $\rho(G) < \rho(G')$ by Lemma 2.2, as required. □

In what follows we assume that G has maximal index among the connected bipartite graphs of fixed order and size.

Lemma 2.4. *Let G be a graph satisfying the above assumptions, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the Perron eigenvector of G. If v, w are vertices in the same colour class such that $x_v \geqslant x_w$ then* $\deg(v) \geqslant \deg(w)$.

Proof. Let U, V be the colour classes of G and suppose, by way of contradiction, that v, w are vertices in V such that $x_v \geqslant x_w$ and $\deg(v) < \deg(w)$. Then $\deg(w) > 1$ and there exists

$u \in U$ such that $v \nsim u \sim w$. By Lemma 2.1, we may rotate uw to uv to obtain a graph G' such that $\rho(G') > \rho(G)$. If uw is a bridge then $\deg(u) = 1$ by Lemma 2.3, and so G' is necessarily connected; but now the maximality of $\rho(G)$ is contradicted, and the proof follows. □

From now on we take the colour classes to be $U = \{u_1, u_2, \ldots, u_p\}$ and $V = \{v_1, v_2, \ldots, v_q\}$, with $x_{u_1} \geqslant x_{u_2} \geqslant \cdots \geqslant x_{u_p}$ and $x_{v_1} \geqslant x_{v_2} \geqslant \cdots \geqslant x_{v_q}$. By Lemma 2.4, this ordering coincides with the ordering by degrees in each colour class, and in the next lemma we note some consequences.

Lemma 2.5. *Let G be a graph satisfying the above assumptions including those on vertex ordering. Then*

(i) *the vertices u_1 and v_1 are adjacent*;
(ii) *u_1 is adjacent to every vertex in V, and v_1 is adjacent to every vertex in U*;
(iii) *if the vertex u is adjacent to v_k then u is adjacent to v_j for all $j < k$, and if the vertex v is adjacent to u_k then v is adjacent to u_j for all $j < k$.*

Proof. First we consider bridges in G: by Lemma 2.3, all bridges are pendant edges. By Lemma 2.2, all pendant edges are attached at the same vertex, and this vertex w is such that x_w is maximal. Without loss of generality, $x_{u_1} \geqslant x_{v_1}$ and $w = u_1$. It follows that the result holds if G is a tree, for then G is a star. Accordingly, we suppose that G is not a tree.

To prove (i), suppose by way of contradiction that $u_1 \nsim v_1$. Then v_1 is adjacent to some vertex $u \in U$, and uv_1 is not a bridge. By Lemma 2.2, we may rotate v_1u to v_1u_1 to obtain a connected bipartite graph G' such that $\rho(G') > \rho(G)$, contradicting the maximality of $\rho(G)$.

To prove (ii), suppose that u is a vertex of U not adjacent to v_1. Then $u \neq u_1$ by (i), uv is not a bridge, and u is adjacent to some vertex v in V other than v_1. Now we can rotate uv to uv_1 to obtain a contradiction as before. Secondly, suppose that v is a vertex of V not adjacent to u_1. Then $v \neq v_1$ by (i), again vu_1 is not a bridge, and a rotation about v yields a contradiction.

To prove (iii), suppose that $u \in U$, $u \sim v_k$ and $u \nsim v_j$ for some $j < k$. Now $u \neq u_1$ by (ii), and so uv_k is not a bridge. Then we can rotate uv_k to uv_j to obtain a contradiction. Finally, suppose that $v \in V$, $v \sim v_k$ and $v \nsim u_j$ for some $j < k$. In this case, vu_k is not a bridge because $k > 1$, and the rotation of vu_k to vu_j yields a contradiction.

This completes the proof. □

The proof of Theorem 2.1. follows now directly from Lemma 2.5 and the definition of a double nested split graph.

We conclude this section with two remarks.

