Renormalization of wave function fluctuations for a generalized Harper equation

Sarah Hulton

University of Stirling

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ABSTRACT

A renormalization analysis is presented for a generalized Harper equation

$$(1+\alpha\cos(2\pi(\omega(i+1/2)+\phi)))\psi_{i+1}+(1+\alpha\cos(2\pi(\omega(i-1/2)+\phi)))\psi_{i-1}$$
$$+2\lambda\cos(2\pi(i\omega+\phi))\psi_i=E\psi_i. \quad (0.1)$$

For values of the parameter ω having periodic continued-fraction expansion, we construct the periodic orbits of the renormalization strange sets in function space that govern the wave function fluctuations of the solutions of the generalized Harper equation in the strong-coupling limit $\lambda\to\infty$.

For values of ω with non-periodic continued fraction expansions, we make some conjectures based on work of Mestel and Osbaldestin on the likely structure of the renormalization strange set.

1. INTRODUCTION

The generalized Harper equation

$$(1 + \alpha \cos(2\pi(\omega(i + 1/2) + \phi)))\psi_{i+1} + (1 + \alpha \cos(2\pi(\omega(i - 1/2) + \phi)))\psi_{i-1} + 2\lambda \cos(2\pi(i\omega + \phi))\psi_i = E\psi_i \quad (1.1)$$

is a discrete model of electron hopping in an applied sinusoidal potential on a one-dimensional integer lattice, taking into account next-nearest neighbour interaction terms.

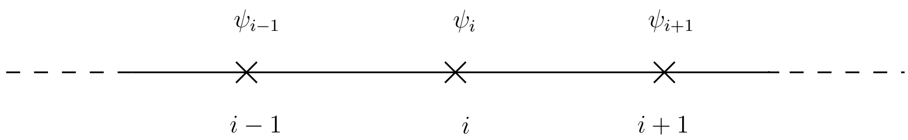


Fig. 1.1: One dimensional integer lattice.

In this discrete Schrödinger equation, the variable ψ_i is the wave function at lattice site $i \in \mathbb{Z}$ (see figure 1.2), and E is the eigenvalue corresponding to the eigenfunction ψ_i. The parameter ω represents the magnetic flux, and λ, α are interaction parameters. The parameter ϕ is the phase, which we set to 0.

The model undergoes a phase transition from electrical conductor to insulator

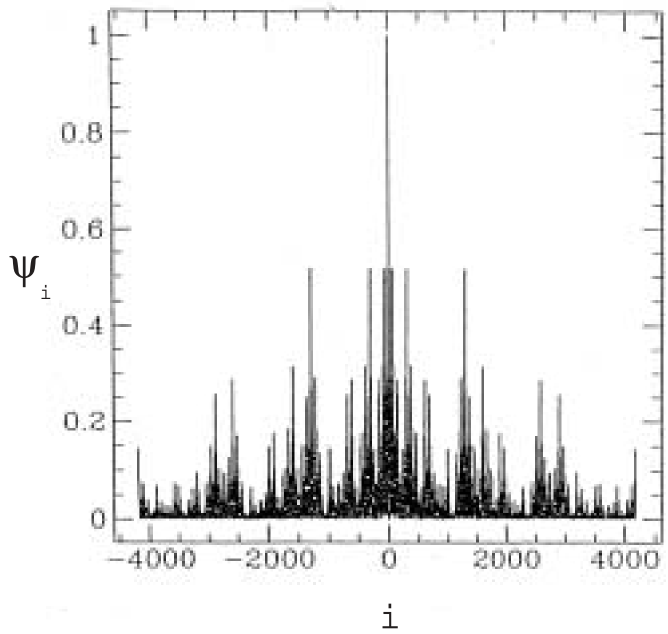


Fig. 1.2: Wave function ψ_i at lattice site $i \in \mathbb{Z}$ [50].

at $\lambda = 1$, at which point there is self-similarity of the spectrum eigenfunctions [61]. This critical behaviour is reflected in the localized regime $\lambda > 1$, where Ketoja and Satija observe [30] (in the Harper equation case ($\alpha = 0$), and for golden mean flux $\omega = (\sqrt{5}-1)/2$) that the exponentially decaying eigenfunctions possess universal self-similar fluctuations which they explain in terms of a universal fixed point of a renormalization operator. For the general case, $\alpha > 0$ the renormalization appears to send the system to a universal strange attractor determined by the strong coupling limit $\lambda \to \infty$, a projection of which is the *orchid* first obtained in [30] and illustrated in figure 1.3. Mestel and Osbaldestin [45] analyse this set and provide a description of its structure in terms of symbolic dynamics.

Such sets occur for other irrational ω (for example, figure 1.4) and the anal-

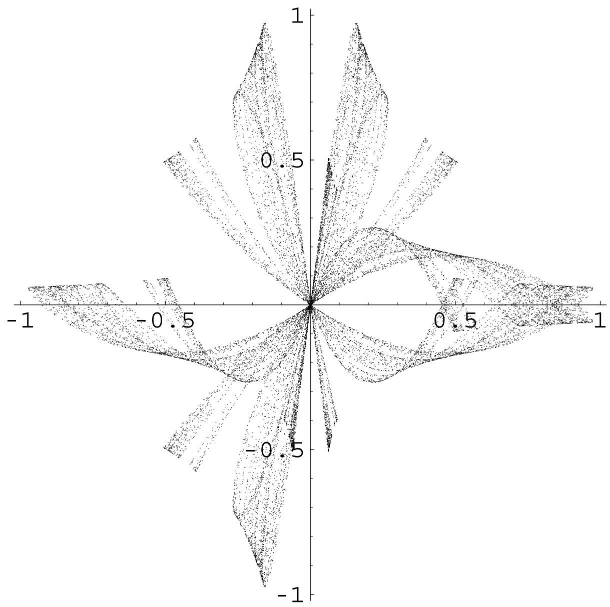


Fig. 1.3: The orchid.

ysis in [45] was extended by Mestel and Osbaldestin [46] to irrationals of the form $\omega = (-a + \sqrt{a^2+4})/2$, $a \in \mathbb{N}$, i.e., those ω with period-1 continued fraction $[a, a, a, \dots]$. The aim of this project is to extend this work further to all periodic continued fractions. Since the orbits under the Gauss map of all quadratic irrationals ω are eventually periodic, i.e., they have periodic tails in their continued-fraction expansions, the theory we develop applies to all quadratic irrationals ω. Our goal in this work is to define a model space for each periodic continued fraction and to construct an embedding of this model space into the space of function pairs on which the renormalization transformation acts. We then demonstrate that the renormalization strange set so constructed corresponds to that observed in the generalized Harper equation, at least up to a scale change. Thereby we present a rigorous structural analysis of the strange sets seen in the generalized Harper equation.

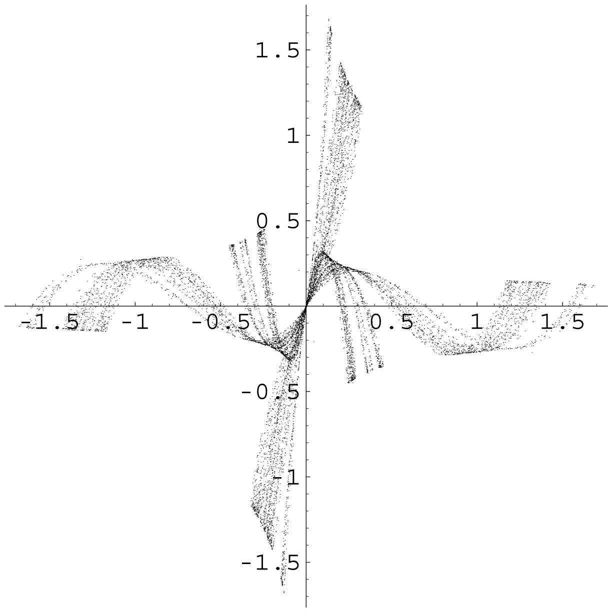


Fig. 1.4: Renormalization strange set for the continued fraction $\omega = [2, 2, 2, \ldots]$.

It turns out that the general periodic continued-fraction case presents considerable additional complications over the period-1 case, in that we must deal with periodic sequences of function pairs, each defined on separate domains, and many parameters change dynamically. For example, it is no longer possible to confine ourselves to a single invariant interval and code space. Instead we must allow the invariant interval and codes to change at each iteration of the renormalization map (see section 3.1.3 below). Thus, a complete analysis using the approach in [45] would present considerable difficulties, and consequently such an analysis is not attempted. Instead, we restrict ourselves to the important special case in which not only are the continued fractions periodic, but also the codes defining the orbits of the strange sets. This restriction allows us to define our function pairs on domains for which the renormalization transformations are compact operators and we may thus use the more elegant techniques presented by Dalton and Mestel in [9] in their study of the strong-coupling fixed point for period-1 continued fractions. Moreover, as the orbits with periodic codes are dense in the code space, our analysis covers a dense subset of the renormalization strange set in function-pair space.

1.1 The Harper equation and generalized Harper equation

In this section we give an overview of the Harper equation and generalized Harper equation and we give a derivation of the these equations from the Hamiltonian formulation in quantum mechanics.

1.2 Previous work related to the Harper and generalized Harper equation

The Harper equation

$$\psi_{n+1} + \psi_{n-1} + 2\lambda \cos(2\pi(\omega_n + \phi))\psi_n = E\psi_n. \tag{1.2}$$

(also known as the almost Mathieu equation) was introduced by Harper in [18] as a tight binding model of electrons on a two-dimensional lattice in a transverse magnetic field. See [62] for a recent discussion.

The model has been extensively studied over the years, principally because it undergoes a metal insulator transition when the coupling parameter $\lambda = 1$.

Most research has concentrated on the spectral analysis of the model. See [58, 59] for a recent review of this work.

The spectrum of the model has an intricate structure as given by the so-called *Hofstadter butterfly*, a fractal discovered by Hofstadter in the 1970's [23].

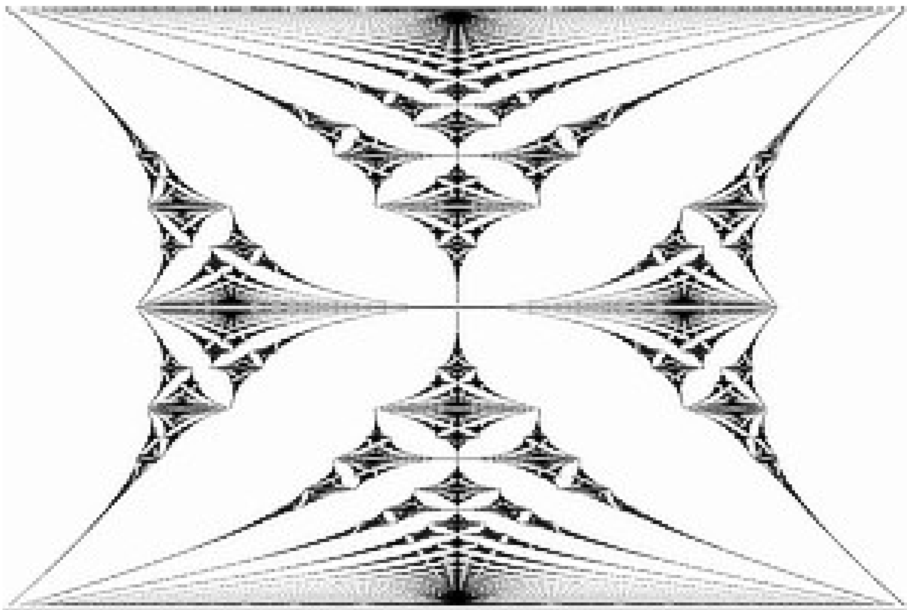


Fig. 1.5: The Hofstadter Butterfly [24].

Hofstadter predicted that the allowed energy level values of an electron in a crystal lattice, as a function of a magnetic field applied to the system, formed a fractal set, that is, the distribution of energy levels for large scale changes in the applied magnetic field repeat patterns seen in the small scale structure. This fractal structure is generally known as *Hofstadter's butterfly*, and is shown in figure 1.5. Analogous structures have also been found to exist for the energy spectrum of the generalized Harper equation [19].

Further important progress was made by Aubry and André [3] who conjecture the metal–insulator transition at $\lambda = 1$, such that, for $\lambda < 1$ there is absolutely continuous spectrum, whilst for $\lambda > 1$ the model exhibits localization. After many years of work by several authors, a modified version of the conjecture was proved by Jitomirskaya in 1999 [26].

Renormalization analysis of the self-similarity of the spectrum eigenfunctions has been studied by Ostlund and Pandit [52] and others authors [12, 28, 29, 34, 35, 53, 64]. See [61] for a review.

The Harper equation is obtained by only including nearest neighbour interactions in the tight binding model [62]. By including next-nearest neighbour interactions a generalized Harper equation is obtained. In section 1.1 we derive the generalized Harper equation from a Hamiltonian formulation introduced in 1994 by Han *et. al.* [16]. However, the generalized Harper equation was obtained earlier by Claro *et. al.* [8] who derived it as a model of an electron in a uniform magnetic field with potential with hexagonal symmetry and obtain the Hofstadter butterfly.

Later Thouless [65] studied the spectrum of the Harper and generalized Harper equation and obtained lower bounds for the measure of the spectrum for rational ω.

1.2.1 Derivation of the Harper equation

Recall from the theory of Quantum Mechanics, that the Eigenvalue equation (stationary Schrödinger equation) is given by

$$\mathcal{H}\psi = E\psi, \tag{1.3}$$

where $\mathcal{H}$ is the Hamiltonian operator. For the Harper equation the Hamiltonian can be written in the form

$$\mathcal{H}(p_x, x) = 2t_a \cos p_x + 2t_b \cos x, \tag{1.4}$$

where $t_a,\ t_b \in \mathbb{R}$ and $p_x = -2\pi i\omega\ d/dx$. Expanding the right hand side of (1.4) we have

$$\begin{aligned}
\mathcal{H}(p_x, x) &= t_a \left(2\cos\left(-2\pi i\omega\ \frac{d}{dx}\right)\right) + 2t_b \cos x \\
&= t_a \left(\exp\left(i\left(-2\pi i\omega\ \frac{d}{dx}\right)\right) + \exp\left(-i\left(-2\pi i\omega\ \frac{d}{dx}\right)\right)\right) + 2t_b \cos x \\
&= t_a \left(\exp\left(2\pi\omega\ \frac{d}{dx}\right) + \exp\left(-2\pi\omega\ \frac{d}{dx}\right)\right) + 2t_b \cos x. \qquad (1.5)
\end{aligned}$$

Applying this operator to $\psi(x)$ equation (1.4) becomes:

$$t_a \left(\psi(x + 2\pi\omega) + \psi(x - 2\pi\omega)\right) + 2t_b \cos(x)\psi(x) = E\psi(x), \tag{1.6}$$

where we have made use of the result $\exp\left(\nu \frac{d}{dx}\right)\psi(x) = \psi(x + \nu)$.

If we fix $x_0 \in \mathbb{R}$ and make the substitution $x = x_0 + 2\pi\omega_n$ we get:

$$t_a\left(\psi(x_0 + 2\pi\omega_n + 2\pi\omega) + \psi(x_0 + 2\pi\omega_n - 2\pi\omega)\right)$$
$$+2t_b \cos(x_0 + 2\pi\omega_n)\psi(x_0 + 2\pi\omega_n) = E\psi(x_0 + 2\pi\omega_n). \quad (1.7)$$

(Here $\omega = \omega_0 = [a_1, a_2, a_3 \ldots]$ and $\omega_n = \{\omega_{n-1}^{-1}\} = [a_{n+1}, a_{n+2}, a_{n+3} \ldots]$.)

Writing $\psi_n = \psi(x_0 + 2\pi\omega_n)$ we have

$$t_a(\psi_{n+1} + \psi_{n-1}) + 2t_b \cos(x_0 + 2\pi\omega_n)\psi_n = E\psi_n, \quad (1.8)$$

Now making the substitutions $x_0 \to 2\pi\phi$, $t_b/t_a \to \lambda$, $E/t_a \to E$ gives the Harper equation:

$$\psi_{n+1} + \psi_{n-1} + 2\lambda\cos(2\pi(\omega_n + \phi))\psi_n = E\psi_n. \quad (1.9)$$

1.2.2 Derivation of the generalized Harper equation

For the generalized Harper equation we replace the Hamiltonian in equation (1.4) by:

$$\mathcal{H}(p_x, x) = 2t_a \cos p_x + 2t_b \cos x + 2t_{\overline{ab}} \cos(p_x - x) + 2t_{ab}\cos(p_x + x), \quad (1.10)$$

where $p_x = -2\pi i\omega \frac{d}{dx}$. The two additional terms correspond to next-nearest neighbour interactions. Then we have

$$
\begin{aligned}
2\cos(p_x \pm x) &= 2\cos(-2\pi i\omega \frac{d}{dx} \pm x) \\
&= \exp(i(-2\pi i\omega \frac{d}{dx} \pm x)) + \exp(-i(-2\pi i\omega \frac{d}{dx} \pm x)) \\
&= \exp(2\pi\omega \frac{d}{dx} \pm ix)) + \exp(-2\pi\omega \frac{d}{dx} \mp ix). \qquad (1.11)
\end{aligned}
$$

To proceed further we need the following result:

Theorem 1. *(Baker-Campbell-Hausdorff [15]). For linear operators, A, B, we have*

$$
\exp(t(A+B)) = \exp(tA)\exp(tB)\exp(-\frac{t^2}{2}[A,B]), \qquad (1.12)
$$

provided that $[A,B]$ commutes with A and B. Here $[A,B] = AB - BA$.

Applying this result to (1.11) with $A = 2\pi\omega \frac{d}{dx}$, $B = \pm ix$, and $[A,B] = \pm 2\pi\omega i$, (which already satisfies the hypothesis of the theorem) and setting $t = 1$, gives

$$
\begin{aligned}
\exp(2\pi\omega \frac{d}{dx} \pm ix) &= \exp(2\pi\omega \frac{d}{dx})\exp(\pm ix)\exp(\mp\pi\omega i) \\
&= \exp(2\pi\omega \frac{d}{dx})\exp(\pm i(x - \pi\omega)). \qquad (1.13)
\end{aligned}
$$

Operating on $\psi(x)$ gives

$$
\begin{aligned}
\exp(2\pi\omega \frac{d}{dx} \pm ix)\psi(x) &= \left(\exp(2\pi\omega \frac{d}{dx})\exp(\pm i(x - \pi\omega)\right)\psi(x) \\
&= \exp(\pm i(x + 2\pi\omega - \pi\omega))\psi(x + 2\pi\omega) \quad (1.14)
\end{aligned}
$$

and

$$\begin{aligned}
\exp(-2\pi\omega\frac{d}{dx} \mp ix)\psi(x) &= \left(\exp(-2\pi\omega\frac{d}{dx})\exp(\mp ix)\exp(\mp\pi\omega i)\right)\psi(x) \\
&= \left(\exp(-2\pi\omega\frac{d}{dx})\exp(\mp i(x+\pi\omega)\right)\psi(x) \\
&= \exp(\mp i(x-2\pi\omega+\pi\omega))\psi(x-2\pi\omega) \qquad (1.15)
\end{aligned}$$

so that (1.11) operating on $\psi(x)$ becomes

$$\begin{aligned}
2\cos(p_x \pm x)\psi(x) &= \exp(\pm i(x+2\pi\omega-\pi\omega))\psi(x+2\pi\omega) \\
&+ \exp(\mp i(x-2\pi\omega+\pi\omega))\psi(x-2\pi\omega). \qquad (1.16)
\end{aligned}$$

Now let us assume $t_{ab} = t_{\overline{ab}}$. Then we have

$$\begin{aligned}
\mathcal{H}(p_x, x)\psi(x) &= (2t_a\cos p_x + 2t_b\cos x + 2t_{ab}\cos(p_x - x) \\
&\qquad +2t_{ab}\cos(p_x + x))\psi(x) \qquad (1.17) \\
&= t_a(\psi(x+2\pi\omega) + \psi(x-2\pi\omega)) + 2t_b\cos(x)\psi(x) \\
&\quad +t_{ab}(\exp(-i(x+2\pi\omega-\pi\omega))\psi(x+2\pi\omega) \\
&\quad +\exp(i(x-2\pi\omega+\pi\omega))\psi(x-2\pi\omega) \\
&\quad +\exp(i(x+2\pi\omega-\pi\omega))\psi(x+2\pi\omega) \\
&\quad +\exp(-i(x-2\pi\omega+\pi\omega))\psi(x-2\pi\omega)) \qquad (1.18) \\
&= 2t_{ab}(\cos(x+\pi\omega)\psi(x+2\pi\omega) + \cos(x-\pi\omega)\psi(x-2\pi\omega)) \\
&\quad +t_a(\psi(x+2\pi\omega) + \psi(x-2\pi\omega)) + 2t_b\cos(x)\psi(x) \qquad (1.19) \\
&= E\psi(x). \qquad (1.20)
\end{aligned}$$

Then, dividing by t_a and letting $2t_{ab}/t_a = \alpha$, $t_b/t_a = \lambda$, and $E = E/t_a$, we have

$$(1 + \alpha\cos(x + \pi\omega))\psi(x + 2\pi\omega) + (1 + \alpha\cos(x - \pi\omega))\psi(x - 2\pi\omega) \\ +2\lambda\cos(x)\psi(x) = E\psi(x). \quad (1.21)$$

Now if we fix ϕ and let $x = 2\pi(\phi + n\omega)$ and $\psi_n = \psi(x) = \psi(2\pi(\phi + n\omega))$ we get the generalized Harper equation

$$(1 + \alpha\cos(2\pi(\omega(n + 1/2) + \phi)))\psi_{n+1} + (1 + \alpha\cos(2\pi(\omega(n - 1/2) + \phi)))\psi_{n-1} \\ +2\lambda\cos(2\pi(n\omega + \phi))\psi_n = E\psi_n. \quad (1.22)$$

1.3 Renormalization Methods

In this section we introduce renormalization methods with the simple example of circle map renormalization and give the sturcture theory of Lanford-Yampolsky for critical circle map renormalization [68]. Our ultimate aim is to construct a similar theory for the generalized Harper equation. In this thesis we make progress towards this goal by constructing the periodic points corresponding to the horseshoe structure.

