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ON MULTIPLE EIGENVALUES OF TREES

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Abstract. Let T be a tree of order $n > 6$ with μ as a positive eigenvalue of multiplicity k. Star complements are used to show that (i) if $k > n/3$ then $\mu = 1$, (ii) if $\mu = 1$ then, without restriction on k, T has $k + 1$ pendant edges that form an induced matching. The results are used to identify the trees with a non-zero eigenvalue of maximum possible multiplicity.

Keywords: Graph, tree, eigenvalue, star complement.

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1 Introduction

Let G be a graph of order $n > 2$ with an eigenvalue μ of multiplicity k. (Thus the corresponding eigenspace of a $(0,1)$-adjacency matrix of G has dimension k.) If $\mu = -1$ then $k \leq n-1$, a bound attained in the complete graph K_n. If $\mu = 0$ and G is connected then $k \leq n-2$, a bound attained in the star $K_{1,n-1}$. If $\mu \neq -1$ or 0 and $n > 4$ then $k \leq n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}}$, a bound attained when $\mu = -2$ and $n = 36$. This last inequality is a reformulation of [1, Theorem 2.3].

For bipartite graphs, reduced upper bounds follow immediately from the fact that the spectrum is symmetric about 0. For example, $k \leq \frac{1}{2}n$ when $\mu \neq 0$; moreover, if μ^2 is not an integer then μ has an algebraic conjugate μ^* such that $\mu, -\mu, \mu^*, -\mu^*$ are distinct eigenvalues of multiplicity k, and so $k \leq \frac{1}{4}n$. We investigate the structure of a tree T for which $k > \frac{1}{3}n$ and $\mu \neq 0$; we may assume that $\mu > 0$. In this case, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T, then $\sum_{i=1}^{n} \lambda_i^2 = 2(n-1)$ and so $\frac{2}{3}n\mu^2 < 2n$; we conclude that $\mu = 1$ or $\sqrt{2}$. We shall see that $\mu = 1$, and this is the motivation for studying the case $\mu = 1$ in general – that is without any restriction on k. It turns out that, with two exceptions, T has $k+1$ endvertices whose neighbours constitute an independent set of size $k+1$. The exceptions are K_2 and Y_6, where Y_6 is the unique tree of order 6 with two (adjacent) vertices of degree 3. As a consequence we are able to identify the trees with a non-zero eigenvalue of maximum possible multiplicity.

We use star complements, defined as follows for any finite graph G. A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. We recall various properties of star complements from [3, Chapter 5].

(SC1) Star sets and star complements exist for any eigenvalue of any graph.

(SC2) If G is connected, and if L is a connected induced subgraph of G without μ as an eigenvalue, then G has a star set X for μ such that $G - X$ is a connected graph containing L.

(SC3) Suppose that G has μ as an eigenvalue of multiplicity k. If X is a star set for μ in G and if S is a proper subset of X then $G - S$ has μ as an eigenvalue of multiplicity $k - |S|$.

(SC4) Let $V(G) = \{1, 2, \ldots, n\}$, and let A be the adjacency matrix of G. Let P be the matrix which represents the orthogonal projection of $I\!R^n$ onto the eigenspace $\mathcal{E}_A(\mu)$ with respect to the standard orthonormal basis $\{\mathbf{e}_1.\mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $I\!R^n$. Then the subset X of $V(G)$ is a star set for μ in G if and only if the vectors $P\mathbf{e}_i$ $(i \in X)$ form a basis for $\mathcal{E}_A(\mu)$.

(SC5) If $\mu \neq -1$ or 0, if X is a star set for μ in G, and if $H = G - X$ then the H-neighbourhoods of vertices in X are non-empty and distinct.

