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Spectral upper bounds for the order of a k-regular induced subgraph

Domingos M. Cardoso[1]
Departamento de Matemática
Universidade de Aveiro
3810-193 Aveiro
Portugal

dcardoso@ua.pt

Peter Rowlinson[2]
Department of Computing Science and Mathematics
University of Stirling
Stirling FK9 4LA
Scotland

p.rowlinson@stirling.ac.uk

Abstract. Let G be a simple graph with least eigenvalue λ, and let S be a set of vertices in G which induce a subgraph with mean degree k. We use a quadratic programming technique in conjunction with the main angles of G to establish an upper bound of the form $|S| \leq \inf\{(k+t)q_G(t) : t > -\lambda\}$, where q_G is a rational function determined by the spectra of G and its complement. In the case $k = 0$ we obtain improved bounds for the independence number of various benchmark graphs.

Keywords: graph, main eigenvalue, independence number, clique number

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[2]Corresponding author

1 Introduction

Let G be a simple graph of order n with $(0,1)$-adjacency matrix A and characteristic polynomial $P_G(x) = \det(xI - A)$. The i-th largest eigenvalue of A is denoted by $\lambda_i(G)$, and we write $\lambda_i = \lambda_i(G)$, $\overline{\lambda_i} = \lambda_i(\overline{G})$, where $\overline{G}$ denotes the complement of G.

Let S be a set of vertices in G which induce a subgraph with mean degree k. We use a quadratic programming technique [2, 3] in conjunction with the main angles of G [8, Section 4.5] to prove that

$$|S| \leq \inf\{h_k^G(t) : t > -\lambda_n(G)\}, \tag{1}$$

where

$$h_k^G(t) = (k+t)\left\{1 - \frac{P_{\overline{G}}(t-1)}{(-1)^n P_G(-t)}\right\}.$$

Thus if we write $H_G(t)$ for the walk-generating function of G (see [4] or [14]) then

$$h_k^G(t) = \left(1 + \frac{k}{t}\right) H_G\left(-\frac{1}{t}\right).$$

We give computational results which demonstrate that the bound (1) is superior to previous bounds. We make use of the functions $f_{k,t}^A : I\!R^n \to I\!R$ defined for $t > 0$ by

$$f_{k,t}^A(\mathbf{x}) = 2\mathbf{j}^\top \mathbf{x} - \frac{1}{k+t}\mathbf{x}^\top (A + tI)\mathbf{x},$$

where $\mathbf{j}$ denotes the all-1 vector in $I\!R^n$. These functions were constructed in [3] to determine upper bounds for the order of a k-regular induced subgraph in terms of eigenvalues. The problem of finding the largest order of such a subgraph is NP-complete [2, Section 2], whereas spectral upper bounds can be computed in polynomial time. We too state our results in terms of k-regular induced subgraphs, but they apply equally to induced graphs with mean degree k (for example, induced unicyclic graphs, with mean degree 2). When $k = 0$ we obtain an upper bound for the independence number $\alpha(G)$; a spectral lower bound for $\alpha(G)$, in terms of n, $\overline{\lambda}_n$ and the mean degree of G, is derived in [13].

We shall first summarize the basic argument in [3]. Recall that the eigenvalue λ of G is a *main* eigenvalue if the eigenspace $\mathcal{E}_A(\lambda)$ is not orthogonal to $\mathbf{j}$. In particular, λ_1 is a main eigenvalue because the Perron-Frobenius theory ensures that A has a corresponding eigenvector whose entries are all non-negative.