1.3.1 General renormalization theory

Certain critical phenomena such as the transition to chaos, magnetism to non-magnetism, liquid gas transition etc, have been observed to occur repeatedly at many different scales. This observation was the motivation for the technique of renormalization which seeks to explain the structure of critical behaviour and in particular scaling exponents and universal behaviour.

Specifically renormalization involves the iteration of a renormalization operator subject to renormalization constants. The system is transformed by aggregating components to form a new system on a different scale and then renormalizing to restore the original scale. The resulting system is one displaying the same interesting phenomena as the original but now encompassing all scales.

1.3.2 Universality classes

Renormalization analysis of the action of the renormalization operator on a suitable function space gives rise to quantitative predictions which may be applied to other systems which share some qualitative features with the system under investigation. Such systems are said to lie in the same *universality class*. In this way simple models can give rise to the derivation of universal constants observed in more complex physical systems within the same universality class.

1.3.3 Circle map renormalization

The renormalization group formalism has led to developments in the understanding of the transition to chaos, the best known examples are period doubling cascades and the breakdown of invariant circles in dissipative and area preserving maps. To introduce the methods involved in renormalization we look first at the illustrative example of circle map renormalization. This section describes some universal properties of critical circle maps with golden mean rotation number. First let us explain what is meant by *rotation number*.

Rotation number

Representing the circle $\mathbb{T}$ by the real numbers $\mathbb{R}$ mod 1 and recalling [1] that every homeomorphism on the circle can be represented by a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ on the reals such that:

$$F(x+1) = F(x) + 1, \tag{1.23}$$

then the circle homeomorphism $f : \mathbb{T} \to \mathbb{T}$ may be represented by a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ on the reals known as the *lift* of f. The rotation number $\rho(f)$ of f can then be defined by:

$$\rho(f) = \left(\lim_{n\to\infty} \frac{F^n(x) - x}{n} \right) \mod 1. \tag{1.24}$$

For homeomorphisms this limit exists and is independent of the choice of lift and the point $x \in \mathbb{T}$.

It is useful to express $\rho(f)$ as the continued fraction

$$\rho(f) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \tag{1.25}$$

where $a_n \in \mathbb{Z}$ for all n. This is abbreviated as $[a_1, a_2, a_3, \dots]$. More about continued fractions can be found in appendix A.

Circle maps

The circle map is a one-dimensional map, mapping the circle onto itself. A prototypical 2 parameter family of circle maps is the Arnold family given by:

$$f_{\kappa,\Omega}: \quad \theta_{n+1} = \theta_n + \Omega - \frac{\kappa}{2\pi}\sin(2\pi\theta_n), \tag{1.26}$$

with θ_n calculated mod 1. The parameter Ω may be interpreted as a forcing frequency while κ controls the amount of nonlinearity. Setting $\kappa = 0$ gives the unperturbed circle map:

$$\theta_{n+1} = \theta_n + \Omega \tag{1.27}$$

If $\Omega = p/q$, $p, q \in \mathbb{N}$ is rational then the rotation number $\rho = \Omega$ and θ_n follows a periodic trajectory since

$$\theta_n^q = \theta_n + p \tag{1.28}$$

$$= \theta_n \text{mod } 1, \tag{1.29}$$

i.e. θ_n returns to the same point after q iterations of the map. If however Ω is irrational then θ_n never returns to the same point, the motion is quasiperiodic and the points fill the circle densely.

For irrational Ω, increasing κ from 0 to 1 gives the transition from quasi-periodicity to chaos as the magnitude of the non-linear term increases. This is the transition we will look at, in particular, how renormalization methods have been used by Ostlund *et. al.* [51] to explain universal scaling constants obtained in the numerical experiments of Shenker [57].

Arnold Tongues

As κ increases from 0 to 1 a plot of the parameter space, Ω against κ, as shown in figure 1.6, reveals tongues that spread out from every rational number, within these tongues the rotation number is rational and corresponds to the value of Ω at $\kappa = 0$. This phenomenon in which rational periodic motion occurs for a finite range of forcing frequencies is known as *mode-locking.* These mode-locked regions surrounding each rational number are known as *Arnold tongues.*

The Arnold tongues $A_{p/q}$ corresponding to rotation number p/q are given by:

$$A_{p/q} = \{(\kappa, \Omega) : f^q_{\kappa,\Omega}(x) = x + p \quad \text{for some } x\}. \tag{1.30}$$

At $\kappa = 0$ the Arnold tongues are an isolated set of measure zero, they widen upwards to a finite width at $\kappa = 1$. At $\kappa = 1$ the map has a cubic critical point at the origin which means that the inverse map is not differentiable.

For $\kappa > 1$ the tongues overlap and the circle map becomes noninvertible.

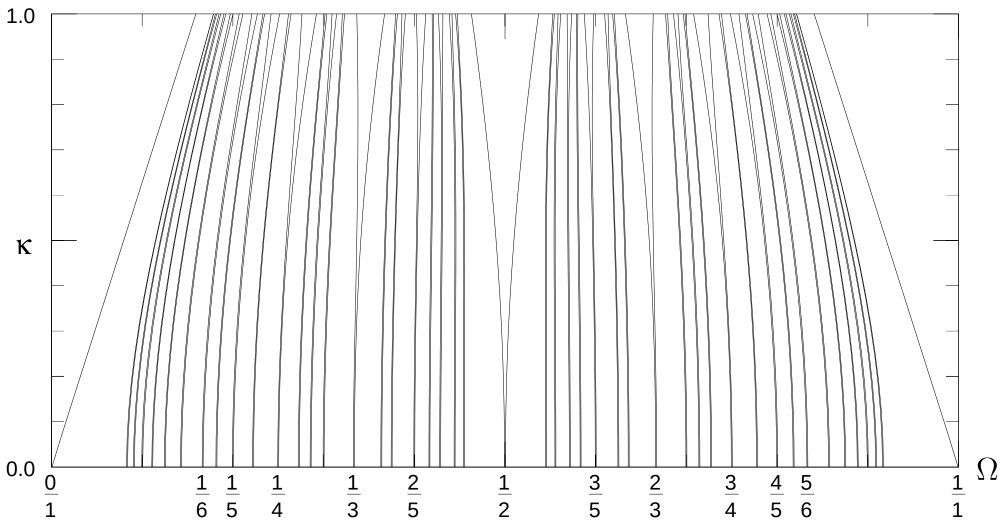


Fig. 1.6: Arnold tongues [6].

Devil's staircase

For $\kappa = 1$, the measure of quasiperiodic states (irrational rotation number) on the Ω-axis has become zero, and the measure of the mode-locked states has become 1. A plot of the rotation number $\rho(\Omega)$ against Ω for the circle map (1.26) with $\kappa = 1$ reveals that at almost all values of Ω, the rotation number is some rational. For $\kappa \leq 1$ the rotation number is monotonic in Ω giving the structure of a devil's staircase being constant on an infinite set of intervals corresponding to every rational rotation and is irrational elsewhere as shown in figure (1.7).

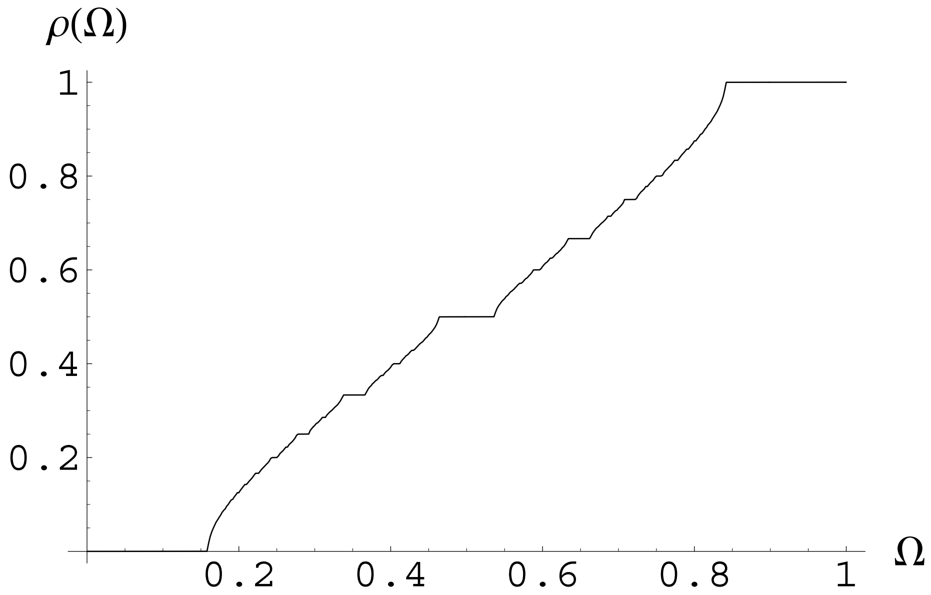


Fig. 1.7: Devil's staircase [66]

Universality

Numerical studies have revealed that circle maps exhibit interesting universal properties, i.e. for large classes of circle maps there exist quantities that are the same for all functions in their class.

Let $|\kappa| < 1$ be fixed so that $f_\Omega = f_{\kappa,\Omega}$ and let Ω_* be the fixed value of Ω such that $\rho(f_\Omega) = \Omega$, i.e. Ω_* is the value of Ω for which the rotation $\rho(f_\Omega)$ is equivalent the fixed rotation Ω. Let Ω_n be the value of Ω closest to Ω_* such that $\rho(f_\Omega) = p_n/q_n$, the nth convergent to Ω_*. If $\kappa = 0$ then $\Omega_n = p_n/q_n$ and $\Omega = \Omega_*$. In general for $|\kappa| < 1$ the following results are given by [22]:

1. If $f = f_\Omega$ then $f^{q_n}(0) - p_n$ decreases as a^n where $a = -\Omega_*$.

2. $a^{-n}(f^{q_n}(a^n x) - p_n)$ converges, up to a scale change, to the rigid rotation

$x \to x + \Omega_*$.

3. $\lim_{n\to\infty}(\omega_n - \omega_{n-1})/(\omega_{n+1} - \omega_n) = \delta$ where $\delta = -\Omega_*^{-2}$

Critical circle maps

A critical circle map is the lift of an analytic homeomorphism with a single critical point that is cubic. For the above two parameter family of circle maps $f_{\kappa,\Omega}$ given by equation (1.26), criticality occurs when $\kappa = 1$.

Setting $\kappa = 1$ we have that $f_\Omega = f_{1,\Omega}$ is critical and numerical experiments of Shenker [57], corresponding to maps with golden mean rotation number, give the following results:

1. $f^{q_n}(0) - p_n$ decreases as a^n where $a = -0.776\ldots = -\Gamma^{0.527\ldots}$
2. $a^{-n}(f^{q_n}(a^n x) - p_n)$ converges to an analytic function ξ of x^3 as $n \to \infty$.
3. $\lim_{n\to\infty}(\omega_n - \omega_{n-1})/(\omega_{n+1} - \omega_n) = \delta$ where $\delta = -2.834\ldots = -\Gamma^{-2.164\ldots}$,

where Γ represents the golden mean $(\sqrt{5} - 1)/2$.

Any 2-parameter family satisfying these conditions is said to be in the golden mean universality class. Similar results hold for other irrational rotation numbers with periodic continued fractions.

Circle map pairs

It is usual to work with an operator acting on a space of pairs of maps. The map of the circle f can be written in terms of a pair of functions (ξ, η) as illustrated in figure 1.8 such that

$$\begin{aligned} f(x) &= \xi(x) \quad \text{for} \quad \eta(0) < x < 0 \\ &= \eta(x) \quad \text{for} \quad 0 < x < \xi(0). \end{aligned} \tag{1.31}$$

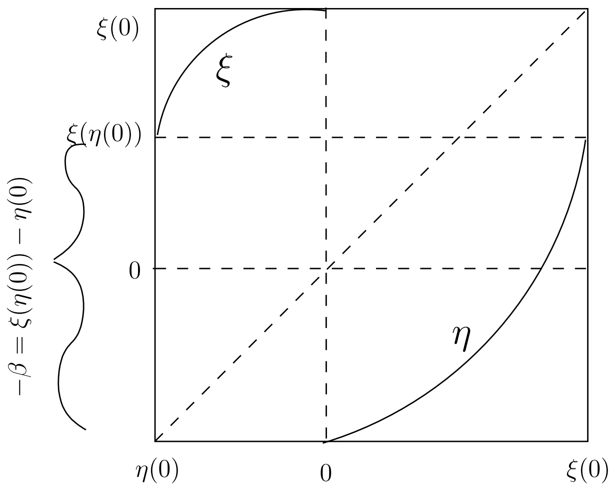


Fig. 1.8: Pairs of maps (ξ, η).

Identifying the points $\eta(0)$ and $\xi(0)$, f is a well defined homeomorphism of the circle $[\eta(0), \xi(0)]$ provided it satisfies the following conditions:

1. $\xi(\eta(0)) = \eta(\xi(0))$

2. $0 < \xi(0) < 1$

3. $\xi(0) = 1 + \eta(0)$ for a circle of length 1.

4. ξ, η are increasing on $[\eta(0), 0]$ and $[0, \xi(0)]$ respectively.

Renormalization scheme

Following the above numerical findings a renormalization theory was developed to explain these properties. The renormalization explanation given by Ostlund *et. al.* [51] is as follows.

The action of the renormalization operator on the pair (ξ, η) where $\rho(\xi, \eta) = [n, \ldots]$ is given by

$$T_n : \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} \mapsto \begin{pmatrix} \beta^{-1}\xi^{n-1}(\eta(\beta x)) \\ \beta^{-1}\xi^{n-1}(\eta(\xi(\beta x))) \end{pmatrix}, \tag{1.32}$$

where $\beta = \xi^{n-1}(\eta(0)) - \xi^{n-1}(\eta(\xi(0)))$.

To illustrate the action on T_n we consider the case $n = 1$ then (1.32) becomes

$$T_1 : \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} \mapsto \begin{pmatrix} \beta^{-1}\eta(\beta x) \\ \beta^{-1}\eta(\xi(\beta x)) \end{pmatrix}, \tag{1.33}$$

with the rescaling $\beta = -(\xi(\eta(0)) - \eta(0))$ chosen to impose a suitable normalization condition, e.g. a circle of length 1.

In general this operator sends rotation numbers $\rho = p_n/q_n$ to $\rho = p_{n-1}/q_{n-1}$ where p_n and q_n are the rational convergents to $\rho(\xi, \eta)$ as given by the recurrence relations (A.3), with the result that it preserves the golden mean.

Renormalization analysis

The renormalization analysis [51] is given in terms of two fixed points of T: The simple and the critical fixed points. The simple fixed point determines the scaling behaviour of diffeomorphisms and is the rigid rotation given by the pair of maps

$$\xi(x) = x + \Gamma \tag{1.34}$$

$$\eta(x) = x + \Gamma - 1, \tag{1.35}$$

where Γ represents the golden mean $(\sqrt{5}-1)/2$.

This fixed point has a one-dimensional unstable manifold with corresponding eigenvalue $-\Gamma^{-2}$. The presence of this fixed point explains the "simple" scaling observed for sub-critical maps with golden mean rotation number.

The critical fixed point determines the scaling behaviour of circle maps with a single cubic critical point at 0 and is observed in the transition from quasiperiodicity to weak turbulence in dissipative dynamical systems. Mestel [41] proves the existence and hyperbolicity of this critical fixed point for cubic critical circle maps with golden mean rotation number.

Structure of renormalization strange set.

In [4] Lanford conjectured the dynamics of renormalization for critical circle maps of fixed degree d (e.g. cubic). Let us define the renormalization transformation $T(\xi, \eta) = T_n(\xi, \eta)$ where the rotation number ρ is given by

$\rho = [n, \ldots]$. Then Lanford conjectures the existence of a renormalization strange set on which T acts hyperbolically with a 1-dimensional expansion and codimension-1 contraction, see figure 1.9.

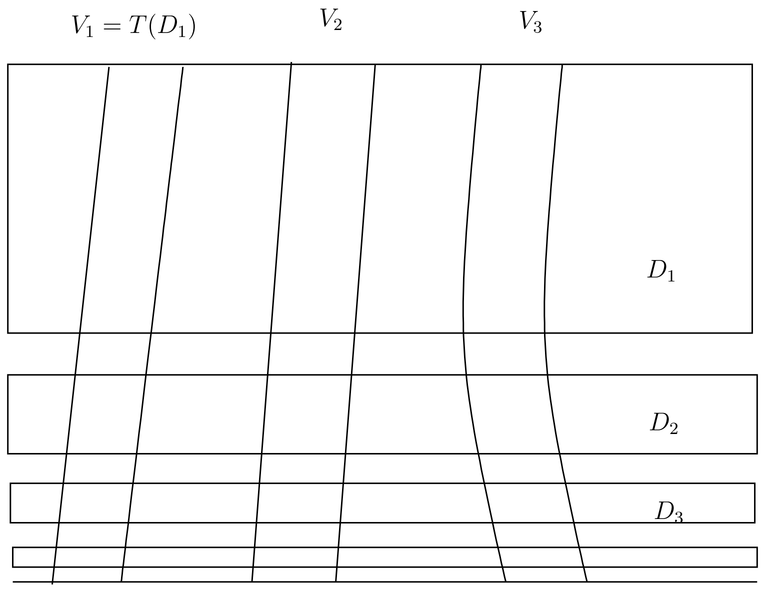


Fig. 1.9: Schematic representation of the action of T on $\cup D_n$ [4].

Yampolsky [68] has proved that this picture is correct when the degree of criticality, d, is an odd integer.

Theorem 2. *(Lanford-Yampolsky) There exists a set I of degree-d critical circle map pairs in the space of critical circle map pairs which is invariant under the renormalization transformation T and such that the action of T on I is topologically conjugate to a two-sided shift on the symbol space $\hat{\Sigma}$ consisting of biinfinite sequences of symbols in $\mathbb{N} \cup \{\infty\}$ with the left shift map σ. The conjugacy map $i : I \mapsto \tilde{\Sigma}$ satisfies $i \circ T \circ i^{-1} = \sigma$ and if a pair $\zeta = i^{-1}(\ldots, a_{-k}, \ldots, a_{-1}, a_0, a_1, \ldots, a_k, \ldots)$, then $\rho(\zeta) = [a_0, \ldots, a_k, \ldots]$.*

The set I has compact closure A and for any critical circle map pair ζ with *irrational rotation number,* $T^n\zeta \longrightarrow A$ *as* $n \longrightarrow \infty$*. Moreover for any* ζ*,* $\zeta' \in I$ *with* $\rho(\zeta) = \rho(\zeta')$ *the distance* $d(T^n\zeta, T^n\zeta') \longrightarrow 0$ *as* $n \longrightarrow \infty$*.*

A more precise statement of this theorem and results on hyperbolicity may be found in [68].