First, with the notation of Lemma 2.5, let $d_i = \deg(u_i)$ $(i = 1, \ldots, p)$ and $e_j = \deg(v_j)$ $(j = 1, \ldots, q)$. Let Π_U be the integer partition $m = d_1 + d_2 + \cdots + d_p$, and let Π_V be the integer partition $m = e_1 + e_2 + \cdots + e_q$. We have $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ and $e_1 \geqslant e_2 \geqslant \cdots \geqslant e_q$; moreover, the structure of a double nested graph ensures that Π_U and Π_V are conjugate, i.e. the Ferrers diagram for Π_U is the transpose of the Ferrers diagram for Π_V.

Secondly, we can give an algorithm for constructing the double nested graphs of order n and size m. For each integer partition $\Pi : m = d_1 + d_2 + \cdots + d_p$ with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ and $d_1 + p = n$, we can construct the double nested graph with $U = \{u_1, u_2, \ldots, u_p\}$, $V = \{v_1, v_2, \ldots, v_q\}$, $q = d_1$ and $\Pi_U = \Pi$ as follows. Considering the vertices $u_1, u_2, \ldots, u_p$ in succession, we join u_k to the first d_k of the vertices $v_1, v_2, \ldots, v_q$.

3. The behaviour of the least eigenvalue of extremal connected bipartite graphs

We may summarize the results of this section as follows.

Theorem 3.1. *For fixed $n \geqslant 7$, let G_m be a graph whose least eigenvalue is minimal (equivalently, whose index is maximal) among the connected bipartite graphs of order n and size $m < \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. Then*

(i) *if $m \neq t(n-t)$ for all $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ then $\rho(G_m) < \rho(G_{m+1})$;*
(ii) *if $m = t(n-t)$ for some $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ then $\rho(G_m) > \rho(G_{m+1})$ unless G_{m+1} has the form $D(p, q; r, s)$, where*

$$\{t, n-t\} = \{p+q, r+s\}, t(n-t) = pr + ps + qr - 1 \leqslant pqrs.$$

The proof follows from sequence of lemmas in which we discuss how $\rho\ (G_m)$ varies with (for fixed n).

Lemma 3.2. *Under the above assumptions we have*:

(i) *$\rho(G_m) \leqslant \sqrt{m}$, with equality if and only if G_m is a complete bipartite graph $K_{t,n-t}$, for some $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$;*
(ii) *$\rho(G_m) < \rho(G_{m+1})$ whenever $t(n-t) + 1 \leqslant m < (t+1)(n-t-1)$, where $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$.*

Proof. Let $\lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} > \lambda_n$ be the eigenvalues of a connected bipartite graph G. Since G is bipartite we have

$$m = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \lambda_i^2. \tag{1}$$

It follows that $\rho(G_m) \leqslant \sqrt{m}$, with equality if and only if $\lambda_1^2 = m$ and $\lambda_2^2 = \cdots = \lambda_{\lfloor \frac{n}{2} \rfloor} = 0$. In this case, $G_m = K_{t,n-t}$ for some t (see, e.g. [2, Theorem 6.5]), and this completes the proof of (i).

In (ii), $m \neq t(n-t)$ for all t, and so G_m is not a complete bipartite graph. Thus G_m is a proper spanning subgraph of some complete bipartite graph K (of order n). Accordingly we may add to G some edge of K to obtain a connected bipartite graph G' of order n for which $\rho(G_m) < \rho(G')$. Since $\rho(G') \leqslant \rho(G_{m+1})$, the proof of (ii) is complete. □

Remark. Computational data obtained by F. Marić shows that if $m = t(n-t)$ for some $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ then both possibilities (namely $\rho(G_m) < \rho(G_{m+1})$ and $\rho(G_m) > \rho(G_{m+1})$) can arise. For $n = 9$ we have the situation presented in Fig. 1, where points at which $m = t(n-t) + 1$ for some t are indicated by vertical lines.

In considering the situation left unresolved by Lemma 3.2, we let $m = t(n-t)$ for some $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$. Then $G_m = K_{t,n-t}$, while G_{m+1} is a double nested graph $D(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$.