(SC6) Suppose that G has μ as an eigenvalue of multiplicity k. Let X be a set of k vertices in the graph G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and

$$\mu I - A_X = B^\top (\mu I - C)^{-1} B.$$

The matrix P of (SC4) is a polynomial in A [3, p.4] and so $\mu P\mathbf{e}_v = AP\mathbf{e}_v = PA\mathbf{e}_v = \sum_{u\sim v} P\mathbf{e}_u$, where we write $u \sim v$ to mean that vertices u and v are adjacent. More generally, for any μ-eigenvector $\mathbf{x} = (x_1, \ldots, x_n)^\top$, we have $\mu x_j = \sum_{i\sim j} x_i$ $(i = 1, \ldots, n)$, and these equations are called the *eigenvalue equations* for $\mathbf{x}$. We shall also require the following observation:

Lemma 1.1. *If u, v are adjacent vertices in a star set for G then the edge uv is not a bridge of G.*

Proof. Suppose by way of contradiction that G is obtained from disjoint graphs H, K by joining the vertex u of H to the vertex v of K. Then the characteristic polynomial $P_G(x)$ of G is given by the following formula of Heilbronner [4]:

$$P_G(x) = P_H(x)P_K(x) - P_{H-u}(x)P_{K-v}(x). \tag{1}$$

We also have:

$$P_{G-u}(x) = P_{H-u}(x)P_K(x),\ P_{G-v}(x) = P_H(x)P_{K-v}(x),\ P_{G-u-v}(x) = P_{H-u}(x)P_{K-v}(x).$$

(Here we take the characteristic polynomial of an empty graph to be 1.) If μ is an eigenvalue of G of multilplicity $m_G(\mu) = k$, and u, v lie in a star set for μ, we deduce from (SC3) that

$$k - 1 = m_{G-u}(\mu) + m_K(\mu),\ k - 1 = m_H(\mu) + m_{K-v}(\mu),\ k - 2 = m_{H-u}(\mu) + m_{K-v}(\mu). \tag{2}$$

It follows from (2) that $m_{H-u}(\mu) = m_H(\mu) - 1$ and $m_{K-v}(\mu) = m_K(\mu) - 1$. Hence $k = m_H(\mu) + m_K(\mu)$, and from Equation (1) we have the contradiction $(x - \mu)^k | P_{H-u}(x)P_{K-v}(x)$. □

2 Star complements in trees

Suppose that T is a tree of order n with μ as a non-zero eigenvalue of multiplicity k. Let X be a star set for μ such that $T - X$ is connected. Thus the star complement $T - X$ is a tree H of order $n - k$. Since T has no cycles, we can deduce the following in turn using property (SC5). First, each vertex u in X is adjacent to a unique vertex u' of H. Secondly, if u, v are distinct vertices of X then $u' \neq v'$. Thirdly, X is an independent set. It follows that the vertices in X are endvertices. For each $u \in X$, we have $\mu P\mathbf{e}_u = P\mathbf{e}_{u'}$, and so by (SC4), the vertices u' $(u \in X)$ also form a star set for μ. Since every edge of T is a bridge, it follows from Lemma 1.1 that the vertices u' $(u \in X)$ are independent. Thus the k pendant edges uu' $(u \in X)$ constitute an induced matching (that is, their vertices induce kK_2). Explicitly, we have:

Proposition 2.1 *Let T be a tree with μ as a non-zero eigenvalue of multiplicity k. If X is a star set for μ in T such that $T - X$ is connected, then each vertex in X has degree 1, and the neighbours of vertices in X constitute an independent set of size k in $T - X$.*

We first use Proposition 2.1 to prove:

Theorem 2.2. *Let T be a tree of order n with μ as a positive eigenvalue of multiplicity k. If $k > \frac{1}{3}n$ then $\mu = 1$.*

Proof. Applying (SC2) with L a trivial graph, we see that T has a star set X for μ such that $T - X$ is connected. We use the notation of (SC6). By Proposition 2.1, we have $A_X = O$, and so

$B^\top(\mu I - C)^{-1}B = \mu I$; moreover, vertices may be labelled so that B has the form $\begin{pmatrix} I \\ O \end{pmatrix}$ and C has the form $\begin{pmatrix} O & M^\top \\ M & N \end{pmatrix}$. Hence $(\mu I - C)^{-1}$ has the form $\begin{pmatrix} \mu I & E^\top \\ E & F \end{pmatrix}$ and we have

$$\begin{pmatrix} \mu I & -M^\top \\ -M & \mu I - N \end{pmatrix} \begin{pmatrix} \mu I & E^\top \\ E & F \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}.$$