1.4 Decimation theory and the renormalization equations

1.4.1 Renormalization of fluctuations

Let us commence our renormalization analysis of the generalized Harper equation,

$$(1 + \alpha \cos(2\pi(\omega(i + 1/2) + \phi)))\psi_{i+1} + (1 + \alpha \cos(2\pi(\omega(i - 1/2) + \phi)))\psi_{i-1} + 2\lambda \cos(2\pi(i\omega + \phi))\psi_i = E\psi_i, \quad (1.36)$$

by outlining heuristically the derivation of the renormalization equations using the Ketoja-Satija decimation approach [30, 28]. We are interested in the insulator regime, $\lambda \geq 1$, in the zero-phase case $\phi = 0$, and we also take the strong coupling limit, $E \sim 2\lambda$, with $\lambda \to \infty$. This regime is characterised by an exponentially decaying wave function $\psi_i \sim e^{-\gamma|i|}$, where

$$\gamma = \log\left(\frac{\lambda}{\alpha} + \sqrt{\left(\frac{\lambda}{\alpha}\right)^2 - 1}\right). \quad (1.37)$$

The reason for taking $E \sim 2\lambda$ is that

$$\gamma \sim \log\left(\frac{2\lambda}{\alpha}\right), \tag{1.38}$$

so

$$e^{-\gamma} \sim \frac{\alpha}{2\lambda}, \tag{1.39}$$

so to ensure convergence we take $E \sim 2\lambda$ giving rise to the strong coupling limit.

Let us write

$$\psi_i = e^{\gamma|i|}\eta_i \tag{1.40}$$

where η_i is the fluctuation at site i shown in figure (1.10). In terms of the fluctuations η_i, the generalized Harper equation (1.1) becomes, for $i > 0$,

$$e^{-2\gamma}\left(1 + \alpha\cos(2\pi(\omega(i+1/2)+\phi))\right)\eta_{i+1} + \left(1 + \alpha\cos(2\pi(\omega(i-1/2)+\phi))\right)\eta_{i-1} + 2e^{-\gamma}\lambda\cos(2\pi(i\omega+\phi))\eta_i = e^{-\gamma}E\eta_i\,. \tag{1.41}$$

We may now consider the strong-coupling limit. Setting $E = 2\lambda$ and using

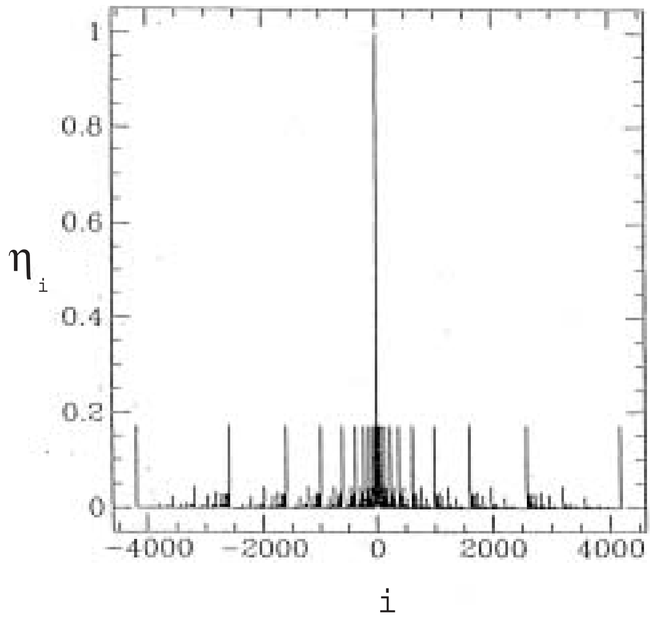


Fig. 1.10: The fluctuation η_i at lattice site $i \in \mathbb{Z}$.

equations (1.38 - 1.39), the equation (1.42) becomes

$$\frac{\alpha^2}{4\lambda^2}\left(1+\alpha\cos(2\pi(\omega(i+1/2)+\phi))\right)\eta_{i+1}+\left(1+\alpha\cos(2\pi(\omega(i-1/2)+\phi))\right)\eta_{i-1} \\ +\alpha\cos(2\pi(i\omega+\phi))\eta_i=\alpha\eta_i\,, \tag{1.42}$$

taking the limit $\lambda \to \infty$ this gives

$$\eta_{i-1}+\frac{\alpha(\cos(2\pi(i\omega+\phi))-1)}{1+\alpha\cos(2\pi(\omega(i-1/2)+\phi)))}\eta_i=0. \tag{1.43}$$

Our goal is now the renormalization analysis for the fluctuations η_i. The analysis is an extension of the decimation method of Ketoja-Satija [30, 28], simplified somewhat by the strong-coupling hypothesis $\lambda \to \infty$.

Let $\omega \in (0,1)$ have continued fraction expansion $\omega = \omega_0 = [a_1, a_2, a_3 \ldots]$ and $\omega_n = \{\omega_{n-1}^{-1}\} = [a_{n+1}, a_{n+2}, a_{n+3} \ldots]$, where $\{\cdot\}$ denotes the fractional part and $a_i \in \mathbb{N}$. The denominators of the rational convergents $c_n = p_n/q_n$ to ω satisfy, for $n \geq 1$, $q_n = a_n q_{n-1} + q_{n-2}$, with $q_{-1} = 0$, $q_0 = 1$, and $p_n = a_n p_{n-1} + p_{n-2}$, with $p_{-1} = 1$, $p_0 = 0$, and we have the important relation:

$$q_{n-1}\omega - p_{n-1} = (-1)^{n-1}\gamma_0^n, \quad n \geq 1 \tag{1.44}$$

where $\gamma_0^n = \prod_{i=0}^{n-1} \omega_i$.

1.4.2 Decimation theory

Following the method of Ketoja and Satija, a *decimation* may be defined by the relation:

$$\eta_{i+q_{n-1}} = \hat{t}_n(i)\eta_i \,. \tag{1.45}$$

The function $\hat{t}_n(i)$ encodes the relationship between $\eta_{i+q_{n-1}}$ and η_i, and thus is an appropriate object to study to understand the fluctuations.

A recurrence for $\hat{t}_n$ may be obtained in the following way. Evaluating (1.45) with i set equal to i, $i + q_{n-1}$, $i + 2q_{n-1}$, $\ldots$, $i + (a_n - 1)q_{n-1}$, gives the following set of equations:

$$\begin{aligned}
\eta_{i+q_{n-1}} &= \hat{t}_n(i)\eta_i && (1.46)\\
\eta_{i+2q_{n-1}} &= \hat{t}_n(i+q_{n-1})\eta_{i+q_{n-1}} && (1.47)\\
&\vdots \\
\eta_{i+a_n q_{n-1}} &= \hat{t}_n(i+(a_n-1)q_{n-1})\eta_{i+(a_n-1)q_{n-1}} \,. && (1.48)
\end{aligned}$$

Now evaluating (1.45) with n set equal to $n-1$ and i set equal to $i+a_nq_{n-1}$ gives

$$\eta_{i+a_nq_{n-1}+q_{n-2}} = \hat{t}_{n-1}(i+a_nq_{n-1})\eta_{i+a_nq_{n-1}}\,. \tag{1.49}$$

Eliminating $\eta_{i+q_{n-1}},\ldots,\eta_{i+a_nq_{n-1}}$ between these equations, we make use of the recurrence for the q_{n-1}, to obtain $\eta_{i+q_n} = \hat{t}_{n+1}(i)\eta_i$ where

$$\hat{t}_{n+1}(i) = \left(\prod_{j=0}^{a_n-1} \hat{t}_n(i+jq_{n-1})\right)\hat{t}_{n-1}(i+a_nq_{n-1})\,. \tag{1.50}$$

Setting the phase $\phi = 0$, and evaluating (1.45) at $n=0$ and $n=1$, we obtain $\eta_i = \hat{t}_0(i)\eta_i$, $\eta_{i+1} = \hat{t}_1(i)\eta_i$, which, on comparing with (1.43) at $i+1$, with $\phi = 0$, gives initial conditions

$$\hat{t}_0(i) = 1, \quad \hat{t}_1(i) = \frac{1+\alpha\cos(2\pi(\omega(i+1/2)))}{\alpha(1-\cos(2\pi(i+1)\omega))}\,. \tag{1.51}$$

Following [30] we now transform from the discrete variable i to a continuous variable x, by writing $x = (-1)^n(\gamma_0^n)^{-1}\{i\omega\}$ and setting $t_n(x) = \hat{t}_n(i)$. Then for $n > 1$ we have

$$\begin{aligned}
t_{n+1}(x) &= t_{n+1}((-1)^{n+1}(\gamma_0^{n+1})^{-1}\{i\omega\}) = \hat{t}_{n+1}(i) && (1.52)\\
&= \left(\prod_{j=0}^{a_n-1} \hat{t}_n(i + jq_{n-1})\right) \hat{t}_{n-1}(i + a_n q_{n-1}) && (1.53)\\
&= \left(\prod_{j=0}^{a_n-1} t_n((-1)^n(\gamma_0^n)^{-1}\{(i + jq_{n-1})\omega\})\right)\\
&\quad t_{n-1}((-1)^{n-1}(\gamma_0^{n-1})^{-1}\{(i + a_n q_{n-1})\omega\}) && (1.54)\\
&= \left(\prod_{j=0}^{a_n-1} t_n(-(-1)^{n+1}(\gamma_0^{n+1})^{-1}\omega_n\{i\omega + j(-1)^{n-1}\gamma_0^n\})\right)\\
&\quad t_{n-1}((-1)^{n+1}(\gamma_0^{n+1})^{-1}\omega_n\omega_{n-1}\{(i\omega + a_n(-1)^{n-1}\gamma_0^n\}) && (1.55)\\
&= \left(\prod_{j=0}^{a_n-1} t_n(-\omega_n x - j)\right) t_{n-1}(\omega_n\omega_{n-1}x + \omega_{n-1}a_n), && (1.56)
\end{aligned}$$

In deriving this equation we have implicitly used the periodicity of the function t_n and (1.44). Using the definition of the variable x and the periodicity of the cosine function, we have from (1.51) the initial conditions

$$t_0(x) = 1, \quad t_1(x) = \frac{1 + \alpha\cos(2\pi(-\omega x + \omega/2))}{\alpha(1 - \cos(2\pi(-\omega x + \omega)))}\,. \qquad (1.57)$$

We have therefore reduced the renormalization theory for the fluctuations η_i to the study of a second-order functional recurrence (1.56) under the initial conditions (1.57).

1.4.3 Derivation of renormalization equations

We now reformulate the second-order recurrence (1.56) as a first order operator on function pairs (u_n, t_n). Writing $u_n(x) = t_{n-1}(-\omega_{n-1}x)$, we obtain

$$u_{n+1}(x) = t_n(-\omega_n x), \quad t_{n+1}(x) = \left(\prod_{j=0}^{a_n-1} t_n(-\omega_n x - j)\right) u_n(-\omega_n x - a_n). \tag{1.58}$$

Thus we may express the recursion (1.56) in terms of the renormalization operator

$$R_n \begin{pmatrix} u_n(x) \\ t_n(x) \end{pmatrix} \mapsto \begin{pmatrix} u_{n+1}(x) \\ t_{n+1}(x) \end{pmatrix}, \tag{1.59}$$

where

$$\begin{pmatrix} u_{n+1}(x) \\ t_{n+1}(x) \end{pmatrix} = \begin{pmatrix} t_n(-\omega_n x) \\ \left(\prod_{j=0}^{a_n-1} t_n(-\omega_n x - j)\right) u_n(-\omega_n x - a_n) \end{pmatrix}. \tag{1.60}$$

From the initial condition (1.57) we get $u_1 = 1$.

It is the operator R_n that is the object of our study. It is closely connected to the operators studied by Mestel and Osabldestin and co-workers [45, 46] but differs significantly in that it is dependent on n, and thus we are in fact considering a sequence of operators.

Several properties of the operator R_n are immediately apparent. First, R_n is multiplicative. Consequently, it is convenient to define unary and binary operations and functions coordinatewise on function pairs, so that, for example, multiplication is defined by $(u, t)(u', t') = (uu', tt')$. The multiplicative property of R_n can then be written as $R_n((u,t)(u',t')) = R_n(u,t)R_n(u',t')$. We can, at least formally, also define the logarithm of a pair $\log(u,t) = (\log u, \log t)$ in which case R_n is a linear operator. It is clear that zeros and singularities of the functions u and t are obstructions to converting R_n into a linear operator, and, indeed, the dynamics of R_n are determined to a large extent by the zeros and poles of these functions. A further consequence is that the dynamics of R_n are dependent on the initial condition (1.57), and in particular on the zero sets and the poles. In fact, the symmetries of the cosine function in the initial condition are important in determining the evolution of R_n

Our second observation is that R_n is determined by the sequence (a_n), which itself determines the sequence of frequencies (ω_n). Although we are interested in n increasing (or forward iteration), it turns out that for a full analysis that we should take $n \in \mathbb{Z}$, and that backward iteration is important too.

Let us write $\mathbb{N}^{\mathbb{Z}}$ for the space of bi-infinite sequences of positive integers, so that $\mathbb{N}^{\mathbb{Z}} = \{\mathbf{a} = (a_k)_{k\in\mathbb{Z}} \mid a_k \in \mathbb{N}\}$, with left shift operator $\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ given by $\sigma(\mathbf{a})_k = \sigma(a_k) = a_{k+1}$.

1.5 Statement of main results

In this section we give the principal results of the thesis. Our aim is to construct an invariant set in function-pair space comprising the periodic points of a renormalization strange set. Specifically, for a given periodic continued fraction, $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$, we wish to construct a sequence of sets $(\mathcal{O}_k)_{k\in\mathbb{Z}}$ in a space of function-pairs such that $R_k(\mathcal{O}_k) = \mathcal{O}_{k+1}$ and such that the dynamics on $\mathcal{O}_k$ can be understood in terms of a shift map on a symbol space together with dynamics on sign pairs. Furthermore, we require that iteration of the renormalization map with initial conditions corresponding to the generalized Harper equation converges to the sequence of sets $\mathcal{O}_k$.

Let us denote by $\mathbb{N}^{\mathbb{Z},Per}$ the subspace of $\mathbb{N}^{\mathbb{Z}}$ consisting of periodic sequences, i.e. satisfying $a_{n+p} = a_n$ for all $n \in \mathbb{Z}$ for some fixed $p \in \mathbb{N}$ and let $\omega = [\overline{a_1, \ldots, a_p}]$, $a_1, \ldots, a_p \in \mathbb{N}$, have a periodic continued fraction expansion of period p. Then, for $k \in \mathbb{Z}$, we set $a_k = a_{k \bmod p}$ and $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$. Thus we may identify $\boldsymbol{\omega}$ with a unique $\mathbf{a} \in \mathbb{N}^{\mathbb{Z},Per}$.

The following theorem is the key to the construction of the strange set.

Theorem 3. *Let* $\mathbf{a}$ *be a fixed sequence in* $\mathbb{N}^{\mathbb{Z},Per}$ *corresponding to a periodic continued fraction. For each* $n \in \mathbb{Z}$*, there exists a continuous map* $\beta_n : \Sigma^{Per} \to \mathcal{F}_n$ *such that for* $\mathbf{c} \in \Sigma^{Per}$*,*

$$R_n\beta_n(\mathbf{c}) = L_{b_n}(+1,+1)\beta_{n+1}(\mathbf{c})\,, \tag{1.61}$$

where $L_{b_n} : \{+1,-1\}^2 \to \{+1,-1\}^2$ *is given by*

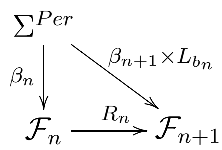
$$L_{b_n}(s_n^u, s_n^t) = (s_n^t, (-1)^{b_n}(s_n^t)^{a_n}s_n^u)\,, \tag{1.62}$$

and multiplication is carried out coordinatewise. Here $b_n = c_n + \tilde{c}_n$ *where* $\tilde{\mathbf{c}} = (\tilde{c}_k)_{k\in\mathbb{Z}}$ *denotes the partner code to* $\mathbf{c}$ *(to be defined in section 3.1.4 below),* Σ^{Per} *is a subspace of periodic codes to be defined in section 3.1.3 below and* $\mathcal{F}_n$ *is the function-pair space defined in section 3.3. The equation* (4.63) *generalizes to*

$$R_n((s^u, s^t)\beta_n(\mathbf{c})) = L_{b_n}(s^u, s^t)\beta_{n+1}(\mathbf{c}) \tag{1.63}$$

where $(s^u, s^t) \in \{+1, -1\}^2$ *is an arbitrary sign pair. The map* β_n *is two-to-one in the sense that* $\beta_n(\mathbf{c}) = \beta_n(\mathbf{c}')$ *if, and only if,* $\mathbf{c}' = \mathbf{c}$ *or* $\mathbf{c}' = \tilde{\mathbf{c}}$.

The property (4.63) may be summed up in the following commutative diagram.



Using the maps β_n we may define strange sets for each $n \in \mathbb{Z}$ as given by the following.

Theorem 4. *There exists a sequence* $\mathcal{O} = (\mathcal{O}_k)_{k\in\mathbb{Z}}$, $\mathcal{O}_k \subseteq \mathcal{F}_k$, *such that for all* $n \in \mathbb{Z}$, $R_n(\mathcal{O}_n) = \mathcal{O}_{n+1}$. *The sets* $\mathcal{O}_n$ *consists of images* $(s^u_n, s^t_n)\beta_n(\mathbf{c})$ *as* $\mathbf{c}$ *ranges over* Σ^{Per} *and* $(s^u_n, s^t_n) \in \{+1, -1\}^2$ *is a sign-pair depending on* $\mathbf{c}$ *(to be defined in section 4.3 below).*

Our final result concerns the generalized Harper equation itself. We show that for a dense set of initial conditions (corresponding to periodic codes) the dynamics under the renormalization transformation converges to a scaled version of the renormalization strange set. Specifically, we have:

Theorem 5. *Let* $\omega = \omega_0$ *have periodic continued fraction expansion* $[\overline{a_1, \dots a_p}]$ *with period* $p \geq 1$, *and let* $(\omega_n)_{n\in\mathbb{Z}}$ *be the associated p-periodic sequence*

$\boldsymbol{\omega}$= $(\omega_k)_{k\in\mathbb{Z}}$ with periodic continued fraction $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$. Then there exist a dense set of α in the range $\alpha > 1$ such that iteration of the associated renormalization operator R_n, for $n \geq 0$, converges to $\mathcal{O}_n^*$, where $\mathcal{O}_n^*$ is the set $\mathcal{O}_n$ scaled by a function-pair (u_n^*, t_n^*) of period a multiple of p.

We prove theorem 1 in section 4.2, theorem 2 is proven in section 4.3, and theorem 3 in section 4.3.5.

1.6 Organisation of thesis

In chapter 2 we give an overview of previous work on the Harper and generalized Harper equation, in particular the work of Mestel and Osbaldestin on renormalization analysis of the fluctuations for the generalised Harper equation. In chapter 3 we begin our renormalization analysis for the case of periodic continued fractions. In chapter 4 we construct the map β_n as given above in terms of $\mathcal{E}_n$ also constructed in this chapter and carry out a numerical study. In chapter 5 we consider the case of general continued fractions and give conjectures based on work of Mestel and Osbaldestin on the likely structure of the renormalization strange set. Chapter 6 contains the conclusions of the thesis and a discussion of future work. In appendix A we introduce continued fractions and give some number theoretic results which are useful in the thesis. Appendix B contains an brief overview of dynamical systems while in appendix C we look at shift spaces, and in appendix D give a brief synopsis of the spectral theory of compact linear operators on Banach spaces which we use in the thesis. In appendix E we prove lemma 2 which is stated in section 4.1.1 and concerns the spectral properties of the renormalization operator. In appendix F we construct the projection operator defined in section 4.1.3 for the periodic continued fractions with period 1, 2, and p and also for the case of a general continued fraction. In appendix G we prove lemma 5 given in section 4.1.3. Appendix H gives the details the construction of $\mathcal{E}_n$ which is done in section 4.1.3. Finally the proof of lemma 9 is found in appendix I.

2. REVIEW OF PREVIOUS WORK

In this chapter we give an overview of the previous work on the Harper and generalized Harper equation, and, in particular, on the application of renormalization theory to the Harper and generalized Harper equations.