If $t \geq -\lambda_n$ then $f_{k,t}^A$ is concave, that is,

$$f_{k,t}^A(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \geq \theta f_{k,t}^A(\mathbf{x}) + (1-\theta) f_{k,t}^A(\mathbf{y})$$

whenever $0 \leq \theta \leq 1$. (To see this, express $\mathbf{x}, \mathbf{y}$ as sums of eigenvectors of A; alternatively, note that the Hessian matrix of $f_{k,t}^A(t)$ is $\frac{-2}{k+t}(A + tI)$, which is negative semi-definite when $t \geq -\lambda_n$.) Accordingly, $f_{k,t}^A$ has a global maximum at $\mathbf{x}^*$ if and only if $\nabla f_{k,t}^A(\mathbf{x}^*) = \mathbf{0}$, that is,

$$\mathbf{j} - \frac{1}{k+t}(A + tI)\mathbf{x}^* = \mathbf{0}.$$

Then $f_{k,t}^A(\mathbf{x}^*) = \mathbf{j}^\top \mathbf{x}^*$. If $\mathbf{x}_S$ is the characteristic vector of S then $\mathbf{x}_S^\top A \mathbf{x}_S = k|S|$ and so $|S| = f_{k,t}^A(\mathbf{x}_S) \leq f_{k,t}^A(\mathbf{x}^*)$. Note that $f_{k,t}^A(\mathbf{x}_S) = f_{k,t}^A(\mathbf{x}^*)$ if and only if $(A+tI)\mathbf{x}_S = (k+t)\mathbf{j}$, equivalently S is a $(k, k+t)$-regular set (that is, S induces a k-regular subgraph, while each vertex outside S is adjacent to $k+t$ vertices inside S).

Let J denote an all-1 matrix. If $G \neq \overline{K_n}$ and $\overline{\lambda}$ is a main eigenvalue of $\overline{G}$ such that $\overline{\lambda} \geq -\lambda_n - 1$, then we may take $t = \overline{\lambda} + 1$ and

$$\mathbf{x}^* = \frac{k+t}{\mathbf{j}^\top \mathbf{u}} \mathbf{u},$$

where $\mathbf{u}$ is an eigenvector of $J - I - A$ corresponding to $\overline{\lambda}$ such that $\mathbf{j}^\top \mathbf{u} \neq 0$. (Note that then $(A+tI)\mathbf{u} = J\mathbf{u} = \mathbf{j}\,\mathbf{j}^\top \mathbf{u}$.) The Courant - Weyl inequalities imply that

$$\lambda_2(\overline{G}) + \lambda_n(G) \leq \lambda_2(K_n) = -1 = \lambda_n(K_n) \leq \lambda_1(\overline{G}) + \lambda_n(G).$$

Thus we may always take $\overline{\lambda} = \overline{\lambda}_1$, and the remaining possibility is $\overline{\lambda} = -\lambda_n - 1$ when $-\lambda_n - 1$ is a main eigenvalue of $\overline{G}$. Since $f_{k,t}^A(\mathbf{x}^*) = \overline{\lambda} + k + 1$, we obtain:

Theorem 1.1 (cf. [3, Section 3]). *Let G be a graph of order n, and let S be a set of vertices which induces a k-regular subgraph of G $(0 \leq k \leq n-1)$. Then*

$$|S| \leq \overline{\lambda}_1 + k + 1. \tag{2}$$

If $-\lambda_n - 1$ is a main eigenvalue of $\overline{G}$ then

$$|S| \leq -\lambda_n + k \tag{3}$$

Two remarks are in order:

(i) When $k = 0$ we obtain from (2) the well-known upper bound $\overline{\lambda}_1 + 1$ for the independence number $\alpha(G)$. This bound is attained when, for example, G is a complete graph or a complete bipartite graph.

(ii) If $-\lambda_n - 1$ is a main eigenvalue of $\overline{G}$ then λ_n is a non-main eigenvalue of G, and $-\lambda_n - 1$ is a multiple eigenvalue of $\overline{G}$. This is a particular case of the following observation, essentially Theorem 2.12 of [5], for which we give a direct proof.