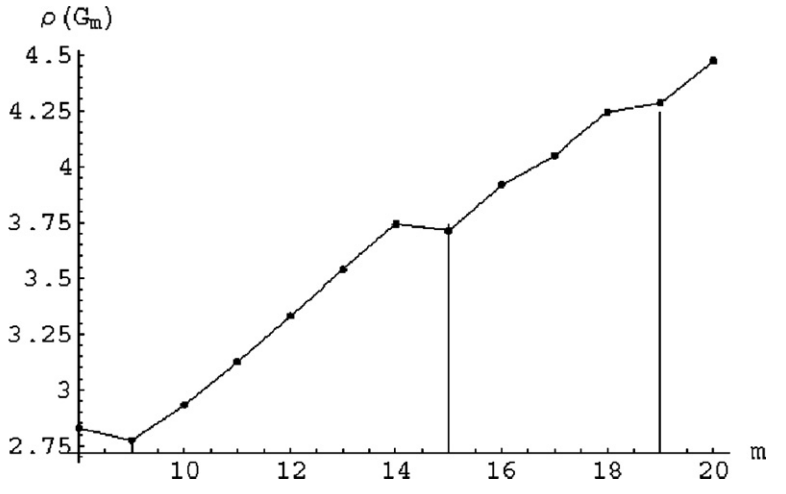

Fig. 1. The behavior of $\rho\ (G_m)$ when $n = 9$.

In the next two lemmas and Theorem 3.1, we assume that $n \geqslant 7$; when $n < 7$, we may refer to the tables of eigenvalues in [2,3].

Lemma 3.3. *Suppose that* $m = t(n - t)$ *and* $n \geqslant 7$. *If* $h \geqslant 3$ *then* $\rho(G_m) > \rho(G_{m+1})$.

Proof. We write $G = G_m$ and $G' = G_{m+1}$. Let $\lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} > \lambda_n$ and $\lambda'_1 > \lambda'_2 \geqslant \cdots \geqslant \lambda'_{n-1} > \lambda'_n$ be the eigenvalues of G and G', respectively.

From (1) we have immediately:

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (\lambda'_i)^2 - \lambda_1^2 = 1.$$

From this it follows that

$$\rho(G)^2 - \rho(G')^2 = \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} (\lambda'_i)^2 - 1. \tag{2}$$

In considering the relation (2), we distinguish two cases.

Case 1: $h \geqslant 4$. In this case, G' has an induced subgraph D_1, where $D_1 = D(1, 1, 1, 1; 1, 1, 1, 1)$, and we have $\lambda'_2 \geqslant \lambda_2(D_1)$. But $\lambda_2(D_1) > 1$, and so $\rho(G)^2 > \rho(G')^2$ by (2).

Case 2: $h = 3$. In this case, G' contains, as an induced subgraph, one of the graphs $D_2 = D(1, 1, 1; 1, 1, 2)$, $D_3 = D(1, 1, 1; 1, 2, 1)$ and $D_4 = D(1, 1, 1; 2, 1, 1)$. Since $\lambda_2(D_i) > 1$ $(i = 2, 3, 4)$, we have $\rho(G)^2 > \rho(G')^2$ as before.

This completes the proof. □

Remark. Note that the graphs D_i $(i = 1, 2, 3, 4)$ appearing in the above lemma are not the smallest induced subgraphs which can be used to obtain the required inequality.

When $h = 1$, G_{m+1} is itself a complete bipartite graph, $n = 2t + 2$ and $\rho(G_m) < \rho(G_{m+1})$. The next lemma deals with the remaining case, $h = 2$.

Lemma 3.4. *Suppose that* $m = t(n - t)$ *and* $G_{m+1} = D(p, q; r, s)$ (*so that* $m + 1 = pr + ps + qr$). *Then we have*:

(i) $\rho(G_m) < \rho(G_{m+1})$ *if* $m > pqrs$;
(ii) $\rho(G_m) = \rho(G_{m+1})$ *if* $m = pqrs$;
(iii) $\rho(G_m) > \rho(G_{m+1})$ *if* $m < pqrs$.