2.1 The strong coupling fixed point

In common with the Harper equation, the generalized Harper model (1.1) undergoes a phase transition from electrical conductor to insulator at $\lambda = 1$, at which point there is self-similarity of the spectrum eigenfunctions [61]. This critical behaviour is reflected in the localized regime $\lambda > 1$, where for $\alpha = 0$ and golden mean flux $\omega = (\sqrt{5} - 1)/2$, Ketoja and Satija observe [30] that the exponentially decaying eigenfunctions possess universal self-similar fluctuations which they explain in terms of a universal fixed point of a renormalization operator.

The existence of this fixed point was proved by Mestel, Osbaldestin and Winn in [42]. This paper is concerned with the functional recurrences occurring in the study of quasiperiodic systems. In particular they prove the existence of the strong-coupling fixed point for golden-mean renormalization of of fluctuations in the Harper equation, thereby establishing a firm foundation for the work of Ketoja and Satija. Below is a summary of their results:

Let

$$\phi_1(z) = -\omega z\,, \qquad \phi_2(z) = \omega^2 z + \omega\,, \tag{2.1}$$

where $\omega = (\sqrt{5}-1)/2$ is the golden mean which satisfies $\omega^2 + \omega = 1$.

Theorem 6. *Let $n \in \mathbb{N}$ be given. Then there exists a unique, real analytic, entire function $t : \mathbb{C} \to \mathbb{C}$ satisfying the fixed point equation*

$$t(z) = t(\phi_1(z))t(\phi_2(z))\,, \tag{2.2}$$

with

1. $t(1) = 0$
2. $t^{(j)}(1) = 0$ *for* $j = 1, \ldots, n-1$, $t^{(n)}(1) \neq 0$, *so that* t *has a zero of order* n *at* $z = 1$*; and*
3. $t(z) > 0$ *for* $z \in (-\omega^{-1}, 1)$.

Moreover

$$t(z) = t_*(z)^n\,, \tag{2.3}$$

where t_* *is the entire function given by*

$$t_*(z) = \frac{1-z}{1-\omega} \prod_{k=1}^{\infty} \prod_{\substack{i_1,\ldots,i_k \\ i_1=1}} \frac{1-\phi_{i_1}\circ\cdots\circ\phi_{i_k}(z)}{1-\phi_{i_1}\circ\cdots\circ\phi_{i_k}(\omega)}\,. \tag{2.4}$$

In addition to existence Mestel *et. al.* derive properties of the fixed point function as follows:

Theorem 7. *The function t_* in theorem 6 satisfies:*

1. *the zeros of t_* are the points 1 and $\phi_{i_k}^{-1} \circ \cdots \circ \phi_{i_1}^{-1}(1)$, where $k \geq 1$, $i_1 = 1$ and $i_2, \ldots, i_k \in \{1, 2\}$ (We note that $-\omega^{-1}$ is of this form.);*
2. *$t_*(\omega) = 1$, $t_*(-\omega) = \omega^{-2}$, $t_*(\omega^2) = \omega^{-1}$;*
3. *t_* has a unique maximum at z_c on $(-\omega^{-1}, 1)$ with $z_c \in (-\omega, 0)$.*

The graph of t_* is given in figure 2.1.

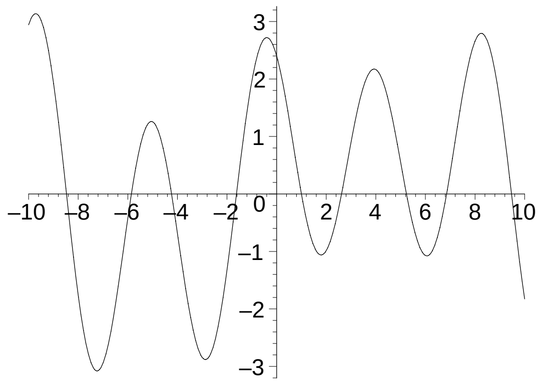


Fig. 2.1: The function t_*.

A Banach space F is defined on which the derivative of the operator

$$\mathcal{R} : (u(x), t(x)) \mapsto (t(-\omega x), t(-\omega x)u(-\omega x - 1)) \tag{2.5}$$

acts, and following theorem is proved.

Theorem 8. *Let $n \in \mathbb{N}$ and let t be the solution of (2.2) given by theorem 6. Let $u(z) = t(-\omega z)$. Then*

1. *The derivative of* $\mathcal{R}$ *at* (u,t), $L = d\mathcal{R}_{(u,t)}$, *is a compact operator on* F.

2. *The spectrum of* L *consists of* 0 *together with eigenvalues*

$$\lambda = \pm\omega^{-n}, \pm\omega^{-(n-1)}, \ldots, \pm\omega^{-1}, -1, \pm\omega, \pm\omega^2, \ldots . \tag{2.6}$$

Each of these eigenvalues is simple except for ω^{-1} *which is a double eigenvalue with a one-dimensional eigenspace and further one-dimensional generalized eigenspace.*

This work was extended to quadratic irrationals with a period-1 continued fraction in [9], where the methods are also simplified.

2.2 Previous work on the Orchid

For the general case, $\alpha > 0$ renormalization appears to send the system to a universal strange set determined by the strong coupling limit $\lambda \to \infty$, a projection of which is the *orchid* illustrated in figure 1.3. The orchid was first obtained by Ketoja and Satija in [30]. In this paper they study both the Harper and generalized Harper equation in the insulator regime from a renormalization standpoint using their decimation approach.

Writing $\psi_i = e^{-\gamma|i|}\eta_i$, Ketoja and Satija define functions $f_n(i)$ and $e_n(i)$ by the relation

$$f_n(i)\eta_{i+F_{n-1}} = \eta_{i+F_n} + e_n\eta_i. \tag{2.7}$$

This is a more general relation than the one we have used in equation (1.45).

The relation used in equation (1.45) corresponds to that of equation (2.7) in the strong coupling limit, but is simpler to analyse. Both lead to the same recurrence (up to a difference in sign). In fact, numerically at least, $f_n(i) \to 0$ and the recurrence for the $e_n(i)$ is (up to a difference in sign) our recurrence for the $t_n(i)$ in the golden mean case. Ketoja and Satija then introduce a transformation from a discrete coordinate to a continuous coordinate leading to the recurrence for the $t_n(x)$.

From numerical experiments Ketoja and Satija make the following conclusions:

- For the Harper equation, $t_n(x)$ tends to a fixed point, the strong-coupling fixed point.
- For the generalized Harper equation $t_n(x)$ converges to a strange set with the golden-mean orchid projection.
- Periodic orbits in the renormalization strange set have period a multiple of three.

Mestel and Osbaldestin give in [45] the first rigorous results on the orchid, giving the work of Ketoja and Satija a firm foundation. Considering the golden-mean case $\omega = (\sqrt{5}-1)/2$, they study the renormalization theory for the fluctuations in the wave function in the generalized Harper equation

$$(1 + \alpha\cos(2\pi(\omega(i + 1/2) + \phi)))\psi_{i+1} + (1 + \alpha\cos(2\pi(\omega(i - 1/2) + \phi)))\psi_{i-1} + 2\lambda\cos(2\pi(i\omega + \phi))\psi_i = E\psi_i. \quad (2.8)$$

In [45], the main topic is the study of the recurrence:

$$t_{n+1}(x) = t_n(-\omega x)t_{n-1}(\omega^2 x + \omega)\,, \tag{2.9}$$

with initial conditions derived from the generalized Harper equation. This is written in terms of a renormalization transformation:

$$(u_{n+1}, t_{n+1}) = R(u_n, t_n)\,, \tag{2.10}$$

where

$$R(u_n, t_n)(x) = (t_n(\theta_0(x)), t_n(\theta_0(x))u_n(\theta_1(x)))\,, \tag{2.11}$$

and θ_0, θ_1 are the linear contractions

$$\theta_0(x) = -\omega x\,, \qquad \theta_1(x) = -\omega x - 1\,. \tag{2.12}$$

and $u_n(x) = t_{n-1}(-\omega x)$.

The transformation R is studied first of all by understanding the iterated function systems defined by θ_0 and θ_1, which has an invariant set on the interval $[-\omega^{-1}, 1]$ with subintervals $[-\omega^{-1}, -\omega]$ and $[-\omega, 1]$. This leads naturally to code spaces: $\hat{\Sigma}$ the *subshift of finite type* consisting of bi-infinite sequences $\mathbf{c} = (c_k)_{k\in\mathbb{Z}}$, $c_k \in \{0, 1\}$, satisfying $c_k c_{k+1} = 0$, i.e., sequences for which no two consecutive terms of the sequence have digit 1. A partnering operation is defined in terms of blocks of symbols $A = 010$, $B = 00$, $C = 01$ by $A \to A$, $B \to C$, $C \to B$. Discarding those codes for which there is not a unique decomposition in terms of blocks, a code space Σ is obtained.

The principal analytical work in the paper is the construction of a pseudo-conjugacy β_n from a (modified) code space Σ to a space $\mathcal{F}$ of function pairs satisfying the equation

$$R\left((s^u, s^t)\beta(\mathbf{c})\right) = L_{b_0}(s^u, s^t)\beta(\sigma(\mathbf{c}))\,, \tag{2.13}$$

where $b_0 = c_0 + \tilde{c}_0 \bmod 2$ and $L_b(s^u, s^t) = (s^t, (-1)^b s^u s^t)$ is a map on sign-pairs. An analysis of the dynamics of the map L_b leads to the following transition diagram (figure 2.2). Here the sign pairs occurring at the start of blocks are enclosed in boxes. The arrows show the possible sign transitions within the blocks A or B/C to the sign pair at the beginning of the next block, which may be $(+1, +1)$, $(+1, -1)$ or $(-1, -1)$, indicated by a box. The sign pair $(-1, +1)$ may be traversed within a block but does not occur at the start of a block. The block to block transitions under the map L_b starting with the sign pair $(-1, +1)$ are found to be invariant.

The three-fold symmetry of the orchid transition diagram (figure 2.2) is the explanation for the three-fold symmetry in the orchid (figure 1.3). Indeed, the structure of the orchid is then obtained by combining the dynamics on the code space Σ with the dynamics induced on the sign pairs.

Further analysis in [45] shows that for the initial condition given by the generalized Harper equation, the renormalization transformation R converges to the a set $\mathcal{O}$ the projection of which is the orchid. Finally, the authors show that the $\mathcal{O}$ is strange in the sense of Devaney [11], although it is not an attractor, having two non-stable directions in function-pair space. Finally, in

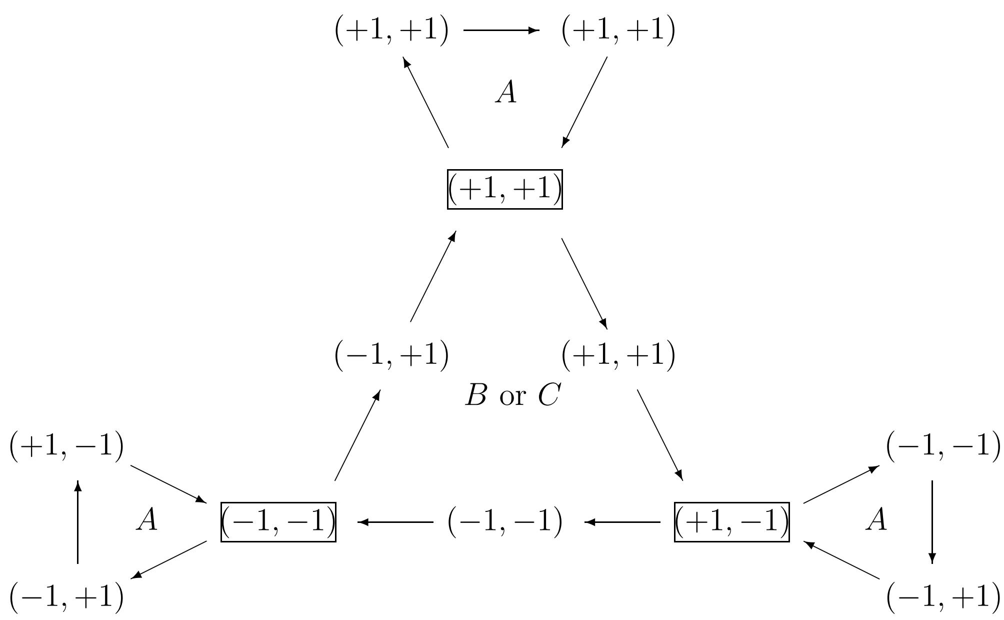


Fig. 2.2: The orchid transition diagram [45].

an analysis of the periodic orbit structure of $\mathcal{O}$ the authors prove a conjecture that all periodic orbits have period a multiple of 3.

A non-rigorous extension of the orchid work to quadratic irrationals of the form $[a, a, \dots]$ is provided in [46].

2.3 Previous work on related topics with similar renormalization analyses

Functional recurrences of the form (2.9) have application in several other quasi-periodic models in dynamical systems, in particular, in the renormalization analysis of correlation functions.

2.3.1 Strange non-chaotic attractors

Feudel *et. al.* [13] study the model

$$x_{t+1} = 2\lambda \tanh(x_t) \sin 2\pi\theta_t \,, \quad \theta_{t+1} = \theta_t + \xi \quad (mod\ 1) \tag{2.14}$$

where ξ is irrational. To simplify the analysis they study the discrete variable $y_t = -sign(x_t)$. Renormalization analysis of the correlation function of y_t: $C(t) = \langle y_n, y_{n+t} \rangle$ leads (for the case ξ the golden mean) to the recurrence

$$Q_n(y) = Q_{n-1}(-\xi y) Q_{n-2}(\xi^2 y + \xi) \tag{2.15}$$

but with a discontinuous initial conditions. They exhibit a numerically obtained period-6 orbit as shown in figure 2.3.

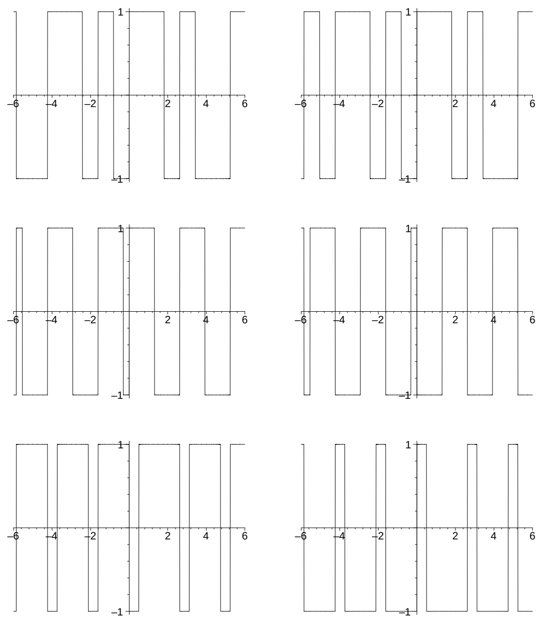


Fig. 2.3: Period-6 orbit of (2.15) showing y along the x axis.

Mestel and Osbaldestin [43] study the Multiplicative golden-mean recurrence:

$$Q_n(x) = Q_{n-1}(-\omega x)Q_{n-2}(\omega^2 x + \omega)\,, \tag{2.16}$$

for piecewise constant functions and apply the theory to justify the above work of Feudel *et. al.* [13].

Kuznetsov *et. al.* [36] study a quasiperiodically forced non-linear systems near the birth of a strange nonchaotic attractor. A renormalization analysis for golden-mean rotation number yields an extension of the golden-mean second-order functional recurrence to a recurrence on two functions Q_n, H_n:

$$Q_{n+2}(y) = Q_{n+1}(-\omega y)Q_n(\omega^2 y + \omega) \tag{2.17}$$

$$H_{n+2}(y) = H_{n+1}(-\omega y) + Q_n(-\omega y)^2 H_n(\omega^2 y + \omega) \tag{2.18}$$

It is likely that our methods will be applicable in this case and this is a prime candidate for further work.

2.3.2 Correlation in quasiperiodic quantum two level systems

In [14] Feudel *et. al.* study a two-level quantum mechanical system in a time-dependent field with Hamiltonian given by

$$H(t) = \frac{1}{2}\omega\sigma_\chi + \frac{1}{2}S(t)\sigma_x \tag{2.19}$$

where σ_χ and σ_x are the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_\chi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.20}$$

In the case when

$$S(t) = \sum_{n=-\infty}^{\infty} R_n \delta(t - 2\pi n/\omega) \tag{2.21}$$

and $R_n = \kappa\Phi(\phi_n)$, $\phi_{n+1} = \phi_n + \Omega$ $(mod\ 1)$, Ω, the golden mean, and ϕ periodic or period 1, the problem reduces to a skew sum mapping

$$\phi_{n+1} = \phi_n + \Omega\,, \quad \theta_{n+1} = \theta_n + \kappa\Phi(\phi_n). \tag{2.22}$$

Feudel *et. al.* study the case when Φ is a discontinuous modulation function and consider the correlation function $K_B(t) = \langle \cos[\kappa Q_t(\phi)] \rangle$ where $Q_t(\phi) = \sum_{l=0}^{t-1} \Phi(\phi + l\Omega), \quad Q_0 = 0$. Writing $Z_m(y) = Q_{F_m}(y(-\Omega)^m)$, they find Z_m satisfies the recurrence $Z_m(y) = Z_{m-1}(-y\Omega) + Z_{m-2}(y\Omega^2 + \Omega)$.

These results are analysed in [44] where Mestel and Osbaldestin study the additive golden-mean recurrence:

$$Z_n(x) = Z_{n-1}(-\omega x) + Z_{n-2}(\omega^2 x + \omega)\,, \tag{2.23}$$

where $\omega = (\sqrt{5} - 1)/2$, and show the existence of a piecewise continuous period-6 orbit and calculate the correlation function. This work is extended in [47] to the case where ω is a quadratic irrational of the form $[a, a, a, \dots]$.

2.3.3 Application to billiards

Chapman and Osbaldestin [7] adapt the renormalization analysis of the functional recurrence

$$Q_n(x) = Q_{n-1}(-\omega x)Q_{n-2}(\omega^2 x + \omega) \tag{2.24}$$

to the problem of symmetric barrier billiards with golden mean trajectories. Let θ_x, θ_y be angles corresponding to a point unit mass in the square chamber $[0,1] \times [0,1]$ with time evolution given by

$$\theta_x(t) = \theta_{x,0} + \omega_x t \quad (\text{mod } 1), \tag{2.25}$$

and

$$\theta_y(t) = \theta_{y,0} + \omega_y t \quad (\text{mod } 1). \tag{2.26}$$

Then the problem of the motion of a point mass in a rectangular chamber with a vertical barrier (see figure 2.4) is described in terms of the skew-product system

$$\theta_{n+1} = \theta_n + \omega \tag{2.27}$$

$$s_{n+1} = s_n \Phi(\theta_n), \tag{2.28}$$

where $\theta_n = \theta_y(n/\omega_x)$ is the angle of the trajectory, ω is the golden mean rotation number, $s_n = \pm 1$ is the sign of x, and $\Phi(\theta) = B(y)$ where $B(y)$ determines if there is a barrier present at y.

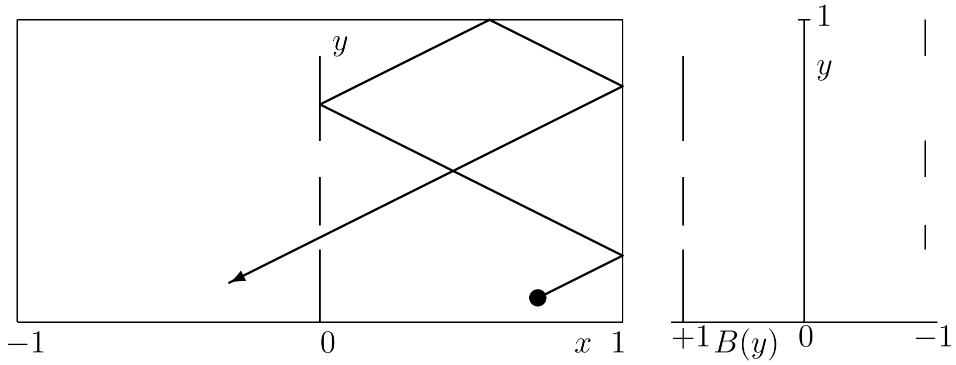


Fig. 2.4: A symmetric barrier billiard and its corresponding barrier function $B(y)$.