Proposition 1.2. *If λ is an eigenvalue of G such that $-\lambda - 1$ is a main eigenvalue of $\overline{G}$, then λ is a non-main eigenvalue of G; moreover, if λ has multiplicity d as an eigenvalue of G then $-\lambda - 1$ has multiplicity $d+1$ as an eigenvalue of $\overline{G}$.*

Proof. Let $(J - I - A)\mathbf{y} = (-\lambda - 1)\mathbf{y}$, where $\mathbf{j}^\top \mathbf{y} \neq 0$. Let $\mathbf{x} \in \mathcal{E}_A(\lambda)$. Then $(J - A)\mathbf{y} = -\lambda \mathbf{y}$ and $\mathbf{x}^\top A = \lambda \mathbf{x}^\top$. Hence $\mathbf{x}^\top (J - A)\mathbf{y} = -\lambda \mathbf{x}^\top \mathbf{y}$ and $\mathbf{x}^\top A \mathbf{y} = \lambda \mathbf{x}^\top \mathbf{y}$. Adding, we have $\mathbf{x}^\top J \mathbf{y} = 0$, that is, $\mathbf{x}^\top \mathbf{j}\,\mathbf{j}^\top \mathbf{y} = 0$. Hence $\mathbf{x}^\top \mathbf{j} = 0$ for all $\mathbf{x} \in \mathcal{E}_A(\lambda)$; in other words, λ is a non-main eigenvalue of G. Now $\mathcal{E}_{J-I-A}(-\lambda - 1) \cap \mathbf{j}^\perp = \mathcal{E}_A(\lambda)$, and the second assertion follows. $\square$

2 Further bounds

Here we introduce improved bounds by involving the main angles of G. We write $\mu_1, \ldots, \mu_s$ for the main eigenvalues of G in decreasing order. Then $\mathbf{j}$ is expressible as

$$\mathbf{j} = \mathbf{u}_1 + \cdots + \mathbf{u}_s \quad (\mathbf{u}_i \in \mathcal{E}_A(\mu_i)).$$

Thus $\mu_1 = \lambda_1$, and the non-zero main angles of G are $\beta_1, \ldots, \beta_s$ where $\sqrt{n}\beta_i = \|\mathbf{u}_i\|$ $(i = 1, \ldots, s)$.

Theorem 2.1. *Let G be a graph of order n, and let S be a set of vertices which induces a k-regular subgraph of G $(0 \leq k \leq n-1)$. If $t > -\lambda_n$ then*

$$|S| \leq n \sum_{i=1}^{s} \frac{t+k}{t+\mu_i} \beta_i^2; \tag{4}$$

equivalently,

$$|S| \leq h_k^G(t), \tag{5}$$

where

$$h_k^G(t) = (k+t)\left\{1 - \frac{P_{\overline{G}}(t-1)}{(-1)^n P_G(-t)}\right\}. \tag{6}$$

Proof. If $t > -\lambda_1$ then the function $f_{k,t}^A$ is concave and attains its maximum at

$$\mathbf{x}^* = \sum_{i=1}^{s} \frac{k+t}{\mu_i + t} \mathbf{u}_i.$$

Hence

$$|S| = f_{k,t}^A(\mathbf{x}_S) \leq \mathbf{j}^\top \mathbf{x}^* = n \sum_{i=1}^{m} \frac{t+k}{t+\mu_i} \beta_i^2.$$

The equivalent bound (5) is obtained by setting $x = t - 1$ in the formula [7, p.90]

$$P_{\overline{G}}(x) = (-1)^n P_G(-x-1)\left\{1 - \sum_{i=1}^{s} \frac{n\beta_i^2}{x+1+\mu_i}\right\}. \tag{7}$$

□

When λ_n is a main eigenvalue of G, the graph of $y = h_k^G(t)$ has $t = -\lambda_n$ as an asymptote, and so we state our main result as follows. Here the second assertion follows from our remarks in Section 1.

Corollary 2.2. *If S induces a k-regular subgraph of G then*

$$|S| \leq \inf\{h_k^G(t) : t > -\lambda_n(G)\}.$$

We have $|S| = h_k^G(t_0)$ if and only if S is a $(k, k+t_0)$-regular set.