Proof. We write $G = G_m$, $G' = G_{m+1}$ as before, and we use the divisor technique (see [2, Chapter 4]) to compute the eigenvalues of G'. Note that $V_{G'}$ has $U_1 \dot\cup U_2 \dot\cup V_1 \dot\cup V_2$ as an equitable partition, and the corresponding divisor has adjacency matrix

$$A_D = \begin{pmatrix} 0 & 0 & r & s \\ 0 & 0 & r & 0 \\ p & q & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

We find easily that $\phi_{A_D^2}(x) = (x^2 - m'x + pqrs)^2$, where $m' = m + 1$.

The vertices in each of the four cells of the equitable partition are duplicate vertices of G', and together they give rise to $n - 4$ eigenvalues equal to 0. We deduce that there are just four non-zero eigenvalues in G', namely $\pm\lambda'_1$, $\pm\lambda'_2$ where

$$\lambda'^2_{1,2} = \frac{1}{2}\left(m' \pm \sqrt{m'^2 - 4pqrs}\right).$$

Now the result follows from (2). □

On the basis of Lemmas 3.3 and 3.4 the proof of Theorem 3.1 readily follows.

In case (ii) of Theorem 3.1, we can use a program written in *Mathematica* to check, for each 4-tuple (p, q, r, s), whether the corresponding graph exists. If at least one such graph exists then $\rho(G_m) \leqslant \rho(G_{m+1})$ by Lemma 3.4. We show that, in this situation, at least two of the parameters p, q, r, s are subject to an absolute bound.

By Lemma 3.4, we have the following basic requirement:

$$pr + ps + rq \geqslant 1 + pqrs. \tag{3}$$

In addition to this, we can assume

$$p + r \geqslant 3, \quad r \geqslant p. \tag{4}$$

The first condition in (4) follows from the fact that $D(p, q, r, s)$ is not a tree, while the second follows from the fact that we may interchange U and V if necessary. We consider the following three cases:

(a) $ps = 1$ (equivalently, $p = s = 1$);
(b) $qs = 1$ (equivalently, $q = s = 1$);
(c) $ps \neq 1$ and $qs \neq 1$.

Note that $rq \neq 1$, by (4).

In cases (a) and (b), respectively, we obtain immediately:

(a′) $p = 1, q \geqslant 1, r \geqslant \max\{2, p\}$ and $s = 1$;
(b′) $p \geqslant 1, q = 1, r \geqslant \max\{2, p\}$ and $s = 1$.

In case (c) we can prove the following:

Proposition 3.5. *If* (c) *holds, then p, q and s are bounded above; indeed, we have*

(c′) $p \leqslant 2, q \leqslant 2, r \geqslant p$ *and* $s \leqslant 3$.

Additionally, if $s = 1$ then $q \leqslant 2$; and if $2 \leqslant s \leqslant 3$ then $q = 1$.

Proof. We can rewrite (3) in the form

$$\frac{1}{qs} + \frac{1}{ps} + \frac{1}{rq} \geqslant 1 + \frac{1}{pqrs}. \tag{5}$$

If q is not bounded, then by letting $q \to +\infty$ we see that $ps \leqslant 1$, a contradiction to (c). Similarly, s is bounded, for otherwise $rq = 1$. Next, if p (and hence also r) is unbounded, then by letting $p, r \to +\infty$ we find that $qs \leqslant 1$, contradicting (c) again.

We now determine the upper bounds for p, q and s. First, if $s = 1$ then from (5) we obtain

$$q \leqslant \frac{1}{r} + \frac{p}{p-1} \leqslant \frac{5}{2}.$$

Here the second inequality holds because $p \geqslant 2$ (by (c)), while $r \geqslant 2$ (by (4)). Thus $q = 2$ (by (c)). Now from (4) and (3) (with $q = 2$ and $s = 1$) we find that $p < 3$, and hence that $p = 2$.