For irrational rotation numbers they study the behaviour of the sign s_n by looking at the autocorrelation function

$$C(t) = \langle s_n s_{n+t} \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s_n s_{n+t}. \tag{2.29}$$

They find that the self similarity of the autocorrelation functions can be explained in terms of periodic orbits of the recurrence (2.24).

For the special case of the half-barrier they give a rigorous calculation of the asymptotic values in the autocorrelation function at Fibonacci numbers.

3. RENORMALIZATION ANALYSIS

In this chapter we begin our renormalization analysis for the case of periodic continued fractions.

3.1 Dynamics of zeros and singularities

The dynamics of the renormalization operator (1.59) depend to a large extent on the zeros of the functions (u_n, t_n) contained in the invariant set of an iterated function system (IFS) which we shall now define.

3.1.1 Iterated function system

If we let, for $i = 0, \ldots, a_n$,

$$\theta_i^n(x) = -\omega_n x - i \tag{3.1}$$

then we may rewrite (1.60) as

$$\begin{pmatrix} u_{n+1}(x) \\ t_{n+1}(x) \end{pmatrix} = \begin{pmatrix} t_n(\theta_0^n(x)) \\ \left(\prod_{i=0}^{a_n-1} t_n(\theta_i^n(x))\right) u_n(\theta_{a_n}^n(x)) \end{pmatrix}. \tag{3.2}$$

Here the functions $\{\theta_0^n, \theta_1^n, \theta_2^n, \ldots, \theta_{a_n}^n\}$ form an Iterated Function System (IFS), whose fixed-point set is the interval I^n, which we refer to as the *fundamental interval.* The interval $I^n = [-\omega_n - a_n, 1]$ splits into subintervals I_i^n given by $I_0^n = [-\omega_n, 1]$ and $I_i^n = [-\omega_n - i, -\omega_n - i + 1]$ for $i = 1, \ldots a_n$.

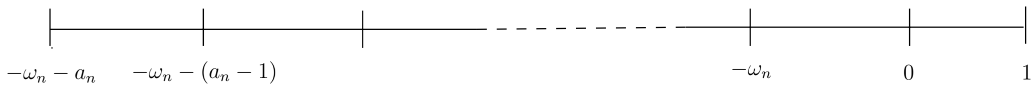


Fig. 3.1: The interval I^n.

3.1.2 The map G_n

We now define the map $G_n : I^n \to I^{n+1}$ by

$$G_n(x) = -\omega_n^{-1} x - i\omega_n^{-1}, \text{ for } x \in I_i^n \,. \tag{3.3}$$

Then $G(I_i^n) = [1 - \omega_n^{-1}, 1] = I_{a_n-1}^{n+1} \cup \cdots \cup I_0^{n+1}$ for $i = 1, \ldots, a_n$ and $G(I_0^n) = [-\omega_n^{-1}, 1] = I^{n+1}$. The functions u_n and t_n are then defined on

$$I_{a_n}^n = [-\omega_n - a_n, -\omega_n - (a_n - 1)] = [-\omega_{n-1}^{-1}, 1 - \omega_{n-1}^{-1}] \tag{3.4}$$

and

$$I_{a_n-1}^n \cup \cdots \cup I_0^n = [-\omega_n - (a_n - 1), 1] = [1 - \omega_{n-1}^{-1}, 1] \tag{3.5}$$

respectively. The significance of the maps G_n is that they determine the dynamics of the zero sets of (u_n, t_n).

3.1.3 Shift spaces

Recall that $\mathbb{N}^{\mathbb{Z},Per}$ denotes the subspace of $\mathbb{N}^{\mathbb{Z}}$ consisting of periodic sequences, i.e. satisfying $a_{n+p} = a_n$ for all $n \in \mathbb{Z}$ for some fixed $p \in \mathbb{N}$. Then for $\mathbf{a} \in \mathbb{N}^{\mathbb{Z},Per}$ we define the code spaces

$$\Sigma = \{\mathbf{c} = (c_k)_{k\in\mathbb{Z}} : c_k \in \{0, 1, \ldots, a_k\},\ c_k = a_k \implies c_{k-1} = 0\}\,. \tag{3.6}$$

This is a space of biinfinite codes of symbols $\{0, \ldots, a_n\}$ with the single restriction that the symbol a_n must be preceded by the symbol 0. The reason for this restriction is as follows. The symbols $0, \ldots, a_n$ correspond to the subintervals $I^n_0, \ldots, I^n_{a_n}$. The map G_n determines the possible transitions between symbols so that the symbol i is permitted to be followed by j if, and only if, $I^n_j \subseteq G_{n-1}(I^{n-1}_i)$. Since

$$G_{n-1}(I^{n-1}_0) = [-\omega^{-1}_{n-1}, 1] = I^n, \tag{3.7}$$

and

$$G_{n-1}(I^{n-1}_i) = [1 - \omega^{-1}_{n-1}, 1] = I^n_{a_n-1} \cup \ldots \cup I^n_0, \tag{3.8}$$

we see that only $G_{n-1}(I^{n-1}_0)$ contains $I^n_{a_n}$, hence the symbol a_n must be preceded by the symbol 0, as claimed. Also note that there are no more restrictions since $G_{n-1}(I^{n-1}_i)$ contains I^n_j for all $j = 0, \ldots, a_n - 1$. This can be seen in figure (3.2).

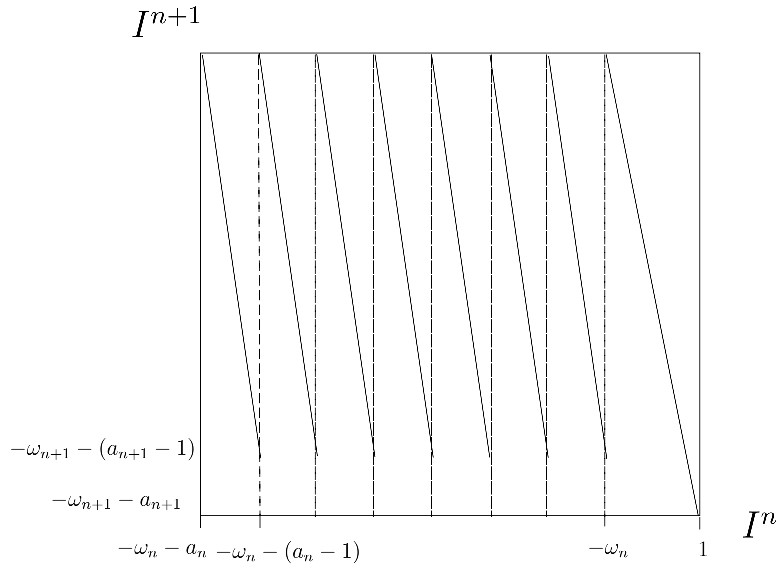


Fig. 3.2: The map G_n illustrated for $n = 7$.

Let us now define the subspace of periodic codes

$$\Sigma^{Per} = \{\mathbf{c} \in \Sigma : \mathbf{c} \text{ is periodic.}\}. \tag{3.9}$$

It turns out that it is convenient to exclude from Σ^{Per} the periodic codes consisting of all 0s and the two codes of the form $\ldots 0a_{n-2}0a_n0a_{n+2}\ldots$ for n odd and even. This is because the partnering relation (to be defined below in section 3.1.4) is not well defined for these three codes since the codes

$$\ldots a_{-3}0a_{-1}0a_10a_30\ldots \tag{3.10}$$

and

$$\ldots 0a_{-2}0a_00a_20a_4\ldots \tag{3.11}$$

both have partner code

$$\dots 00000000 \dots \tag{3.12}$$

and the partnering is not unique. Henceforth, we assume that Σ^{Per} has these codes omitted.

Let us denote by σ the left-shift map, defined for a code $\mathbf{c} = (c_k)_{k\in\mathbb{Z}}$ by $\sigma(\mathbf{c})_k = c_{k+1}$.

We now introduce a map $e_n : \Sigma^{Per} \to I^n$ which connects directly the code space Σ^{Per} with the maps G_n. For $n \in \mathbb{Z}$, let $e_n : \Sigma^{Per} \to [-\omega_n^{-1}, 1]$ denote the *evaluation map* at time n given by

$$e_n(\mathbf{c}) = -\sum_{k=n}^{\infty} c_k(-1)^{k-n}\gamma_n^k \tag{3.13}$$

where $\gamma_s^t = \omega_s \dots \omega_{t-1}$. It is straightforward to verify that the following properties of $e_n(\mathbf{c})$ hold for $\mathbf{c} \in \Sigma^{Per}$.

1. $e_n(\mathbf{c})$ is a continuous function of $\mathbf{c} \in \Sigma^{Per}$.
2. $e_n(\mathbf{c}) \in int(I^n_{c_n})$, the interior of the interval $I^n_{c_n}$.
3. $G_n(e_n(\mathbf{c})) = e_{n+1}(\mathbf{c})$.
4. For $\mathbf{c}, \mathbf{c}' \in \Sigma^{Per}$, the equality $e_n(\mathbf{c}) = e_n(\mathbf{c}')$ if, and only if, $\mathbf{c} = \mathbf{c}'$.

The first statement follows immediately from the geometric decay of γ_n^k as $k \to \infty$. We now prove the second statement, for which we need to use the identity

$$\sum_{r=1}^{\infty} a_{n+2r-1}\gamma_n^{n+2r-1} = 1\,. \tag{3.14}$$

(See section 3.1.4 below.) We consider first the case $c_n \in \{1, \ldots, a_n\}$. Then

$$e_n(\mathbf{c}) = -\sum_{k=n}^{\infty} c_k(-1)^{n-k}\gamma_n^k \tag{3.15}$$

$$< -c_n - \omega_n + \sum_{r=1}^{\infty} a_{n+2r-1}\gamma_n^{n+2r-1} \tag{3.16}$$

$$= -c_n - \omega_n + 1\,, \tag{3.17}$$

where we have used the identity (3.14) and the inequality $c_{n+1} \leq a_{n+1} - 1$. Strict inequality follows from the periodicity on $\mathbf{c}$ and $\mathbf{a}$, since $c_{n+1+2\ell} = c_{n+1} < a_{n+1} = a_{n+1+2\ell}$ for some $\ell \geq 1$. Similarly, we have

$$e_n(\mathbf{c}) > -c_n - \sum_{r=1}^{\infty} a_{n+2r}\gamma_n^{n+2r} \tag{3.18}$$

$$= -c_n - \omega_n \sum_{r=1}^{\infty} a_{n+1+2r-1}\gamma_n^{n+1+2r-1} = -c_n - \omega_n\,, \tag{3.19}$$

where we have again used (3.14) with n replaced by $n+1$. Strict inequality follows from the fact that $c_{n+2r} < a_{n+2r}$ for some $r \geq 1$ since $x_{n+2r-1} \neq 0$ for at least one $r \geq 1$ and we have excluded codes of the form $(\ldots a_n 0 a_{n+2} 0 \ldots)$. The case when $c_n = 0$ is similar, and we omit it for the sake of brevity.

To prove the third statement, we calculate

$$
\begin{aligned}
G_n(e_n(\mathbf{c})) &= -\omega_n^{-1}\left(-\sum_{k=n}^{\infty} c_k(-1)^{k-n}\gamma_n^k\right) - i\omega_n^{-1} && (3.20)\\
&= -\sum_{k'=n+1}^{\infty} c_{k'}(-1)^{k'-(n+1)}\omega_{n+1}\ldots\omega_{k'} && (3.21)\\
&= -\sum_{k'=n+1}^{\infty} c_{k'}(-1)^{k'-(n+1)}\gamma_{n+1}^{k'} && (3.22)\\
&= e_{n+1}(\mathbf{c}) && (3.23)\\
&= e_n(\sigma(\mathbf{c})). && (3.24)
\end{aligned}
$$

This is illustrated in the following commutative diagram.

$$
\begin{array}{ccc}
\Sigma^n & \xrightarrow{e_n} & I^n \\
\sigma\big\downarrow & & \big\downarrow G_n \\
\Sigma^{n+1} & \xrightarrow{e_n} & I^{n+1}
\end{array}
$$

The fourth statement follows immediately from the previous ones.

3.1.4 Evaluation map and partnering

Following [46] we now introduce a *partnering* operation $\tilde{S}$ on codes in Σ^{Per}. The purpose of this operation is to take into account the symmetries of the zeros of the initial conditions (1.57), in particular the reflection symmetry of the zeros of the function $1+\cos 2\pi x$ about $x = 1/2$ (the case $\alpha = 1$ in (1.57)).

The partnering operation is related to the following identities. Recalling the notation $\gamma_n^m = \prod_{k=n}^{m-1} \omega_k$, the identity

$$\omega_n = [a_{n+1}, a_{n+2}, \ldots] = \frac{1}{a_{n+1} + \omega_{n+1}}, \tag{3.25}$$

gives in turn

$$a_{n+1}\gamma_n^{n+1} + \gamma_n^{n+2} = 1. \tag{3.26}$$

Repeatedly permuting the indices in this equation, multiplying through by $-\omega_n$, and adding the resulting equation to (3.26) results in the identities

$$\begin{aligned}
1 &= a_{n+1}\gamma_n^{n+1} + \gamma_n^{n+2}, && (3.27)\\
1 &= (a_{n+1}+1)\gamma_n^{n+1} - (a_{n+2}-1)\gamma_n^{n+2} - \gamma_n^{n+3}, && (3.28)\\
1 &= (a_{n+1}+1)\gamma_n^{n+1} - a_{n+2}\gamma_n^{n+2} + (a_{n+3}-1)\gamma_n^{n+3} + \gamma_n^{n+4}, && (3.29)\\
1 &= (a_{n+1}+1)\gamma_n^{n+1} - a_{n+2}\gamma_n^{n+2} + a_{n+3}\gamma_n^{n+3} - (a_{n+4}-1)\gamma_n^{n+4} - \gamma_n^{n+5},\\
&\vdots && (3.30)
\end{aligned}$$

which are equivalent to the observation that 1 is a fixed point of the following contractions, which we make use of in the proof of propostion 1 below.

$$
\begin{aligned}
\kappa_2^n(x) &= a_{n+1}\gamma_n^{n+1} + \gamma_n^{n+2}x, && (3.31)\\
\kappa_3^n(x) &= (a_{n+1}+1)\gamma_n^{n+1} - (a_{n+2}-1)\gamma_n^{n+2} - \gamma_n^{n+3}x, && (3.32)\\
\kappa_4^n(x) &= (a_{n+1}+1)\gamma_n^{n+1} - a_{n+2}\gamma_n^{n+2} + (a_{n+3}-1)\gamma_n^{n+3} + \gamma_n^{n+4}x, && (3.33)\\
\kappa_5^n(x) &= (a_{n+1}+1)\gamma_n^{n+1} - a_{n+2}\gamma_n^{n+2} + a_{n+3}\gamma_n^{n+3} - (a_{n+4}-1)\gamma_n^{n+4} - \gamma_n^{n+5}x,\\
&\vdots && (3.34)
\end{aligned}
$$

Repeating this process *ad infinitum* gives the identity

$$
1 = \gamma_n^{n+1} + a_n - \sum_{k=n}^{\infty}(-1)^{k-n}a_k\gamma_n^k = \omega_{n-1}^{-1} - \sum_{k=n}^{\infty}(-1)^{k-n}a_k\gamma_n^k\,, \qquad (3.35)
$$

which we shall use in the definition of partnering below. A further identity that follows in an analogous manner is

$$
\sum_{r=1}^{\infty} a_{n+2r-1}\gamma_n^{n+2r-1} = 1\,, \qquad (3.36)
$$

which may be obtained in the limit from (3.26) by repeatedly advancing the indices in (3.26) by 2, multiplying through by $\gamma_n^{n+2} = \omega_n\omega_{n+1}$ and adding.

We now define a substitution operation $\tilde{S}$ on the biinfinite periodic codes in $\mathbf{c} \in \Sigma^{Per}$ (from which we excluded the periodic code consisting of all zeros for convenience). For a given periodic code $\mathbf{c}$, we have two cases: case (i) all $c_n \neq 0$ and case (ii) $c_n = 0$ and $c_m \neq 0$ for infinitely many n, m.

In case (i), we note that from (3.6) that $1 \leq c_n \leq a_n - 1$ for all $n \in \mathbb{Z}$. We may therefore define $\tilde{S}(\mathbf{c}) = \tilde{\mathbf{c}} = (\tilde{c}_n)_{n\in\mathbb{Z}}$ by $\tilde{c}_n = a_n - c_n$. It is immediate that $1 \le \tilde{c}_n \le a_n - 1$ and $\tilde{\mathbf{c}}$ is again a code in Σ^{Per} with no zeros.

The definition in case (ii) is more complicated. We split the code $\mathbf{c}$ into finite sequences which we call *elementary blocks* and then define $\tilde{S}$ on each of these elementary blocks in turn. As $\tilde{S}$ preserves this elementary block structure, we may recombine the blocks to form the partner code $\tilde{\mathbf{c}}$. The procedure is as follows.

Firstly $\mathbf{c}$ is split into finite blocks beginning with a single 0, i.e., into blocks of the form

$$0d_0d_1\dots d_k0^r\,, \tag{3.37}$$

where $k \ge 0$, $d_j \ne 0$, $j = 0,\dots,k$, $r \ge 0$. Further the trailing run of 0s is split so that this block is written

$$0d_0d_1\dots d_k0(00)^{(r-1)/2}\,, \qquad r \text{ odd}\,, \tag{3.38}$$

$$0d_0d_1\dots d_k(00)^{r/2}\,, \qquad r \text{ even}\,, \tag{3.39}$$

and thus we subdivide each of these blocks into elementary blocks of the form $0d$, $0d_0d_1\dots d_kd$ where $k \ge 0$, $d_j \ne 0$, for $j = 0,\dots,k$ and $0 \le d$. On these resulting elementary blocks the operation $\tilde{S}$ is then defined as follows. Let us suppose that an elementary block has initial zero starting at index $n \in \mathbb{Z}$, so that, in view of (3.6), we have either a block of length two, of the form $0d$, where $0 \le d \le a_{n+1}$, or a block of length three, or more, of the form $0d_0d_1\dots d_kd$, where $k \ge 0$, $1 \le d_0 \le a_{n+1}$, $1 \le d_j \le a_{n+j+1} - 1$ $(1 \le j \le k)$, and $0 \le d \le a_{n+k+2} - 1$. (Note that when $k = 0$ we have a block of length three of the form $0d_0d$.) The operation $\tilde{S}$ can now be defined on the elementary blocks by

$$\begin{aligned}
\tilde{S}(0d) &= 0(a_{n+1} - d), && (3.40)\\
\tilde{S}(0d_0d_1 \dots d_kd) &= 0(a_{n+1} + 1 - d_0)(a_{n+2} - d_1) \dots && (3.41)\\
&\quad (a_{n+(k+1)} - d_k)(a_{n+(k+2)} - d - 1), \quad k \geq 0. && (3.42)
\end{aligned}$$

It is straightforward to check that $\tilde{S}$ gives a new elementary block of the same length satisfying the above conditions. The action of $\tilde{S}$ is to take digit complements according to the *elementary block structures* of lengths 2, 3, ..., which are

$$\begin{aligned}
&0a_{n+1} && (3.43)\\
&0(a_{n+1} + 1)(a_{n+2} - 1) && (3.44)\\
&0(a_{n+1} + 1)a_{n+2}(a_{n+3} - 1) && (3.45)\\
&0(a_{n+1} + 1)a_{n+2}a_{n+3}(a_{n+4} - 1) && (3.46)\\
&\vdots
\end{aligned}$$

For example, suppose $\omega = [\overline{5, 5, 5, \dots}]$. We split the code

$$\mathbf{c} = \dots (011234)(05)(00)(03412330)(00)051230 \dots \qquad (3.47)$$

as shown. Then

$$\tilde{S}(\mathbf{c}) = (054320)(00)(05)(03143224)(05)\tilde{S}(051230 \dots). \qquad (3.48)$$

Having defined $\tilde{S}$ on each elementary block, all of the elementary blocks are then combined to form a new partner code $\tilde{\mathbf{c}} = \tilde{S}(\mathbf{c})$. This defines a map $\tilde{S} : \Sigma^{Per} \to \Sigma^{Per}$ which is easily seen to be a continuous involution, i.e., $\tilde{S}^2 = id$, the identity map. Continuity follows from the observation that in case (ii) for each index n the elementary block containing c_n is constant for all codes sufficiently close to $\mathbf{c}$ in Σ^{Per}. We note that case (i) can be thought of as a single elementary block stretching to infinity in both directions and, indeed, continuity in case (i) follows from the fact that as a periodic code approaches $\mathbf{c}$ in Σ^{Per}, for each index n the elementary block containing c_n grows in length to the left and right so the partner code also approaches $\tilde{S}(\mathbf{c})$. It is also clear that $\tilde{S}$ commutes with the left shift map σ on Σ^{Per}.