When λ_n is a non-main eigenvalue of G, we have $G \neq \overline{K_n}$ and we may take $t = -\lambda_n$ to obtain the following reformulation of [3, Theorem 3.4]:

Theorem 2.3. *Let G be a graph of order n, and let S be a set of vertices which induces a k-regular subgraph of G $(0 \leq k \leq n-1)$. If λ_n is a non-main eigenvalue of G then*

$$|S| \leq n \sum_{i=1}^{s} \frac{-\lambda_n + k}{-\lambda_n + \mu_i} \beta_i^2. \tag{8}$$

In Equation (5) we should cancel factors common to $P_{\overline{G}}(t-1)$ and $P_G(-t)$. To this end, let $M_G(x) = (x-\mu_1)\cdots(x-\mu_s)$, and $M_{\overline{G}}(x) = (x-\overline{\mu}_1)\cdots(x-\overline{\mu}_s)$, where $\overline{\mu}_1,\ldots,\overline{\mu}_s$ are the main eigenvalues of $\overline{G}$ (cf. [14]). By Proposition 1.2 applied to G and $\overline{G}$, or by Equation (8) of [14], we have

$$\frac{P_{\overline{G}}(t-1)}{(-1)^n P_G(-t)} = \frac{M_{\overline{G}}(t-1)}{(-1)^s M_G(-t)}; \tag{9}$$

moreover, $M_{\overline{G}}(t-1)$ and $M_G(-t)$ have no common factors. Thus $h_k^G(t) = k+t$ if and only if $t-1$ is a main eigenvalue of $\overline{G}$. In particular, we may take $t = 1+\overline{\lambda}_1$ to obtain the bound (1). In the case that $-1-\lambda_n$ is a main eigenvalue of $\overline{G}$, we take $t = -\lambda_n$ in (4) and (6) to deduce:

Proposition 2.4. *When $-\lambda_n - 1$ is a main eigenvalue of $\overline{G}$, the upper bounds* (3) *and* (8) *coincide.*

To discuss the improvements on (2) afforded by Corollary 2.2, we write $h_k(t)$ for $h_k^G(t)$. If either

$$-\lambda_n < \overline{\lambda}_1 + 1 \text{ and } h_k'(1+\overline{\lambda}_1) \neq 0$$

or

$$-\lambda_n = \overline{\lambda}_1 + 1 \text{ and } h_k'(1+\overline{\lambda}_1) < 0,$$

then an improvement on (1) is assured in a neighbourhood of $1+\overline{\lambda}_1$. We have

$$h_k'(1+\overline{\lambda}_1) = 1-(k+1+\overline{\lambda}_1)(-1)^s\left\{\frac{M_{\overline{G}}'(\overline{\lambda}_1)}{M_G(-1-\overline{\lambda}_1)}\right\},$$

but it is more revealing to inspect two small examples.

Example 2.5. Let $G = 3K_1\dot{\cup}K_2\dot{\cup}K_3$. Then $P_G(x) = (x-2)(x-1)x^3(x+1)^3$. Using the computer package GRAPH, we find that $P_{\overline{G}}(x) = (x^3-2x^2-21x-24)\,x^3(x+1)^2$; moreover, 0 $(= -\lambda_8 - 1)$ is not a main eigenvalue of $\overline{G}$. We have $\overline{\lambda}_1 \approx 6.0930$, and so the bound (1) yields $|S| \leq 7$ when $k=0$. Here $\mu_s = 0 = k$ and $y = h_0(t)$ does not have $t=0$ as an asymptote. We have

$$h_0(t) = \frac{2(2t+3)(2t+1)}{(t+1)(t+2)},$$

a function which increases monotonically on $[-\lambda_8, \infty)$. Whenever $h_k(t)$ has this property, and $\mu_s > \lambda_n$, the best bound arises when $t = -\lambda_n$, giving a formula that coincides with (8). In this example, we obtain $|S| \leq 5$ (a sharp upper bound since $\alpha(G) = 5$). □