Secondly, if $s \geqslant 2$, we first use the relation

$$\frac{1}{qs} + \frac{1}{ps} + \frac{1}{rq} > 1 \tag{6}$$

to obtain

$$s < \frac{1 + \frac{1}{p}}{1 - \frac{1}{rq}} \leqslant 4.$$

Thus $s \in \{2, 3\}$, as required. From (6) we find that

$$q < \frac{1 + \frac{s}{r}}{s - \frac{1}{p}} < 2.$$

Thus $q = 1$. If $s = 2$, then from (4) and (3) (with $q = 1, s = 2$), we find that $p \leqslant 2$. Similarly, if $s = 3$ then we find that $p = 1$.

This completes the proof. □

4. The behaviour of the least eigenvalue of extremal connected graphs

In this section, we establish several propositions which serve to prove the following theorem.

Theorem 4.1. *Let G be a graph whose least eigenvalue is minimal among the connected graphs of order n and size m. Then*

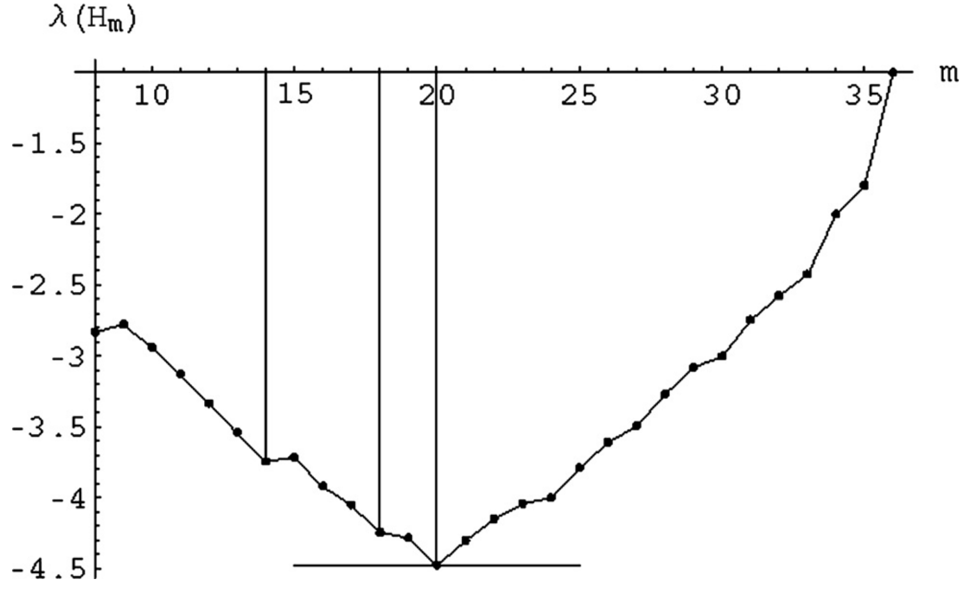

Fig. 2. The behavior of $\rho\ (H_m)$ when $n = 9$.

(i) *if* $n - 1 \leqslant m \leqslant \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ *and* $m \neq t(n-t)+1$ *for all* $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, *then G is bipartite and hence a double nested graph*;

(ii) *if* $m \leqslant \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ *and* $m = t(n-t)+1$ *for some* $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, *then G is either bipartite or the non-bipartite graph* $K_{t,n-t} + e$, *where e is an edge joining two vertices of degree* $\min\{t, n-t\}$ *in* $K_{t,n-t}$;

(iii) *if* $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil < m < \binom{n}{2}$ *then G is non-bipartite and hence the join of two nested split graphs.*

The bipartite graphs which appear in the case (ii) of Theorem 4.1 are more precisely described in Theorem 3.1(ii); see also Lemma 3.4 and Proposition 3.5.

We fix n and take H_m to be a graph whose least eigenvalue is minimal among the connected graphs of order n and size m. Fig. 2 shows the behaviour of $\lambda = \lambda(H_m)$ for $n = 9$ (obtained by direct calculation).