We next define the *sum map*

$$\mathbf{S} : \Sigma^{Per} \to \{\mathbf{S} = (S_k)_{k\in\mathbb{Z}} : S_k \in \{1, -\omega_{k-1}^{-1}, 1 - \omega_{k-1}^{-1}\}\}, \tag{3.49}$$

the space of biinfinite sequences with terms at index n taken from $\{1, -\omega_{n-1}^{-1}, 1- \omega_{n-1}^{-1}\}$. The sum map is important because it motivates the reason for the choice of partner operation $\tilde{S}$. For a code $\mathbf{c}$ is case (i) above we set $\mathbf{S}(\mathbf{c})_n = 1 - \omega_{n-1}^{-1}$ for all $n \in \mathbb{Z}$. In case (ii), $\mathbf{S}$ is defined in terms of the elementary block structures given above. Let $\mathbf{c} = (c_k)_{k\in\mathbb{Z}}$ be a code with partner $\tilde{S}(\mathbf{c})$, denoted by $\tilde{\mathbf{c}}$. $\mathbf{c}$ and $\tilde{\mathbf{c}}$ are divided into blocks and $\mathbf{S}$ is defined on the elementary block structures (3.43) - (3.46) as follows. Let the first entry in the block be at index n. Then,

$$\begin{aligned}
\mathbf{S}(0a_{n+1}) &= 1(-\omega_n^{-1}) && (3.50)\\
\mathbf{S}(0(a_{n+1}+1)(a_{n+2}-1)) &= 1(-\omega_n^{-1})(1-\omega_{n+1}^{-1}) && (3.51)\\
\mathbf{S}(0(a_{n+1}+1)a_{n+2}(a_{n+3}-1)) &= 1(-\omega_n^{-1})(1-\omega_{n+1}^{-1})(1-\omega_{n+2}^{-1}) && (3.52)\\
\mathbf{S}(0(a_{n+1}+1)a_{n+2}a_{n+3}(a_{n+4}-1)) &= 1(-\omega_n^{-1})(1-\omega_{n+1}^{-1})(1-\omega_{n+2}^{-1})(1-\omega_{n+3}^{-1})\\
&\vdots && (3.53)
\end{aligned}$$

extending $\mathbf{S}$ to the whole of $\mathbf{c}$. By construction, we have that $\mathbf{S}(\mathbf{c}) = \mathbf{S}(\tilde{\mathbf{c}})$ and $\mathbf{S}(\sigma(\mathbf{c})) = \sigma(\mathbf{S}(\mathbf{c}))$, provided that the biinfinite sequence $\mathbf{a}$ is also left shifted. Again it is straightforward to check that $\mathbf{S}$ is continuous. We have the following result which specifies precisely how $\mathbf{y} = \mathbf{e}(\mathbf{c})$ and $\tilde{\mathbf{y}} = \mathbf{e}(\tilde{\mathbf{c}})$ are related, and explains the terminology 'sum map':

Proposition 1. *Let $\mathbf{y} = \mathbf{e}(\mathbf{c})$ and $\tilde{\mathbf{y}} = \mathbf{e}(\tilde{\mathbf{c}})$. Then $\mathbf{y} + \tilde{\mathbf{y}} = \mathbf{S}(\mathbf{c})$, where, the sum is to be calculated termwise.*

To prove this proposition, let us consider a bi-infinite code $\mathbf{c}$ and its partner code $\tilde{\mathbf{c}}$, given by the above substitution rules. We write $y_k = e_k(\mathbf{c})$, $\tilde{y}_k = e_k(\tilde{\mathbf{c}})$ and set $S_k = y_k + \tilde{y}_k$. We first of all consider case (i) in which all $c_n \neq 0$ for all $n \in \mathbb{Z}$. Then from the definition of $e_k(\mathbf{c})$ in (3.13), the identity (3.35) above, and the partnering definition above, we have

$$S_k = e_k(\mathbf{c}) + e_k(\tilde{\mathbf{c}}) = -\sum_{k=n}^{\infty}(-1)^{k-n}a_k\gamma_k^n = 1 - \omega_{n-1}^{-1}\,, \qquad (3.54)$$

as required. In case (ii), we proceed somewhat differently. Let us write the codes in terms of the above block structures, (3.43) - (3.46). Then we claim that if $n \in \mathbb{Z}$ starts a block (i.e., $c_n = \tilde{c}_n = 0$, the first zero of a block), then $y_n + \tilde{y}_n = 1$. Indeed, suppose the block starting at n is of total length $j_1 \geq 2$. Then from (3.13) we have $S_n = \kappa_{j_1}^n(S_{n+j_1})$ so that, making use of the identities (3.27) - (3.30) and (3.31) - (3.34),

$$\begin{aligned} |S_n - 1| &= |\kappa_{j_1}^n(S_{n+j_1}) - \kappa_{j_1}^n(1)| && (3.55) \\ &= \gamma_n^{n+j_1}|S_{n+j_1} - 1|. && (3.56) \end{aligned}$$

Since the next block, of total length $j_2 \geq 2$ starts at $n+j_1$, we have, similarly,

$$|S_n - 1| = \gamma_n^{n+j_1}\gamma_{n+j_1}^{n+j_1+j_2}|S_{n+j_1+j_2} - 1|. \quad (3.57)$$

Continuing in this way,

$$|S_n - 1| = \gamma_n^{n+j_1+j_2+\ldots+j_k}|S_{n+j_1+j_2+\ldots+j_k} - 1|, \quad (3.58)$$

we notice that the S_k are bounded and obtain the limit $S_n = 1$, as claimed.

We now consider $k \in \mathbb{Z}$ within a block. Consider an elementary block structure and let $n \in \mathbb{Z}$ correspond to the start of the block. Now from (3.13) we know that,

$$\begin{aligned} y_{n+i} = e_{n+i}(\mathbf{c}) &= -\sum_{k=n+i}^{\infty} c_k(-1)^{k-(n+i)}\gamma_{n+i}^k \\ &= -c_{n+i} + \omega_{n+i} \sum_{k=n+(i+1)}^{\infty} c_k(-1)^{k-(n+(i+1))}\gamma_{n+(i+1)}^k \\ &= -c_{n+i} - \omega_{n+i} y_{n+(i+1)}, \end{aligned} \tag{3.59}$$

and similarly,

$$\tilde{y}_{n+i} = -\tilde{c}_{n+i} - \omega_{n+i}\tilde{y}_{n+(i+1)}, \tag{3.60}$$

then,

$$\begin{aligned} S_{n+i} &= y_{n+i} + \tilde{y}_{n+i} \\ &= -(c_{n+i} + \tilde{c}_{n+i}) - \omega_{n+i} S_{n+(i+1)}. \end{aligned} \tag{3.61}$$

If we now consider the elementary block structure $0a_{n+1}$ of length 2, we have

$$\begin{aligned} S_n &= -(c_n + \tilde{c}_n) - \omega_n S_{n+1} \\ &= -\omega_n S_{n+1}, \end{aligned} \tag{3.62}$$

then if $S_n = 1$,

$$S_{n+1} = -\omega_n^{-1}. \tag{3.63}$$

Thus $\mathbf{S}(0a_{n+1}) = 1(-\omega_n^{-1})$ corresponds to the sum $\mathbf{y} + \tilde{\mathbf{y}}$ on the block.

Similarly, for the elementary block structure $0(a_{n+1}+1)(a_{n+2}-1)$ of length

3 we have, again, $S_n = 1$, and $S_{n+1} = -\omega_n^{-1}$, where now

$$S_{n+1} = -(c_{n+1} + \tilde{c}_{n+1}) - \omega_{n+1} S_{n+2}, \tag{3.64}$$

which implies

$$-\omega_n^{-1} = -(a_{n+1} + 1) - \omega_{n+1} S_{n+2}, \tag{3.65}$$

so that, using (3.25)

$$S_{n+2} = 1 - \omega_{n+1}^{-1}, \tag{3.66}$$

and $\mathbf{S}(0(a_{n+1}+1)(a_{n+2}-1)) = 1(-\omega_n^{-1})(1-\omega_{n+1}^{-1})$ corresponds again to the sum $\mathbf{y} + \tilde{\mathbf{y}}$ on the block.

Finally, for an elementary block structure $0(a_{n+1}+1)a_{n+i}^j(a_{n+2+j}-1)$ with $i = 2, \ldots, j+1$ of length $j+3$ for $j \geq 1$, we have again, $S_n = 1$, $S_{n+1} = -\omega_n^{-1}$, $S_{n+2} = 1 - \omega_{n+1}^{-1}$, and

$$S_{n+i} = -(c_{n+i} + \tilde{c}_{n+i}) - \omega_{n+i} S_{n+(i+1)}, \tag{3.67}$$

which implies

$$1 - \omega_{n+(i-1)}^{-1} = -(a_{n+i} + 1) - \omega_{n+i} S_{n+(i+1)}, \tag{3.68}$$

so that

$$S_{n+(i+1)} = 1 - \omega_{n+i}^{-1}, \tag{3.69}$$

and similarly for the final block,

$$S_{n+j+1} = 1 - \omega_{n+j}^{-1} = -a_{n+j+1} - \omega_{n+j+1} S_{n+2+j}, \tag{3.70}$$

which implies

$$S_{n+2+j} = 1 - \omega_{n+j+1}^{-1}. \tag{3.71}$$

Giving,

$$\mathbf{S}(0(a_{n+1}+1)a_{n+i}^{j}(a_{n+2+j}-1)) = 1(-\omega_n^{-1})(1-\omega_{n+1}^{-1})(1-\omega_{n+2}^{-1})\ldots(1-\omega_{n+j+1}^{-1}). \tag{3.72}$$

This completes the proof of the proposition.

Let us now explain briefly the significance of these definitions for the problem in hand. The function $t_1(x)$ in the initial conditions (1.57) has two zeros whose sum is either 1 or $1 - \omega_0^{-1}$ corresponding to either the start or the middle of an elementary block defined above (see section **??** below). Thus when we map Σ^{Per} into the function-pair space $\mathcal{F}_n$ at index n, we make use of the partnering operator to obtain a zero set for the resulting function pair that correspond to that of t_1.

3.2 Function-pair spaces

In order to give a precise definition of the function space and of the renormalization transformation R_n, we must specify the domains of the functions U and T. Let $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$ and the corresponding $\boldsymbol{\omega} = (\omega_k)_{k\in\mathbb{Z}}$ be fixed.

For $c^n \in \mathbb{C}$ and $r^n > 0$, let $D(c^n, r^n)$ denote the disc centered at c^n with radius r^n. Let $V_1^n = D(c_1^n, r_1^n)$ and $V_0^n = D(c_0^n, r_0^n)$ be the discs in $\mathbb{C}$ where

$$c_0^n = 1 - \frac{\omega_{n-1}^{-1}}{2}, \quad r_0^n = \frac{\omega_{n-1}^{-1}}{2} + \delta, \quad c_1^n = \frac{1}{2} - \omega_{n-1}^{-1}, \quad r_1^n = \frac{1}{2} + \delta, \tag{3.73}$$

where $\delta > 0$, as illustrated in figure 3.3.

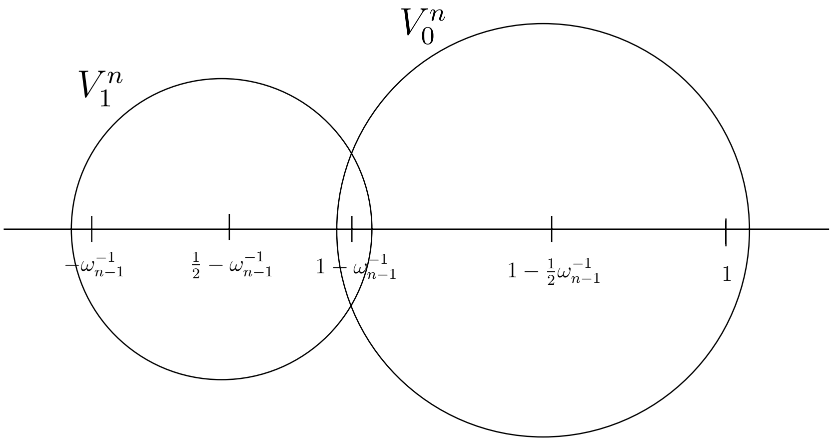


Fig. 3.3: The domains V_0^n and V_1^n.

The choice of $\delta > 0$ is dependent on the particular code $\mathbf{c} \in \Sigma^{Per}$ for which we construct $\mathcal{E}_n(\mathbf{c})$ such that $\beta_n(\mathbf{c}) = \mathcal{E}_n(\mathbf{c})\mathcal{E}_n(\tilde{\mathbf{c}})$ in section (4.1.3) below. Indeed for a given $\mathbf{c}$ the values of $y_n = e_n(\mathbf{c})$ and $\tilde{y}_n = e_n(\tilde{\mathbf{c}})$ form periodic sequences, which (as we have seen in section 3.1 above) are bounded away from the boundaries of the intervals I_j^n. Let us choose $\delta = \delta(\mathbf{c})$ so that the distance of each of the points y_n and $\tilde{y}_n$ is bounded away from ∂V_0^n and ∂V_1^n by at least $\delta > 0$. It is clear that such a choice of δ is always possible and that $\delta(\mathbf{c})$ may be chosen to be locally constant, i.e., there is a neighbourhood N of $\mathbf{c}$ in Σ^{Per} such that δ is constant on N. Unfortunately, Σ^{Per} is not compact so that it is not possible to choose δ independently of $\mathbf{c}$ on the whole of Σ^{Per}. Our approach is therefore to define $\mathcal{E}_n$ locally and then to patch together the local definitions to obtain a map defined on the whole of Σ^{Per}. Since δ may be arbitrarily small the final construction is a map from Σ^{Per} to the function space obtained by taking $\delta = 0$.

Letting $\delta > 0$, we now prove the following set inclusions, which show that the operator R_n is so-called analyticity improving.

Lemma 1. *The following inclusions hold for all $\delta > 0$.*

$$\overline{\theta_0^n(V_1^{n+1})} \subseteq V_0^n \tag{3.74}$$

$$\overline{\theta_i^n(V_0^{n+1})} \subseteq V_0^n, \quad i = 0, \ldots, a_n - 1\,, \tag{3.75}$$

$$\overline{\theta_{a_n}^n(V_0^{n+1})} \subseteq V_1^n\,. \tag{3.76}$$

Proof. Since θ_0^n is a similarity it is sufficient to show that θ_0^n maps the end points of a diameter of V_1^{n+1} inside V_0^n. Now, $DV_1^{n+1} = [-\delta - \omega_n^{-1}, 1 - \omega_n^{-1} + \delta]$ is a diameter of V_1^{n+1}, and

$$\theta_0^n(DV_1^{n+1}) = [-\omega_n + 1 - \omega_n\delta, 1 + \delta\omega_n] \subset (1 - \omega_{n-1}^{-1} - \delta, 1 + \delta) \subseteq V_0^n \tag{3.77}$$

(using $\omega_{n-1}^{-1} = \omega_n + a_n$), so that equation (3.74) holds.

Similarly, for $i = 0, \ldots, a_n - 1$, $DV_0^{n+1} = [1 - \omega_n^{-1} - \delta, 1 + \delta]$ is a diameter of V_0^{n+1} and

$$\theta_i^n([1 - \omega_n^{-1} - \delta, 1 + \delta]) = [-\omega_n - \omega_n\delta - i, 1 - \omega_n + \omega_n\delta - i], \tag{3.78}$$

so that

$$\begin{aligned} \theta_0^n(DV_0^{n+1}) \cup \ldots \cup \theta_{a_n-1}^n(DV_0^{n+1}) &= [-\omega_n - \omega_n\delta - (a_n - 1), 1 - \omega_n + \omega_n\delta] \\ &\subseteq (1 - \omega_n - a_n - \delta, 1 + \delta) \subseteq V_0^n \end{aligned} \tag{3.79}$$

and again we can conclude that equation (3.75) also holds.

Finally for (3.76):

$$\theta_{a_n}^n(DV_0^{n+1}) = [-\omega_n - \omega_n\delta - a_n, 1 - \omega_n + \omega_n\delta - a_n] \tag{3.80}$$

$$\subseteq (-\omega_{n-1}^{-1} - \delta, 1 - \omega_{n-1}^{-1} + \delta) \tag{3.81}$$

$$\subseteq V_1^n\,. \tag{3.82}$$

□

3.3 Definition of the renormalization transformations

We may now formally define the renormalization maps R_n and related additive versions:

$$R_n(u(x),t(x)) = (t(\theta_0^n(x)), \left(\prod_{i=0}^{a_n-1} t(\theta_i^n(x))\right) u(\theta_{a_n}^n(x))) \tag{3.83}$$

$$R_{0,n}(U(x),T(x)) = (T(\theta_0^n(x)), \left(\sum_{i=0}^{a_n-1} T(\theta_i^n(x))\right) + U(\theta_{a_n}^n(x))) \tag{3.84}$$

where $\theta_i^n(x) = -\omega_n x - i$.

For $\delta > 0$, we define the function space:

$$\begin{aligned}\mathcal{F}_n^\delta \quad = \quad & \{(U,T) : U : V_1^n \to \mathbb{C},\ T : V_0^n \to \mathbb{C}, \quad U,T \text{ real analytic} \\ & \text{and } \|(U,T)\| = \|U\|_1 + \|T\|_1 < \infty\},\end{aligned} \tag{3.85}$$

so that we have from Lemma 1

$$R_n : \mathcal{F}_n^\delta \to \mathcal{F}_{n+1}^\delta, \quad R_{0,n} : \mathcal{F}_n^\delta \to \mathcal{F}_{n+1}^\delta. \tag{3.86}$$

We also have $R_n(\exp(U,T)) = \exp(R_{0,n}(U,T))$, with exp defined coordinatewise. For the case **a** periodic with period p, say, it follows that $\mathcal{F}_{n+p}^\delta = \mathcal{F}_n^\delta$ for all $n \in \mathbb{Z}$. Let us denote by $\mathcal{F}_n$ the function space obtained by setting $\delta = 0$. It is clear from the above inclusions that R_n is well defined on $\mathcal{F}_n$ and we also have that $\mathcal{F}_n^\delta$ may be continuously embedded in $\mathcal{F}_n$.

4. PERIODIC CONTINUED FRACTIONS

In this chapter we shall be concerned with periodic continued fractions, although much of the theory applies in the more general case.

Let $\omega = [\overline{a_1, \ldots, a_p}]$, $a_1, \ldots, a_p \in \mathbb{N}$, have a periodic continued fraction expansion of period p. Then, for $k \in \mathbb{Z}$, we set $a_k = a_{k \bmod p}$ and $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$. Thus we may identify $\boldsymbol{\omega}$ with a unique $\mathbf{a} \in \mathbb{N}^{\mathbb{Z},Per}$. In what follows we shall restrict to this fixed sequence $\mathbf{a}$ with period $p \geq 1$ so that $a_{n+p} = a_n$ for all n. We further set, for $k \in \mathbb{Z}$, $\omega_k = [\overline{a_{k+1}, \ldots, a_{k+p}}]$, and let $\boldsymbol{\omega}= (\omega_k)_{k\in\mathbb{Z}}$. For $k \in \mathbb{Z}$, we also define V_0^k, V_1^k to be the domains in $\mathbb{C}$ given in section 3.2 above. Let us also set a code space $\Sigma = \{\mathbf{c} \in (c_k)_{k\in\mathbb{Z}} \mid c_k \in \{0, \ldots, a_k\}, c_k = a_k \Rightarrow a_{k-1} = 0\}$, and $\Sigma^{Per} = \{\mathbf{c} \in \Sigma \mid \mathbf{c} \text{ is periodic}\}$ (with certain codes removed, as explain in section 3.1.3.) We can now define a space of function pairs for each $k \in \mathbb{Z}$. For domains V_1^k, $V_0^k \subseteq \mathbb{C}$ we have

$$\mathcal{F}_k = \{(U, T) : U : V_1^k \to \mathbb{C},\ T : V_0^k \to \mathbb{C},\ U, T \text{ real analytic}\}, \tag{4.1}$$

with norm $\|(U, T)\| = \|U\|_1 + \|T\|_1$, where $\| \,.\, \|_1$ denotes the standard L^1-norm, given for a function analytic on $D(a, r)$ with Taylor expansion $f(x) = \sum_{i=0}^{\infty} f_i (x-a)^i / r^i$ by $\| f \|_1 = \sum_{i=0}^{\infty} | f_i |$. Then $\mathcal{F}_k$ is a real Banach space of function pairs.