Example 2.6. Let G be the graph on 6 vertices numbered 50 in the table [6], where characteristic polynomials are listed and main angles are identified; the complement of G is numbered 100 in [6]. We have $s = 4$, $\mu_4 = \lambda_6 \approx -2.508$ and $\overline{\lambda}_1 \approx 2.228$. We take $k=0$ again, and then the upper bound (1) for $|S|$ is 3.228. In this case $y = h_0(t)$ has $t = -\lambda_6$ as an asymptote. Explicitly,

$$h_0(t) = \frac{2t(3t^3-9t^2+t+7)}{t^4-9t^2+4t+7}.$$

This function has a unique local minimum on $(-\lambda_6, \infty)$. Using Mathematica, we find that this minimum is 3.132 at $t = 2.834$ (to three places of decimals).

This new upper bound is smaller, but of course both bounds yield $|S| \leq 3$ (a sharp inequality since $\alpha(G) = 3$). $\square$

These examples are provided to illustrate differences in the behaviour of h_k. To demonstrate the superiority of the bound in Corollary 2.2, we should consider larger graphs, and this we do in the the next section. Here we first discuss properties of h_k in the general case.

Proposition 2.7 *The function $h_k(t)$ has at most one local minimum in $(-\mu_s, \infty)$.*

Proof. The result is immediate if $s = 1$ (that is, if G is regular), since then $h_k(t)$ is monotonic. Accordingly we suppose that $s > 1$. We have

$$h_k(t) = n - \sum_{i=1}^{s} \frac{n(\mu_i - k)\beta_i^2}{t + \mu_i}. \tag{10}$$

Suppose first that k is not a main eigenvalue of G, so that the graph $\mathcal{G}$ of $y = h_k(t)$ has asymptotes $t = -\mu_i$ $(i = 1, \dots, s)$. Note also that $h_k(t) \to n$ as $t \to \infty$ and as $t \to -\infty$.

If $\mu_s < k$ then the line $y = d$ cuts $\mathcal{G}$ in (at least) $s-1$ points of $(-\infty, -\mu_s)$ when $d > n$, and (at least) $s-2$ points of $(-\infty, -\mu_s)$ when $d < n$. If $\mu_s > k$ then the line $y = d$ cuts $\mathcal{G}$ in (at least) s points of $(-\infty, -\mu_s)$ when $d > n$, and (at least) $s-1$ points of $(-\infty, -\mu_s)$ when $d < n$.

Now suppose that $h_k(t)$ has a local minimum at $t_0 \in (-\mu_s, \infty)$. Then $h_k'(t) \geq 0$ for all $t \geq t_0$, for otherwise $h_k(t)$ has a local maximum at some point $t_1 \in (t_0, \infty)$. If $h_k(t_1) > n$ then for some $d > n$, the line $y = d$ cuts $\mathcal{G}$ in (at least) 3 points in $(-\mu_s, \infty)$. If $h_k(t_1) \leq n$ then for some $d < n$, the line $y = d$ cuts $\mathcal{G}$ in (at least) 4 points in $(-\mu_s, \infty)$. In any case, the function $h_k(t) - d$ has more than s zeros in $\mathbb{R}$. This is a contradiction because $h_k(t) - d$ $(d \neq n)$ has the form $p(t)/q(t)$, where $p(t), q(t)$ are polynomials of degree s.

If k is a main eigenvalue of G, then the same arguments apply to a graph with $s-1$ vertical asymptotes.

It follows that $h_k(t)$ has no more than one local minimum in $(-\mu_s, \infty)$. $\square$

Corollary 2.8 *For a non-regular graph G, we have:*
(i) *if $\mu_s < 0$ then $h_0^G(t)$ has a unique local minimum in $(-\mu_s, \infty)$,*
(ii) *if $\mu_s = 0$ then $h_0^G(t)$ is increasing on $(-\mu_{s-1}, \infty)$,*
(iii) *if $\mu_s > 0$ then $h_0^G(t)$ is increasing on $(-\mu_s, \infty)$.*