It was observed that, for $m \leqslant 20$, H_m is always a bipartite graph; of course, for $m > 20$ this is impossible. In the following proposition, we give a partial result which explains this phenomenon in a more general setting.

Proposition 4.2. *If* $m \leqslant \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ *and* $m \neq t(n-t)+1$, *where* $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, *then* H_m *is a bipartite graph.*

Proof. Assume the contrary, and let $H = H_m$ where m is the least integer for which the assertion is false. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be a unit eigenvector of H corresponding to $\lambda(H)$. From Proposition 1.2, we know that H contains an edge $e = vw$ such that $x_v x_w \geqslant 0$ and $H - e$ is connected. Writing $H^* = H - e$, we have

$$\lambda(H^*) \leqslant \mathbf{x}^{\mathrm{T}} A_{H^*} \mathbf{x} = \mathbf{x}^{\mathrm{T}} A_H \mathbf{x} - 2x_v x_w \leqslant \mathbf{x}^{\mathrm{T}} A_H \mathbf{x} = \lambda(H). \tag{7}$$

Now H_{m-1} is bipartite (by the choice of m), and so we have

$$\lambda(G_{m-1}) = \lambda(H_{m-1}) \leqslant \lambda(H^*) \leqslant \lambda(H) \leqslant \lambda(G_m).$$

On the other hand, since $m-1 \geqslant s(n-s)+1$, we have $\lambda(G_m) < \lambda(G_{m-1})$ by Lemma 3.2. This contradiction completes the proof. □

Remark. Note that the arguments in the above proof cannot always be used when $m = t(n-t)+1$ for some t, since then we may have $\lambda(G_{m-1}) < \lambda(G_m)$ (see Lemma 3.4).

When $n = 9$, we can see that, for $m > 20$, $\lambda(H_m)$ increases strictly with m (up to -1). This property is easily established in the general case:

Proposition 4.3. *For fixed n, and for $m > \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, $\lambda(H_m)$ increases strictly with m (to a maximum of -1).*

Proof. We use the notation of Proposition 4.2, with $H = H_m$, $H^* = H - e$, $e = vw$ and $\mathbf{x}$ a unit eigenvector of H corresponding to λ. By Proposition 1.2 we may choose v, w such that $x_v x_w \geqslant 0$ and $x_v \neq 0$. Now Eq. (7) holds, and we deduce that $\lambda(H^*) \leqslant \lambda(H)$. If $\lambda(H^*) = \lambda(H)$ then $\mathbf{x}$ is an eigenvector of H^* corresponding to λ; but then the eigenvalue equations for w in H and H^* are inconsistent since $x_v \neq 0$. Thus $\lambda(H^*) < \lambda(H)$, and since $\lambda(H_{m-1}) \leqslant \lambda(H^*)$, the proof is complete. □

Remark. Let $\hat{H}_m$ be a graph whose least eigenvalue is minimal among the connected non-bipartite graphs of order n and size m. If n is fixed and $m \leqslant \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ then $\lambda(\hat{H}_m)$ does not necessarily increase with m.

Finally, we resolve the situation not covered by Proposition 4.2.

Proposition 4.4. *If $m \leqslant \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and H_m is a non-bipartite graph, then $m = t(n-t)+1$ for some $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ and $H_m = K_{t,n-t} + e$, where e is an edge joining two vertices of degree $\min\{t, n-t\}$ in $K_{t,n-t}$.*

Proof. First, by Proposition 4.2 we have $m = t(n-t)+1$ for some $t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$. On the other hand, from Theorem 1.1 we know that H_m has a complete bipartite graph $B = K_{u,n-u} (u \leqslant \lfloor \frac{n}{2} \rfloor)$ as a proper spanning subgraph. Thus $u \leqslant t$, and it suffices to show that $u = t$. We suppose by way of contradiction that $u < t$.