4.1 Construction of the map $\mathcal{E}_n$

The construction of the renormalization strange sets is achieved through a map $\mathcal{E}_n$ which gives a function pair for each periodic sequence $\mathbf{c} \in \Sigma^{Per}$. Our aim in this section is the construction of the map $\mathcal{E}_n$ for periodic ω. We then give an outline of the spectral properties of R_n and define a projection operator P_n that annihilates the expanding directions in R_n. Further details are relegated to the appendix. Finally, we give an outline of the construction of $\mathcal{E}_n$ in section 4.1.3.

4.1.1 Spectral properties of the renormalization operator

In this section we study the spectral properties of the renormalization maps $R_{0,n}$ for $\delta > 0$. Introducing the notation $R_n^\ell = R_{n+\ell-1}R_{n+\ell-2}\dots R_n$ and similarly for $R_{0,n}^\ell$, we will investigate the spectral properties of $R_{0,n}^p : \mathcal{F}_n^\delta \to \mathcal{F}_n^\delta$. The spectral properties of the linear maps $R_{0,n}$ will be important in what follows, and in this section we prove the following lemma, which characterizes the spectral properties of the operator $R_{0,n}^p$ for $\mathbf{a}$ periodic of period-p.

Lemma 2. *Let* $\mathbf{a}$ *be periodic of period* $p \geq 1$. *Then the operator* $R_{0,n}^p : \mathcal{F}_n^\delta \to \mathcal{F}_n^\delta$ *is compact. Its spectrum consists of simple eigenvalues*

$$\{(-1)^{p(m+1)}(\gamma_0^p)^{m+1}, (-1)^{pm}(\gamma_0^p)^{m-1}, m = 0, 1, 2, \dots\}, \tag{4.2}$$

where $\gamma_0^p = \omega_0 \dots \omega_{p-1}$. *In particular there are two eigenvalues of modulus greater than or equal to* 1, *viz.,* $(-1)^p$ *and* $(\gamma_0^p)^{-1}$.

The proof of this lemma is given in appendix E. Our aim now is to define projection operators $P_{0,n}$ so that the non-stable eigenvalues, $(\gamma_0^p)^{-1}$ and $(-1)^p$ (which are given by setting $m = 0, 1$ in lemma 2) are killed off by the projection $P_{0,n}$ and so that the largest remaining eigenvalue has absolute value $0 < \gamma_0^p < 1$. Then we can conclude that there exists $\rho \in (\gamma_0^p, 1)$ such that $\|R_{0,n}^k P_{0,n}\| \leq K\rho^k$ for $k \geq 0$ and for some constant K .

4.1.2 Projection operator

In order to construct the maps β_n, we first define a projection operator that 'kills off' the expanding eigendirections in the function space $\mathcal{F}_n^\delta$. In other words, we project down to the stable manifold of the renormalization strange set. This is done in order to obtain a convergent series. Specifically, we have the following lemma.

Lemma 3. *Let* **a** *be periodic of period* $p \geq 1$*. Then for each* $n \in \mathbb{Z}$*, there exists a projection map* $P_{0,n} : \mathcal{F}_n^\delta \to \mathcal{F}_n^\delta$*, satisfying* $R_n P_{0,n} = P_{0,n+1} R_n$ *and such that the spectral radius of* $R_{0,n}^p P_{0,n} : \mathcal{F}_n^\delta \to \mathcal{F}_n^\delta$ *is strictly less than* 1*. Furthermore, there exists constants* $K > 0$ *and* $0 < \rho < 1$ *such that, for all* $n \in \mathbb{Z}$*,* $k \geq 0$*,*

$$\|R_{0,n}^k P_{0,n}\| \leq K\rho^k \, . \tag{4.3}$$

Proof. In order to prove the lemma, let us define

$$P_{0,n}(U_n, T_n) = (U_n, T_n) - \Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n, \tag{4.4}$$

where, for $\ell \geq 0$,

$$\Delta_\ell^n(U_n, T_n) = \int_{-\omega_{n-1}^{-1}}^{1-\omega_{n-1}^{-1}} U_n^{(\ell)}(x)dx + \int_{1-\omega_{n-1}^{-1}}^{1} T_n^{(\ell)}(x)dx, \tag{4.5}$$

and $\mathbf{v}_0^n,\ \mathbf{v}_1^n$ are function pairs satisfying

$$R_{0,n}\mathbf{v}_0^n = \omega_n^{-1}\mathbf{v}_0^{n+1}, \tag{4.6}$$

$$R_{0,n}\mathbf{v}_1^n = -\mathbf{v}_1^{n+1}, \tag{4.7}$$

subject to the conditions:

$$\Delta_0^n(\mathbf{v}_0^n) = 1, \tag{4.8}$$

$$\Delta_0^n(\mathbf{v}_1^n) = 0, \tag{4.9}$$

$$\Delta_1^n(\mathbf{v}_0^n) = 0, \tag{4.10}$$

$$\text{and} \qquad \Delta_1^n(\mathbf{v}_1^n) = 1, \tag{4.11}$$

The pairs $\mathbf{v}_0^n$ and $\mathbf{v}_1^n$ satisfying these conditions are given in appendix F below. With this definition, the projection $P_{0,n}$ kills off the non-contracting directions $\mathbf{v}_0^n,\ \mathbf{v}_1^n$. Moreover, $P_{0,n}$ commutes with $R_{0,n}$ in the following way:

Proposition 2.

$$R_{0,n}P_{0,n}(U_n, T_n) = P_{0,n+1}R_{0,n}(U_n, T_n) \tag{4.12}$$

Proof.

$$\begin{aligned}
R_{0,n}P_{0,n}(U_n,T_n) &= R_{0,n}\left((U_n,T_n)-\Delta_0^n(U_n,T_n)\mathbf{v}_0^n-\Delta_1^n(U_n,T_n)\mathbf{v}_1^n\right)\\
&= R_{0,n}(U_n,T_n)-\Delta_0^n(U_n,T_n)R_{o,n}(\mathbf{v}_0^n)-\Delta_1^n(U_n,T_n)R_{0,n}(\mathbf{v}_1^n)\\
&= R_{0,n}(U_n,T_n)-\Delta_0^n(U_n,T_n)\omega_n^{-1}\mathbf{v}_0^{n+1}-\Delta_1^n(U_n,T_n)(-\mathbf{v}_1^{n+1})\\
&= R_{0,n}(U_n,T_n)-\Delta_0^{n+1}(R_{0,n}(U_n,T_n))\mathbf{v}_0^{n+1}\\
&\quad -\Delta_1^{n+1}(R_{0,n}(U_n,T_n))\mathbf{v}_1^{n+1}\\
&= P_{0,n+1}(R_{0,n}(U_n,T_n)). \qquad (4.13)
\end{aligned}$$

□

Here we have used equations (4.6), (4.7) and the following result:

Proposition 3.

$$\Delta_\ell^{n+1}(R_{0,n}(U_n,T_n))=-(-\omega_n)^{\ell-1}\Delta_\ell^n(U_n,T_n) \qquad (4.14)$$

Proof.

$$\begin{aligned}
\Delta_\ell^{n+1}(R_{0,n}(U_n,T_n)) &= \int_{-\omega_n^{-1}}^{1-\omega_n^{-1}} \frac{d^\ell}{dx^\ell} T_n(\theta_0^n(x))dx \\
&\quad + \int_{1-\omega_n^{-1}}^{1} \frac{d^\ell}{dx^\ell}\left(\sum_{i=0}^{a_n-1} T_n(\theta_i^n(x)) + U_n(\theta_{a_n}^n)\right)dx \qquad (4.15)\\
&= (-\omega_n)^\ell \int_{-\omega_n^{-1}}^{1-\omega_n^1} T_n^{(\ell)}(\theta_0^n(x))dx \\
&\quad +(-\omega_n)^\ell \int_{1-\omega_n^{-1}}^{1} \sum_{i=0}^{a_n-1} T_n^{(\ell)}(\theta_i^n(x)) + U_n^{(\ell)}(\theta_{a_n}^n(x))dx \\
&\qquad (4.16)\\
&= (-\omega_n)^{\ell-1} \int_{1}^{1-\omega_n} T_n^{(\ell)}(y)dy \\
&\quad +(-\omega_n)^{\ell-1} \sum_{i=0}^{a_n-1} \int_{-\omega_n-(i-1)}^{-\omega_n-1} T_n^{(\ell)}(y)dy \\
&\quad +(-\omega_n)^{\ell-1} \int_{-\omega_n-(a_n-1)}^{-\omega_n-a_n} U_n^{(\ell)}(y)dy \qquad (4.17)\\
&= -(-\omega_n)^{\ell-1} \int_{1-\omega_n}^{1} T_n^{(\ell)}(y)dy \\
&\quad -(-\omega_n)^{\ell-1} \int_{-\omega_n-(a_n-1)}^{1-\omega_n} T_n^{(\ell)}(y)dy \\
&\quad -(-\omega_n)^{\ell-1} \int_{-\omega_n-a_n}^{-\omega_n-(a_n-1)} U_n^{(\ell)}(y)dy \qquad (4.18)\\
&= -(-\omega_n)^{\ell-1}\left(\int_{-\omega_{n-1}^{-1}}^{1-\omega_{n-1}^{-1}} U_n^{(\ell)}(y)dy + \int_{1-\omega_{n-1}^{-1}}^{1} T_n^{(\ell)}(y)dy\right) \\
&\qquad (4.19)\\
&= -(-\omega_n)^{\ell-1}\Delta_\ell^n(U_n,T_n). \qquad (4.20)
\end{aligned}$$

□

We conclude that the spectral radius of $R_{0,n}^p P_{0,n}$ is $(\omega_1 \ldots \omega_p)$, and further deduce that there exist constants $K > 0$ and $0 < \rho < 1$ such that

$$\|R_{0,n}^k\| \leq K\rho^k \,, \qquad \text{for} \quad k \geq 0. \tag{4.21}$$

This concludes the proof of lemma 3. □

We may now readily define a multiplicative version P_n of the projection operator $P_{0,n}$ as follows. Letting $(u_n, t_n) \in \mathcal{F}_n^\delta$, we set

$$P_n(u_n, t_n) = (u_n, t_n) \exp(-\Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n) \,, \tag{4.22}$$

where $(U_n, T_n) = \log |(u_n, t_n)|$. It is straightforward to show that P_n has the following properties.

1. $P_n \exp(U_n, T_n) = \exp(P_{0,n}(U_n T_n))$ for $(U_n, T_n) \in \mathcal{F}_n^\delta$.
2. $P_n^2 = P_n$, so that P_n is a projection operator;
3. $R_n P_n = P_{n+1} R_n$;

Proof. 1.

$$\begin{aligned}
P_n(\exp(U_n, T_n)) &= P_n(u_n, t_n) \\
&= (u_n, t_n) \exp(-\Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n) \\
&= \exp(U_n, T_n) \exp(-\Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n) \\
&= \exp((U_n, T_n) - \Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n) \\
&= \exp(P_{0,n}(U_n, T_n)).
\end{aligned} \tag{4.23}$$

2. Let $P_n(u_n, t_n) = (\hat{u}_n, \hat{t}_n)$ then

$$\begin{aligned} P_n^2(u_n, t_n) &= P_n(\hat{u}_n, \hat{t}_n) \\ &= (\hat{u}_n, \hat{t}_n) \exp(-\Delta_0^n(\hat{U}_n, \hat{T}_n)\mathbf{v}_0^n - \Delta_1^n(\hat{U}_n, \hat{T}_n)\mathbf{v}_1^n), \end{aligned} \tag{4.24}$$

where $(\hat{U}_n, \hat{T}_n) = \log |(\hat{u}_n, \hat{t}_n)|$. Now,

$$\begin{aligned} (\hat{U}_n, \hat{T}_n) &= \log |(\hat{u}_n, \hat{t}_n)\| \\ &= \log |P_n(u_n, t_n)| \\ &= \log |(u_n, t_n) \exp(-\Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n)| \\ &= \log |(u_n, t_n)| - \Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n \\ &= (U_n, T_n) - \Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n, \end{aligned} \tag{4.25}$$

and using linearity we have

$$\Delta_\ell^n(\hat{U}_n, \hat{T}_n) = \Delta_0^n(U_n, T_n) - \Delta_0^n(U_n, T_n)\Delta_\ell^n(\mathbf{v}_0^n) - \Delta_1^n(U_n, T_n)\Delta_\ell^n(\mathbf{v}_1^n). \tag{4.26}$$

Taking $\ell = 0$ and using equations (4.8) and (4.9) we have

$$\Delta_0^n(\hat{U}_n, \hat{T}_n) = \Delta_0^n(U_n, T_n) - \Delta_0^n(U_n, T_n)\Delta_0^n(\mathbf{v}_0^n) - \Delta_1^n(U_n, T_n)\Delta_0^n(\mathbf{v}_1^n) = 0, \tag{4.27}$$

similarly for $\ell = 1$ using equations (4.10) and (4.11) we have

$$\Delta_1^n(\hat{U}_n, \hat{T}_n) = \Delta_1^n(U_n, T_n) - \Delta_0^n(U_n, T_n)\Delta_1^n(\mathbf{v}_0^n) - \Delta_1^n(U_n, T_n)\Delta_1^n(\mathbf{v}_1^n) = 0, \tag{4.28}$$

so that equation (4.24) becomes

$$P_n^2(u_n, t_n) = (\hat{u}_n, \hat{t}_n) = P_n(u_n, t_n). \tag{4.29}$$

3. From (4.23) we have that $P_n(u_n, t_n) = \exp(P_{o,n}(U_n, T_n))$, then

$$\begin{aligned} R_n(P_n(u_n, t_n)) &= R_n(\exp((U_n, T_n) - \Delta_0^n(U_n, T_n)\mathbf{v}_0^n - \Delta_1^n(U_n, T_n)\mathbf{v}_1^n)) \\ &= R_n(u_n, t_n) \exp(-\Delta_0^n(U_n, T_n) R_n \mathbf{v}_0^n - \Delta_1^n(U_n, T_n) R_n \mathbf{v}_1^n) \\ &= R_n(u_n, t_n) \exp(-\omega_n^{-1}\Delta_0^n(U_n, T_n)\mathbf{v}_0^{n+1} + \Delta_1^n(U_n, T_n)\mathbf{v}_1^{n+1}) \\ &= R_n(u_n, t_n) \\ &\quad \exp(-\Delta_0^{n+1}(R_n(u_n, t_n))\mathbf{v}_0^{n+1} - \Delta_1^{n+1}(R_n(u_n, t_n))\mathbf{v}_1^{n+1}) \\ &= P_{n+1}(R_n(u_n, t_n)), \end{aligned} \tag{4.30}$$

where we have used the result

$$\Delta_\ell^{n+1}(R_n(u_n, t_n)) = -(-\omega_n)^{\ell-1}\Delta_\ell^n(U_n, T_n). \tag{4.31}$$

□

Lemma 3 has a corollary for the multiplicative projection P_n.

Lemma 4. *Let* $(u_n, t_n) \in \mathcal{F}_n^\delta$ *satisfy*

1. $P_n(u_n, t_n) = (u_n, t_n)$;

2. $(u_n, t_n) \neq 0$ *on* $\overline{V}_1^n \times \overline{V}_0^n$;

3. $(u_n, t_n) > 0$ *on* $\left(\overline{V}_1^n \times \overline{V}_0^n\right) \cap \mathbb{R}^2$.

Then $R_n^\ell(u_n, t_n) \to (1, 1)$ *as* $\ell \to \infty$.

Proof. We have that $(u_n, t_n) = \exp(U_n, T_n)$ where $(U_n, T_n) \in \mathcal{F}_n^\delta$ is well defined and analytic since $(u_n, t_n) \neq 0$ on $\overline{V}_1^n \times \overline{V}_0^n$. Moreover $P_{0,n}(U_n, T_n) = (U_n, T_n)$. Then from Lemma 3, $\|R_{0,n}^\ell(U_n, T_n)\| \to (0, 0)$ as $\ell \to \infty$. We conclude that $R_n^\ell(u_n, t_n) = \exp(R_{0,n}^\ell(U_n, T_n)) \to (1, 1)$ as $\ell \to \infty$. □

We shall make use of the multiplicative projection in our subsequent analysis.

4.1.3 Construction of the map $\mathcal{E}_n$

We aim to construct a biinfinite sequence of maps $\mathcal{E}_n : \Sigma^{Per} \to \mathcal{F}_n^\delta$, such that

$$\mathcal{E}_n = h^n(\mathbf{c}) \,, \tag{4.32}$$

where

$$h^n(\mathbf{c}) = \begin{cases} (y_n - x, y_n - x) & c_{n-1} = 0 \\ (1, y_n - x) & c_{n-1} \neq 0 \end{cases} \tag{4.33}$$

and multiplication is carried out coordinatewise so that $(u_1, t_1)(u_2, t_2) = (u_1 u_2, t_1 t_2)$. Here $y_n = e_n(\mathbf{c})$ is the image of the evaluation map, and $(u_n^1, t_n^1) \in$

$\mathcal{F}_n^\delta$ with

$$u_n^1(x) > 0, \quad x \in (-\omega_n - a_n, -\omega_n - (a_n - 1)) \tag{4.34}$$

$$t_n^1(x) > 0, \quad x \in (-\omega_n - (a_n - 1), 1) \tag{4.35}$$

and $\mathcal{E}_n$ has the important property

$$R_n((s^u, s^t)\mathcal{E}_n(\mathbf{c})) = \kappa_{c_n}(s^u, s^t)\mathcal{E}_{n+1}(\mathbf{c})\,. \tag{4.36}$$

where the sign-pair $(s^u, s^t) \in \{+1, -1\}^2$ and $\kappa_{c_n} : \{+1, -1\}^2 \to \{+1, -1\}^2$ is given by

$$\kappa_{c_n}(s^u, s^t) = (-s^t, -(-1)^{c_n}(s^t)^{a_n} s^u). \tag{4.37}$$

Let us we briefly explain the origin of κ_{c_n}. We show below that $R_n(\mathcal{E}_n(\mathbf{c})) = (-1, -(-1)^{c_n})\mathcal{E}_{n+1}$. Using the multiplicative properties of R_n we have

$$\begin{aligned} R_n((s^u, s^t)\mathcal{E}_n) &= (s^t, (s^t)^{a_n} s^u))(-1, -(-1)^{c_n})\mathcal{E}_{n+1} &(4.38)\\ &= \kappa_{c_n}(s^u, s^t)\mathcal{E}_{n+1}. &(4.39)\end{aligned}$$

When $(s^u, s^t) = (+1, +1)$ the relation (4.36) becomes

$$R_n(\mathcal{E}_n(\mathbf{c})) = \kappa_{c_n}(+1, +1)\mathcal{E}_{n+1} = (-1, -(-1)^{c_n})\mathcal{E}_{n+1}\,. \tag{4.40}$$

Let us define $H^{n,1}(\mathbf{c})$ by the equation:

$$R_n h^n(\mathbf{c}) = \kappa_{c_n}(+1, +1)h^{n+1}(\mathbf{c})\exp(H^{n,1}(\mathbf{c})), \tag{4.41}$$

where $H^{n,1}(\mathbf{c}) \in \mathcal{F}^{\delta}_{n+1}$. We give the construction of $H^{n,1}(\mathbf{c})$ explicitly. There are four cases:

Case (i). $c_{n-1} = 0$, $c_n = 0$. Recalling that

$$R_n(u(x), t(x)) = (t(\theta_0^n(x)), \prod_{i=0}^{a_n-1} t(\theta_i^n(x))u(\theta_{a_n}^n(x))), \tag{4.42}$$

and

$$y_{n+i} = -c_{n+i} - \omega_{n+i} y_{n+(i+1)}, \tag{4.43}$$

and using equation (4.37) we have that

$$\begin{aligned}
R_n(h^n(\mathbf{c})) &= (y_n - \theta_0^n(x), (\prod_{i=0}^{a_n-1} y_n - \theta_i^n(x))(y_n - \theta_{a_n}^n(x))) \\
&= (\theta_0^n(y_{n+1}) - \theta_0^n(x), (\theta_0^n(y_{n+1}) - \theta_0^n(x)) \prod_{i=1}^{a_n}(y_n - \theta_i^n(x))) \\
&= (-1,-1)(y_{n+1} - x, y_{n+1} - x)(\omega_n, \omega_n \prod_{i=1}^{a_n}(y_n - \theta_i^n(x))) \\
&= \kappa_0(+1,+1)h^{n+1}(\mathbf{c}) \exp(H^{n,1}(\mathbf{c})), \qquad (4.44)
\end{aligned}$$

where

$$H^{n,1}(\mathbf{c}) = \log\left(\omega_n, \omega_n \prod_{i=1}^{a_n}(y_n - \theta_i^n(x))\right). \tag{4.45}$$