Proof. We have $h_0(1 + \overline{\lambda}_1) = 1 + \overline{\lambda}_1 < n$ and $1 + \overline{\lambda}_1 \in (-\mu_s, \infty)$. Thus if $\mu_s < 0$ then $h_0(t)$ has a local minimum on $(-\mu_s, \infty)$, and this minimum is unique by Proposition 2.7. If $\mu_s = 0$ then from (10) we see that $h_0'(t) > 0$ for all $t \in (-\mu_{s-1}, \infty)$, and if $\mu_s > 0$ then $h_0'(t) > 0$ for all $t \in (-\mu_s, \infty)$. $\square$

We conclude this section by deriving sharp upper bounds in two special cases. First, if G is r-regular, we may apply Theorem 2.3 to obtain

$$|S| \leq \frac{n(k - \lambda_n)}{r - \lambda_n}.$$

This bound, known as the Hoffman bound when $k = 0$, coincides with that obtained from interlacing (cf. [10, Lemma 9.6.2]). It is attained in some of the regular graphs $\overline{G}$ discussed in Section 3. Other generalizations of the Hoffman bound may be found in [1, Theorem 7] and [9, Corollary 3.2].

Secondly, consider a connected harmonic graph G, that is, a connected graph G for which $A\mathbf{d} = \mu_1\mathbf{d}$, where $\mathbf{d}$ is the vector whose entries are the vertex degrees. We show that if G has e edges then

$$\alpha(G) \leq n - \frac{e}{\mu_1}. \tag{11}$$

The main eigenvalues of G are μ_1 and 0 [14, Proposition 3.3], and so

$$\alpha(G) \leq h_0(-\lambda_n) = n\left\{1 - \frac{\mu_1}{\mu_1 - \lambda_n}\beta_1^2\right\} \leq n(1 - \tfrac{1}{2}\beta_1^2).$$

To determine β_1 when G is connected, note that

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{d}\|^2}(\mathbf{d}^\top\mathbf{j})\mathbf{d}, \quad \text{whence} \quad n\beta_1^2 = \frac{4e^2}{\|\mathbf{d}\|^2}.$$

Since $\mathbf{d} - \mu_1\mathbf{j} \in \mathcal{E}_A(0) \subseteq \mathcal{E}_A(\mu_1)^\perp = \mathbf{d}^\perp$, we have $\|\mathbf{d}\|^2 = 2e\mu_1$, and so $n(1 - \frac{1}{2}\beta_1^2) = n - \frac{e}{\mu_1}$, proving (11). We note that this bound is attained in all Grünewald trees [11, 14]: for such a tree T we have $\lambda_n = -\mu_1$, $e = n - 1 = \mu_1(\mu_1^2 - \mu_1 + 1)$ and $\alpha(T) = (\mu_1 - 1)(\mu_1^2 - \mu_1 + 1) + 1$.

3 Computational results

Here we apply our results to $\overline{G}$ with $k = 0$ to obtain bounds on the clique number $\omega(G) = \alpha(\overline{G})$. We compare old and new bounds for $\omega(G)$ for graphs G from the Second DIMACS Implementation Challenge [12]: these are benchmark graphs used for testing algorithms that determine or estimate $\omega(G)$. The old bounds in the table are given by $1 + \lambda_1(G)$, while the new bounds $h_0^{\overline{G}}(t^*)$ are calculated in accordance with Corollary 2.8: if $\overline{\mu}_s \geq 0$ (in particular, if $\overline{G}$ is regular) then $t^* = -\overline{\lambda}_n$; otherwise $h_0^{\overline{G}}(t^*)$ is the unique local minimum on $(-\overline{\lambda}_n, \infty)$. In practice, t^* is determined to within a computational error, and so

$$h_0^{\overline{G}}(t^*) \approx \inf\{h_0^{\overline{G}}(t) : t > -\lambda_n(\overline{G})\}.$$