It follows that $\mu^2 I - M^\top E = I$. Since $n < 3k$ the number of rows of E is less than k, and so there exists a non-zero vector $\mathbf{x}$ such that $E\mathbf{x} = \mathbf{0}$. Now $\mu^2\mathbf{x} = \mathbf{x}$, and the result follows. □

We now investigate the case $\mu = 1$, without any restriction on k. We write $\mathcal{E}$ for the eigenspace of 1, and $N_d(v)$ for the subgraph induced by vertices at distance at most d from the vertex v.

Theorem 2.3. *Let T be a tree with 1 as an eigenvalue of multiplicity k. If $T \neq K_2$ or Y_6 then T has $k+1$ pendant edges that form an induced matching.*

Proof. Suppose that T is a counterexample to the statement of the theorem. By (SC2), X has a star set for 1 such that the star complement $H = T - X$ is a tree. By Proposition 2.1, each vertex $u \in X$ has degree 1; moreover, if u' denotes the neighbour of u then the vertices u' $(u \in X)$ are distinct and form an independent set in H. We fix $u \in X$. Since $T \neq K_2$, we also have $N_1(u') \neq K_2$; thus $N_1(u')$ is a star without 1 as an eigenvalue. By (SC2), T has a connected star complement $H_1 = T - X_1$ containing $N_1(u')$. By Proposition 2.1, the k vertices i of X_1 are endvertices whose neighbours i' form an independent set of size k. Note that this set avoids u'. By (SC4), the vectors $P\mathbf{e}_i$ $(i \in X_1)$ form a basis for $\mathcal{E}$. Also, $P\mathbf{e}_u \neq \mathbf{0}$, and so there exists $w \in X_1$ such that another basis for $\mathcal{E}$ is obtained when we replace $P\mathbf{e}_w$ with $P\mathbf{e}_u$. Let $X_2 = \{u\} \cup (X_1 \setminus \{w\})$. Each vertex in X_2 has degree 1 and so $P\mathbf{e}_j = P\mathbf{e}_{j'}$ for all $j \in X_2$. Since the vectors $P\mathbf{e}_{j'}$ $(j \in X_2)$ form a basis for $\mathcal{E}$, the vertices j' $(j \in X_2)$ constitute a star set for 1, and hence are independent by Lemma 1.1. It follows that $u' \sim w'$ for otherwise the $k+1$ edges ii' $(i \in X_1 \cup \{u\})$ constitute an induced matching.

If $N_2(u')$ does not have 1 as an eigenvalue then by (SC2), there exists a star set X_3 such that $T - X_3$ is a tree containing $N_2(u')$. But then, with the same notation as above, the $k+1$ edges ii' $(i \in X_3 \cup \{u\})$ constitute an induced matching. Hence 1 is an eigenvalue of $N_2(u')$.

Suppose that u' has r neighbours of degree 1 and t neighbours of degree greater than 1. Note that $r \geq 1$ since $u \sim u'$, and $t \geq 1$ since $w' \sim u'$. Moreover, u' has degree $r + t > 2$ for otherwise $P\mathbf{e}_{w'} = \mathbf{0}$ (since then $P\mathbf{e}_{u'} = P\mathbf{e}_u + P\mathbf{e}_{w'}$, while $P\mathbf{e}_u = P\mathbf{e}_{u'}$). A similar argument shows that w' has degree greater than 2. We let $u_1, \dots, u_t$ be the neighbours of degree greater than 1, and consider separately the two possibilities (a) $N_2(u')$ has a 1-eigenvector $\mathbf{x}$ with u'-entry 1, (b) all 1-eigenvectors of $N_2(u')$ have u'-entry 0.