Let $H = H_m$, and let $\mathbf{x}$ be a unit eigenvector for $\lambda(H)$. Then we have

$$\lambda(H) = \mathbf{x}^{\mathrm{T}} A_H \mathbf{x} = 2 \sum_{vw \in E_H} x_v x_w \geqslant 2 \sum_{vw \in E_B} x_v x_w \geqslant \lambda(B).$$

Now consider a graph $K = K_{t,n-t} + e$, where e is an edge joining two vertices in a colour class. We obtain the contradiction $\lambda(K) < \lambda(H)$ by showing that $\lambda(K) < \lambda(B)$. Note that $\lambda(B) \geqslant -\sqrt{c}$ where $c = (t-1)(n-t+1)$.

First we compute the spectrum of a graph $G = K_{a,b} + e$, where e is added to the colour class of size b. Counting the number of duplicate and co-duplicate vertices of G, we see that at least

$a + b - 3$ eigenvalues are equal to 0 or -1. On the other hand, if $b > 2$, three eigenvalues can be determined from the divisor with adjacency matrix

$$A_D = \begin{pmatrix} 0 & b-2 & 2 \\ a & 0 & 0 \\ a & 0 & 1 \end{pmatrix}.$$

Thus the three remaining eigenvalues are the solutions of $f(x) = 0$, where

$$f(x) = x^3 - x^2 - abx + a(b-2).$$

If $b = 2$ then $A_D = \begin{pmatrix} 0 & 2 \\ a & 1 \end{pmatrix}$, and again the least eigenvalue is a solution of $f(x) = 0$.

Taking $a = t, b = n - t$, we have

$$f(-\sqrt{c}) = \sqrt{c}(n - 2t + 1) + (n - 4t + 1) > (t - 1)(n - 2t + 1) + (n - 4t + 1) \geqslant 0.$$

Hence $\lambda(K) < -\sqrt{c} \leqslant \lambda(B)$, and so $\lambda(K) < \lambda(H)$ as required.

Finally, suppose that $a > b$. If we interchange a and b above, $f(x)$ is replaced by $g(x)$, where $g(x) = f(x) + 2(a - b)$. Since $g(x) > f(x)$, the smallest root of $g(x)$ is less than the smallest root of $f(x)$. Accordingly, $\lambda(K)$ is minimal when e joins two vertices of smaller degrees.

This completes the proof. □

Remark. We give an example due to F. Marić which illustrates Proposition 4.4. If $n = 12$ and $m = 21$ then $H_m = K_{2,10} + e$, where e is an edge joining two vertices of degree 2 in $K_{2,10}$. Actually, now $\lambda(H_m) = -4.38835\ldots$, while any connected bipartite graph of order 12 and size 21 has all eigenvalues greater than $-4.37228\ldots$, as required. Among all graphs G of order 12 and size 21 (not necessarily connected), the minimal value of $\lambda(G)$ is not attained by H_{21} because $\lambda(K_{3,7}\dot{\cup}2K_1) = -\sqrt{21} = -4.58275\ldots$.

In view of Theorem 1.1 and Propositions 4.2, 4.4, the proof of Theorem 4.1 clearly follows.

Remark. Let $\mathcal{G}(n, m)$ be the set of graphs of order n and size m, and define

$$f(n, m) = \min\{\lambda(G) : G \in \mathcal{G}(n, m)\},$$
$$g(n, m) = \min\{\lambda(G) : G \in \mathcal{G}(n, m) \text{ and } G \text{ is connected}\}.$$

We noted in [1] that $f(n, m) = \min\{g(k, m) : k \leqslant n$ and $\mathcal{G}(k, m)$ contains at least one connected graph$\}$. Since $k - 1 \leqslant m \leqslant k(k - 1)/2$, we have

$$\frac{1}{2}(1 + \sqrt{1 + 8m}) \leqslant k \leqslant \min\{n, m + 1\}.$$

To find the value of k for which the minimum of $g(k, m)$ is attained, we need to know the behaviour of $\min\{\lambda(G) : G \in \mathcal{G}(k, m)\}$ as a function of k when m is constant. In principle, this can be deduced from Theorem 4.1 but we do not attempt an explicit formulation.

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