Most of the graphs in the table have $\lambda_n(\overline{G})$ as a main eigenvalue, with $h_0'(1 + \lambda_1(G)) > 0$, where $h_0 = h_0^{\overline{G}}$. Then $\overline{\mu}_s < 0$ and we estimate t^* using successive bisections of intervals, starting with $[-\lambda_n(\overline{G}) + 10^{-6}, \lambda_1(G) + 1]$, where the value of h_0 at the mid point is less than the value at each end point. For an interval $[a, b]$ with mid-point c, let x, y be the mid points of $[a, c], [c, b]$ respectively. If $h_0(x)$ and $h_0(y)$ are both greater than $h_0(c)$ then we replace $[a, b]$ with $[x, y]$. Otherwise, $[a, b]$ is replaced with $[a, c]$ if $h_0(x) \leq h_0(c)$, or with $[c, b]$ if $h_0(x) > h_0(c)$. The process is repeated until we reach an interval where the values of h_0 at the mid point and end points coincide to within four decimal places.

In the graph c-fat200-1.clq, $-\lambda_n(\overline{G}) - 1$ is a main eigenvalue of G and $h_0'(-\lambda_n(\overline{G})) > 0$; thus the best upper bound is that in (3), attained when $t^* = h_0(t^*) = -\lambda_n(\overline{G}) = 17.2675$.

G	n	$\omega(G)$	$\lambda_1(G)+1$	t^*	$h_0^{\bar{G}}(t^*)$	Notes
brock200-1.clq	200	21	149.5707	12.4952	43.3005	(a)
brock200-2.clq	200	12	100.1963	14.0483	26.4234	(a)
brock200-3.clq	200	15	121.8181	13.9645	32.0650	(a)
brock200-4.clq	200	17	132.2037	13.5104	35.3994	(a)
brock400-1.clq	400	27	299.8496	17.2781	62.8351	(a)
brock400-2.clq	400	29	300.1480	17.4017	62.8164	(a)
brock400-3.clq	400	31	299.6317	17.6204	63.9385	(a)
brock400-4.clq	400	33	300.0543	17.5317	63.3207	(a)
c-fat200-1.clq	200	12	17.8135	17.2675	17.2675	
c-fat200-2.clq	200	24	33.6036	32.7001	32.9611	(a)
c-fat200-5.clq	200	58	85.7778	64.7787	72.9051	
hamming6-2.clq	64	32	58		32	(b)
hamming6-4.clq	64	4	23		13.5385	(b)
hamming8-2.clq	256	128	248		128	(b)
hamming8-4.clq	256	16	164		72	(b)
johnson8-2-4.clq	28	4	16		4	(b)
johnson8-4-4.clq	70	14	54		14	(b)
johnson16-2-4.clq	120	8	92		8	(b)
johnson32-2-4.clq	496	16	436		16	(b)
MANN-a9.clq	45	16	41.8039	2.3885	19.7076	
MANN-a27.clq	378	126	374.3035	6.7405	278.9118	
p-hat300-1.clq	300	8	80.7579	16.6554	26.3647	(a)
p-hat300-2.clq	300	25	158.9345	30.3485	78.1328	(a)
p-hat300-3.clq	300	36	225.8307	19.3401	88.3742	(a)
keller4.clq	171	11	111.8552	17.7206	41.1585	
san200-0.7-1.clq	200	30	140.5107	51.6650	94.7681	(a)
san200-0.7-2.clq	200	18	143.5080	68.3020	117.1690	(a)
san200-0.9-1.clq	200	70	180.3256	22.8092	118.7377	(a)
san200-0.9-2.clq	200	60	180.1964	17.4725	98.3736	(a)
san200-0.9-3.clq	200	44	180.1697	14.1434	86.5558	(a)
san400-0.5-1.clq	400	13	202.9588	151.2577	179.3039	(a)
san400-0.7-1.clq	400	40	280.4968	102.2726	184.7757	(a)
san400-0.7-2.clq	400	30	280.5105	98.4703	182.4865	(a)
san400-0.7-3.clq	400	22	280.8343	93.7929	183.7393	(a)

Notes: (a) $-\lambda_n(\overline{G})$ is a main eigenvalue, (b) G is regular.

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