Case (a). If u_i has degree d_i and the u_i-entry of $\mathbf{x}$ is a_i $(i = 1, \dots, t)$, then we find from the eigenvalue equations for $\mathbf{x}$ that

$$1 = r + a_1 + \cdots + a_t, \quad a_i = 1 + (d_i - 1)a_i \quad (i = 1, \dots, t),$$

whence $d_i > 2$ $(i = 1, \dots, t)$ and

$$r = 1 + \frac{1}{d_1 - 2} + \frac{1}{d_2 - 2} - \cdots + \frac{1}{d_t - 2}. \tag{3}$$

Eigenvalue equations also show that $N_2(u')$ has no 1-eigenvector with u'-entry 0, and so 1 is a simple eigenvalue of $N_2(u')$. Hence if $N_2(u') = T$ then $k = 1$, while $N_2(u')$ does not have an induced matching consisting of two pendant edges. Therefore $t = 1$ and it follows from Equation (3) that $d_1 = 3$ and $r = 2$; but then $T = Y_6$, a contradiction. Thus $N_2(u') \neq T$ and without loss of generality T has an edge pq with $p \sim u_t$ and $q \neq u'$. Let L be the induced subgraph of T obtained from $N_2(u')$ by adding the edge pq.

We claim that 1 is not an eigenvalue of L. To see this, suppose that $\mathbf{y}$ is a 1-eigenvector of L with u_i-entry c_i. From the eigenvalue equations we see that the p, q-entries of $\mathbf{y}$ coincide and so $c_t = 0$. We deal first with the case $t = 1$. If the u'-entry of $\mathbf{y}$ is zero then all entries are zero, a contradiction. If the u'-entry of $\mathbf{y}$ is non-zero then $r = 1$ and so u' has degree 2, another contradiction. When $t > 1$, we find again that the u' entry of $\mathbf{y}$ is non-zero, for otherwise $c_i = (d_i - 1)c_i$ $(i = 1, \ldots, t-1)$, whence $c_i = 0$ $(i = 1, \ldots, t)$ and $\mathbf{y} = \mathbf{0}$. Now the eigenvalue equations yield

$$r = 1 + \frac{1}{d_1 - 2} + \frac{1}{d_2 - 2} - \cdots + \frac{1}{d_{t-1} - 2},$$

in contradiction to Equation (3). Thus 1 is not an eigenvalue of L, and so T has a star set X_4 for 1 such that $T - X_4$ is a tree containing L. For each vertex v in X_4, the neighbour v' of v is not adjacent to u', and so the $k+1$ edges jj' $(j \in X_4 \cup \{u\})$ form an induced matching, a contradiction.

Case (b). In this case, let $\mathbf{z}$ be a 1-eigenvector of $N_2(u')$ with u_i-entry e_i $(i = 1, \ldots, t)$. Since $e_i = 0 + (d_i - 1)e_i$, either $d_i = 2$ or $e_i = 0$. We label vertices so that $u_1 = w'$ and $d_i > 2$ if and only if $i = 1, \ldots, s$; note that $s < t$ since $\mathbf{z} \neq \mathbf{0}$. For $j = s+1, \ldots, t$, let u_i'' be the neighbour of u_i different from u'. Let L_1 be the graph obtained from $N_2(u')$ by deleting $u_{s+1}'', \ldots, u_t''$, and let L_2 be the graph obtained from $N_2(u')$ by deleting $u_{s+1}'', \ldots, u_t''$ and $u_{s+1}, \ldots, u_t$. If L_1 has 1 as an eigenvalue then (as above)

$$r + t - s = 1 + \frac{1}{d_1 - 2} - \frac{1}{d_2 - 2} - \cdots - \frac{1}{d_s - 2},$$

while if L_2 has 1 as an eigenvalue then

$$r = 1 + \frac{1}{d_1 - 2} - \frac{1}{d_2 - 2} - \cdots - \frac{1}{d_s - 2}.$$

Accordingly, one of L_1, L_2, say L', does not have 1 as an eigenvalue. Then there exists a star set X_5 for 1 such that $T - X_5$ is a tree containing L'. If v' is the neighbour of a vertex $v \in X_5$ then $v' \neq u_i$ $(i = 1, \ldots, s)$ because v lies outside L', while $v' \neq u_i$ $(i = s+1, \ldots, t)$ because $P\mathbf{e}_{u'} \neq \mathbf{0}$. Now the $k+1$ edges jj' $(j \in X_5 \cup \{u\})$ form an induced matching, a final contradiction. □

Since Y_6 has spectrum $-2, -1, 0, 0, 1, 2$ we have the following as an immediate consequence of Theorems 2.2 and 2.3:

Corollary 2.4. *Let T be a tree of order $n \geq 3$ with μ as a positive eigenvalue of multiplicity k. If $k > \frac{1}{3}n$ then $\mu = 1$ and T has $k+1$ pendant edges that form an induced matching.*

We can now identify the trees with an eigenvalue of maximum possible multiplicity. We write $S(K_{1,h})$ for the tree obtained from the star $K_{1,h}$ by subdividing each edge.

Corollary 2.5. *Let T be a tree of order $n > 6$ with μ as an eigenvalue of multiplicity k.*
(i) *If $\mu = 0$ then $k \leq n - 2$, with equality if and only if $T = K_{1,n-1}$.*
(ii) *If $\mu \neq 0$ and n is odd, then $k \leq \frac{1}{2}(n-3)$, with equality if and only if $\mu = \pm 1$ and $T = S(K_{1,k+1})$.*
(iii) *If $\mu \neq 0$ and n is even, then $k \leq \frac{1}{2}(n - 4)$, with equality if and only if $\mu = \pm 1$ and T is obtained from $S(K_{1,k+1})$ by adding a pendant edge at the central vertex.*

Proof. If $\mu = 0$ and $k \geq n - 2$ then, by interlacing, T has no induced path of length 3 and the first assertion follows. In the remaining cases we may assume that $\mu > 0$. For $n = 7, 8, 9, 10$ the result follows by inspection of the spectra listed in Table 2 of the Appendix to [2]. Accordingly, we suppose that $n > 10$.

If n is odd and $k \geq \frac{1}{2}(n - 3)$ then $k > \frac{1}{3}n$ and we may apply Corollary 2.4. Thus $\mu = 1$ and T has $k+1$ pendant edges that form an induced matching. Then T has just one further vertex u, and so $T = S(K_{1,k+1})$ with u the central vertex. For the converse it suffices to observe that $S(K_{1,k+1})$ has k linearly independent 1-eigenvectors. Note that if (x_i) is a 1-eigenvector then $x_u = 0$ while $x_w = x_{w'}$ whenever w is an endvertex with neighbour w'. For a fixed endvertex v and k choices of $w \neq v$, we obtain k linearly independent eigenvectors by taking $x_v = x_{v'} = 1$, $x_w = x_{w'} = -1$ and all other x_i equal to 0.

If n is even and $k \geq \frac{1}{2}(n - 4)$ then either $k > \frac{1}{3}n$ or $(n, k) = (12, 4)$. In the former case, $\mu = 1$ by Theorem 2.2. In the latter case, we know that μ^2 is an integer (since $k > \frac{1}{4}n$), while $8\mu^2 + 2\lambda_1^2 \leq 22$, where λ_1 is the largest eigenvalue of T. Now the largest eigenvalue of a tree exceeds the mean degree [2, Theorem 3.8] and so here $\lambda_1 > \frac{11}{6}$. Hence always $\mu = 1$ and by Theorem 2.3, T has $k + 1$ pendant edges that form an induced matching, say ww' ($w \in W$) where each vertex w has degree 1. It follows that $n = 2k + 4$ and T has two further vertices u, v such that either (a) $u \sim v$ and each vertex w' is adjacent to precisely one of u, v, or (b) $u \not\sim v$, exactly one vertex w' is adjacent to both u and v, and each of the remaining vertices w' is adjacent to precisely one of u, v. In case (a) we can construct k linearly independent 1-eigenvectors if and only if u or v is adjacent to all vertices w' ($w \in W$); in this situation, G is the graph described in (iii). In case (b), we cannot construct k linearly independent 1-eigenvectors, and so the corollary is proved. □

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