Case (ii). $c_{n-1} \neq 0$, $c_n = 0$. Similarly,

$$\begin{aligned}
R_n(h^n(\mathbf{c})) &= (y_n - \theta_0^n(x), \prod_{i=0}^{a_n-1}(y_n - \theta_i^n(x))) \\
&= (\theta_0^n(y_{n+1}) - \theta_0^n(x), (\theta_0^n(y_{n+1}) - \theta_0^n(x)) \prod_{i=1}^{a_n-1}(y_n - \theta_i^n(x))) \\
&= (-1,-1)(y_{n+1} - x, y_{n+1} - x)(\omega_n, \omega_n \prod_{i=1}^{a_n}(y_n - \theta_i^n(x))) \\
&= \kappa_0(+1,+1)h^{n+1}(\mathbf{c}) \exp(H^{n,1}(\mathbf{c})), \qquad (4.46)
\end{aligned}$$

where

$$H^{n,1}(\mathbf{c}) = \log\left(\omega_n, \omega_n \prod_{i=1}^{a_n-1}(y_n - \theta_i^n(x))\right). \qquad (4.47)$$

Case (iii). $c_{n-1} = 0$, $c_n \neq 0$. Here,

$$\begin{aligned}
R_n(h^n(\mathbf{c})) &= (y_n - \theta_0^n(x), (\prod_{i=0}^{a_n-1} y_n - \theta_i^n(x))(y_n - \theta_{a_n}^n(x))) \\
&= (1, y_n - \theta_{c_n}^n(x)) \\
&\quad (y_n - \theta_0^n(x), (\prod_{i=0}^{c_n-1}(y_n - \theta_i^n(x)))(\prod_{j=c_n+1}^{a_n}(y_n - \theta_j^n(x)))) \\
&= (-1, -(-1)^{c_n})(1, y_{n+1} - x) \\
&\quad (\theta_0^n(x) - y_n, (-1)^{c_n}\omega_n(\prod_{i=0}^{c_n-1}(y_n - \theta_i^n(x)))(\prod_{j=c_n+1}^{a_n}(y_n - \theta_j^n(x)))) \\
&= \kappa_{c_n}(+1,+1)h^{n+1}(\mathbf{c}) \exp(H^{n,1}(\mathbf{c})), \qquad (4.48)
\end{aligned}$$

where

$$H^{n,1}(\mathbf{c}) = \log\left(\theta_0^n(x) - y_n, \omega_n(\prod_{i=0}^{c_n-1}(\theta_i^n(x) - y_n))(\prod_{j=c_n+1}^{a_n}(y_n - \theta_j^n(x)))\right). \tag{4.49}$$

Case (iv). $c_{n-1} \neq 0$, $c_n \neq 0$. Finally,

$$\begin{aligned}
R_n(h^n(\mathbf{c})) &= (y_n - \theta_0^n(x), \prod_{i=0}^{a_n-1}(y_n - \theta_i^n(x))) \\
&= (1, y_n - \theta_{c_n}^n(x)) \\
&\quad (y_n - \theta_0^n(x), (\prod_{i=0}^{c_n-1}(y_n - \theta_i^n(x)))(\prod_{j=c_n+1}^{a_n}(y_n - \theta_j^n(x)))) \\
&= (-1, -(-1)^{c_n})(1, y_{n+1} - x) \\
&\quad (\theta_0^n(x) - y_n, (-1)^{c_n}\omega_n(\prod_{i=0}^{c_n-1}(y_n - \theta_i^n(x)))(\prod_{j=c_n+1}^{a_n-1}(y_n - \theta_j^n(x)))) \\
&= \kappa_{c_n}(+1, +1)h^{n+1}(\mathbf{c})\exp(H^{n,1}(\mathbf{c})),
\end{aligned} \tag{4.50}$$

where

$$H^{n,1}(\mathbf{c}) = \log\left(\theta_0^n(x) - y_n, \omega_n(\prod_{i=0}^{c_n-1}(\theta_i^n(x) - y_n))(\prod_{j=c_n+1}^{a_n-1}(y_n - \theta_j^n(x)))\right). \tag{4.51}$$

The reason for defining $H^{n,1}$ is that it is a function pair that has no zeros on the domains V_1 and V_0, provided δ is chosen sufficiently small, and, we have a bound for $\|H^{n,1}(\mathbf{c})\|$ independent of n. Indeed, we have the following technical lemma:

Lemma 5. *There exist $\delta > 0$ and $L > 0$ (depending on* $\mathbf{a}$ *and* $\mathbf{c}$*) such that* *the function* $H^{n,1} = (u^1_{n+1}, t^1_{n+1}) \in \mathcal{F}^\delta_{n+1}$ *satisfies*

1. $u^1_{n+1}(x) > 0$ *for* $x \in \overline{V}^{n+1}_1 \cap \mathbb{R}$, *and* $t^1_{n+1}(x) > 0$ *for* $x \in \overline{V}^{n+1}_0 \cap \mathbb{R}$;
2. $\|H^{n,1}\| \leq L$.

The proof of this lemma is laborious, if straightforward. We refer the reader to appendix G, where full details will be found. The precise bound L is dependent on the choice of δ, but this merely governs the estimates for the convergence of $\mathcal{E}_n(\mathbf{c})$ and does not affect the function so defined which is independent of δ.

Let us now define $\mathcal{E}_n = h^n \exp(K^n)$ where K^n is constructed so that $K^n = K^n(\mathbf{c}) \in \mathcal{F}^\delta_n$, and $H^{n,1} + R_{0,n}K^n = K^{n+1}$, which is the required functional equation for K.

Defining $H^{n,0} = \log|h^n|$ it is also shown in appendix H that

$$R_{0,n}H^{n,0} = H^{n+1,0} + H^{n,1} \tag{4.52}$$

where $H^{n,1} \in \mathcal{F}^\delta_{n+1}$ is as before.

Using the notation $R^\ell_n = R_{n+\ell-1}R_{n+\ell-2}\ldots R_n$ and similarly for $R^\ell_{0,n}$, we define:

$$H^{n,\ell} = R^{\ell-1}_{0,n+1}H^{n,1}\,, \quad \hat{H}^{n,\ell} = R^{\ell-1}_{0,n+1}P_{0,n+1}H^{n,1}\,. \tag{4.53}$$

Setting $\hat{G}^n = \sum_{k=1}^\infty \hat{H}^{n-k,k}$, the convergence of the series is shown in appendix H where it is also demonstrated that $\hat{G}^n$ satisfies the equation

$$R_{0,n}\hat{G}^n + \hat{H}^{n,1} = \hat{G}^{n+1}\,. \tag{4.54}$$

The function pair K^n is now given by

$$K^n = \hat{G}^n - (I - P_{0,n})H^{n,0} \in \mathcal{F}_n^\delta\,, \tag{4.55}$$

where $P_{0,n}$ is given by (4.4) above. The details are laid out in appendix H.

The construction of $\mathcal{E}_n$ depends on $\mathbf{c}$ in that δ depends on $\mathbf{c}$. δ may be chosen locally constant, however, so that we may define a restricted map $\mathcal{E}_n : N \cap \Sigma^{Per} \to \mathcal{F}_n \subseteq \mathcal{F}_n^\delta$, where the patch N is an open neighbourhood of $\mathbf{c}$. By covering Σ^{Per} by patches of this form, we may define $\mathcal{E}_n : \Sigma^{Per} \to \mathcal{F}_n$.

Let us remark briefly on the continuity of the map $\mathcal{E}_n$. Since the evaluation map e_n and the partnering operation are continuous, the functions h_n depend continuously on $\mathbf{c}$ in $\mathcal{F}_n^\delta$ and $\mathcal{F}_n$. Since the series for $\hat{G}_n$ is uniformly convergent in $\mathcal{F}_n^\delta$ it follows also that $\hat{G}_n$, and, hence, K^n and $\mathcal{E}_n$ depend continuously on $\mathbf{c}$ in each neighbourhood N. It follows immediately that $\mathcal{E}_n : \Sigma^{Per} \to \mathcal{F}$ is continuous.

4.2 The map β_n and symmetry properties

The maps $\mathcal{E}_n$ constructed above form the basic ingredients of the embedding of the model space into the function-pair space $\mathcal{F}_n^\delta$. They map the space Σ^{Per} into $\mathcal{F}_n^\delta$ sending a periodic code $\mathbf{c}$ into a function pair $(u(x), t(x))$ with $t(x)$ having precisely one zero. The initial condition $t_1(x)$ has two zeros in the fundamental interval (see (1.57)) and these zeros are related by the symmetry of the cosine function. We incorporate this by constructing a new map β_n in terms of $\mathcal{E}_n$. This map β_n is that required by Theorem 9.

4.2.1 Definition of β_n

Specifically, we define $\beta_n : \Sigma^{Per} \to \mathcal{F}_n^{\delta}$ for $\mathbf{c} \in \Sigma^{Per}$ by

$$\beta_n(\mathbf{c}) = \mathcal{E}_n(\mathbf{c})\mathcal{E}_n(\tilde{\mathbf{c}})\,, \tag{4.56}$$

where, as usual, multiplication is carried out coordinatewise. Let us now define a map L_{b_n}, for $b_n \in \mathbb{Z}$ on sign-pairs $(s_n^u, s_n^t) \in \{+1, -1\}^2$

$$L_{b_n}(s_n^u, s_n^t) = (s_n^t, (-1)^{b_n}(s_n^t)^{a_n} s_n^u). \tag{4.57}$$

The map is invertible with inverse

$$L_{b_n}^{-1}(s_n^u, s_n^t) = ((-1)^{b_n}(s_n^u)^{a_n} s_n^t, s_n^u). \tag{4.58}$$

The map L_{b_n} is derived from κ_{c_n} as follows. Firstly, from equation (4.36), we have that

$$R_n(\beta_n(\mathbf{c})) = R_n(\mathcal{E}_n(\mathbf{c}))R_n(\mathcal{E}_n(\tilde{\mathbf{c}})) = \kappa_{c_n}(+1,+1)\mathcal{E}_{n+1}(\mathbf{c})\kappa_{\tilde{c}_n}(+1,+1)\mathcal{E}_{n+1}(\tilde{\mathbf{c}}) \tag{4.59}$$

$$= L_{b_n}(+1,+1)\beta_{n+1}(\mathbf{c})\,, \tag{4.60}$$

where $b_n = c_n + \tilde{c}_n$. Similarly, for general sign-pairs (s^u, s^t), we have

$$R_n((s^u, s^t)\beta_n(\mathbf{c})) = L_{b_n}(s^u, s^t)\beta_{n+1}(\mathbf{c})\,, \tag{4.61}$$

as claimed in Theorem 9. For, from equation (4.56),

$$\begin{aligned}
R_n((s^u_n, s^t_n)\beta_n(\mathbf{c})) &= R_n(s^u_n, s^t_n)R_n(\mathcal{E}_n(\mathbf{c}))R_n(\mathcal{E}_n(\tilde{\mathbf{c}})) \\
&= (s^t_n, (s^t_n)^{a_n} s^u_n)\kappa_{c_n}(+1,+1)\mathcal{E}_{n+1}(\mathbf{c})\kappa_{\tilde{c}_n}(+1,+1)\mathcal{E}_{n+1}(\tilde{\mathbf{c}}) \\
&= (s^t_n, (s^t_n)^{a_n} s^u_n)(-1, -(-1)^{c_n})(-1, -(-1)^{\tilde{c}_n})\mathcal{E}_{n+1}(\mathbf{c})\mathcal{E}_{n+1}(\tilde{\mathbf{c}}) \\
&= (s^t_n, (-1)^{b_n}(s^t_n)^{a_n} s^u_n)\beta_{n+1}(\mathbf{c}) \\
&= L_{b_n}(s^u_n, s^t_n)\beta_{n+1}(\mathbf{c}).
\end{aligned} \tag{4.62}$$

The following theorem is the key to the construction of the strange set.

Theorem 9. *Let* $\mathbf{a}$ *be a fixed sequence in* $\mathbb{N}^{\mathbb{Z},Per}$ *corresponding to a periodic continued fraction. For each* $n \in \mathbb{Z}$*, there exists a continuous map* $\beta_n : \Sigma^{Per} \to \mathcal{F}_n$ *such that for* $\mathbf{c} \in \Sigma^{Per}$*,*

$$R_n\beta_n(\mathbf{c}) = L_{b_n}(+1,+1)\beta_{n+1}(\mathbf{c})\,, \tag{4.63}$$

where $L_{b_n} : \{+1,-1\}^2 \to \{+1,-1\}^2$ *is given by*

$$L_{b_n}(s^u_n, s^t_n) = (s^t_n, (-1)^{b_n}(s^t_n)^{a_n} s^u_n)\,, \tag{4.64}$$

and multiplication is carried out coordinatewise. Here $b_n = c_n + \tilde{c}_n$ *where* $\tilde{\mathbf{c}} = (\tilde{c}_k)_{k\in\mathbb{Z}}$ *denotes the partner code to* $\mathbf{c}$ *(to be defined in section 3.1.4* *below),* Σ^{Per} *is a subspace of periodic codes to be defined in section 3.1.3 below and* $\mathcal{F}_n$ *is the function-pair space defined in section 3.3. The equation* (4.63) *generalizes to*

$$R_n((s^u, s^t)\beta_n(\mathbf{c})) = L_{b_n}(s^u, s^t)\beta_{n+1}(\mathbf{c}) \tag{4.65}$$

where $(s^u, s^t) \in \{+1, -1\}^2$ *is an arbitrary sign pair. The map* β_n *is two-to-one in the sense that* $\beta_n(\mathbf{c}) = \beta_n(\mathbf{c}')$ *if, and only if,* $\mathbf{c}' = \mathbf{c}$ *or* $\mathbf{c}' = \tilde{\mathbf{c}}$*.*

Proof. It is immediately clear from (4.56) that $\beta_n(\mathbf{c}) = \beta_n(\tilde{\mathbf{c}})$. Conversely, let us suppose that $\beta_n(\mathbf{c}) = \beta_n(\mathbf{c}')$ for $\mathbf{c}$, $\mathbf{c}' \in \Sigma^{Per}$. Then, by construction, $\mathcal{E}_n(\mathbf{c})\mathcal{E}_n(\tilde{\mathbf{c}}) = \mathcal{E}_n(\mathbf{c}')\mathcal{E}_n(\tilde{\mathbf{c}}')$. Now, by construction, the zero of the second coordinate of $\mathcal{E}_n(\mathbf{c})$ has $y_n = e_n(\mathbf{c})$ and, similarly for $\mathcal{E}_n(\tilde{\mathbf{c}})$, $\mathcal{E}_n(\mathbf{c}')$ and $\mathcal{E}_n(\tilde{\mathbf{c}}')$. It follows immediately that $e_n(\mathbf{c}') = e_n(\mathbf{c})$ or $e_n(\mathbf{c}') = e_n(\tilde{\mathbf{c}})$ and from section 3.1 that $\mathbf{c}' = \mathbf{c}$ or $\mathbf{c}' = \tilde{\mathbf{c}}$. □

We note that β_n is continuous because $\mathcal{E}_n$ is continuous and since the partnering operation is continuous.

4.2.2 Sign dynamics

We now consider the dynamics of a sign-pair $(s^u, s^t) \in \{-1, +1\}^2$ under the map $L_b(s^u, s^t) = (s^t, (-1)^b (s^t)^{a_n} s^u)$. It is the sign dynamics that determine the symmetry structure of the renormalization strange set. Our task is to understand the behaviour of the map L_b on the block structures,

$$(0a_{n+1}) \quad \text{for} \quad m=2; \tag{4.66}$$

$$(0(a_{n+1}+1)(a_{n+2}-1)) \quad \text{for} \quad m=3; \tag{4.67}$$

$$(0(a_{n+1}+1)a_{n+2}\dots a_{n+m-2}(a_{n+m-1}-1)) \quad \text{for} \quad m\geq 4. \tag{4.68}$$

Let us consider a block of length $m \geq 1$ starting at iteration n with sign-pair (s^u_n, s^t_n). We calculate the sign-pair (s^u_{n+m}, s^t_{n+m}) in terms of (s^u_n, s^t_n) and, to this end, we write, for $j \geq 0$,

$$(s^u_{n+j}, s^t_{n+j}) = \left((-1)^{r_{n+j-1}}(s^u_n)^{p_{n+j-1}}(s^t_n)^{q_{n+j-1}}),\ (-1)^{r_{n+j}}(s^u_n)^{p_{n+j}}(s^t_n)^{q_{n+j}})\right)\ , \tag{4.69}$$

where this equation defines the indices p_{n+j}, q_{n+j}, r_{n+j}. Using the relation $(s^u_{n+j+1}, s^t_{n+j+1}) = L_{b_{n+j}}(s^u_{n+j}, s^t_{n+j})$, where b_{n+j} is the jth entry in the block structure starting at n, a straightforward calculation gives the recurrence relations and initial conditions

$$p_{n+j+1} = a_{n+j}p_{n+j} + p_{n+j-1}\ , \quad p_n = 0, \quad p_{n-1} = 1 \tag{4.70}$$

$$q_{n+j+1} = a_{n+j}q_{n+j} + q_{n+j-1}\ , \quad q_n = 1, \quad q_{n-1} = 0 \tag{4.71}$$

$$r_{n+j+1} = a_{n+j}r_{n+j} + r_{n+j-1} + b_{n+j}\ , \quad r_n = 0, \quad r_{n-1} = 0 \tag{4.72}$$

valid for $j = 0, \dots, n+m-1$. Using this notation, we may readily obtain the following results.

Lemma 6. *Calculating modulo* 2, *we have*

1. $r_{n-1} \neq p_{n-1}$, $r_n = p_n$, $r_{n+j} \neq p_{n+j}$, *for* $j = 1, \dots, m-1$, *and* $r_{n+m} = p_{n+m}$*;*

2. *for* $j = 0, \dots, m$,

$$\begin{pmatrix} p_{n+j-1} & q_{n+j-1} \\ p_{n+j} & q_{n+j} \end{pmatrix} = M_{n+j-1} \dots M_n \,. \tag{4.73}$$

The proof of the lemma is a straightforward application of the recurrence relations (4.70 – 4.72) and the block structures (4.66 – 4.68) given above. Calculating modulo 2, gives for $m = 2$, $r_{n+1} = 0 \neq 1 = p_{n+1}$ and $r_{n+2} = a_{n+2} = p_{n+2}$ and for $m = 3$, $r_{n+1} = 0 \neq 1 = p_{n+1}$, $r_{n+2} = a_{n+1} + 1 \neq a_{n+1} = p_{n+2}$, whilst $r_{n+3} = a_{n+2}a_{n+1} + 1 = p_{n+3}$. For $m \geq 4$, we have $r_{n+1} = 0 \neq 1 = p_{n+1}$, $r_{n+2} = a_{n+1} + 1 \neq a_{n+1} = p_{n+2}$ and, by induction, for $j = 1, \dots, n+m-2$, we have $r_{n+j+1} - p_{n+j+1} = a_{n+j}r_{n+j} + r_{n+j-1} + b_{n+j} - (a_{n+j}p_{n+j} + p_{n+j-1})$ $a_{n+j}(r_{n+j} - p_{n+j}) + r_{n+j-1} - p_{n+j-1} + a_{n+1} = a_{n+j} + 1 + a_{n+1} = 1$, using $r_{n+j} - p_{n+j}$, $r_{n+j-1} - p_{n+j-1} = 1$. Similarly, we have $r_{n+m} - p_{n+m} = a_{n+j}r_{n+m-1} + r_{n+m-2} + b_{n+m-1} - (a_{n+m-1}p_{n+m-1} + p_{n+m-2})$ $= a_{n+m-1}(r_{n+m-1} - p_{n+m-1}) + r_{n+m-2} - p_{n+m-2} + a_{n+m-1} - 1 = a_{n+m-1} + 1 + a_{n+m-1} - 1 = 0$, as claimed. Here we have used $r_{n+m-1} - p_{n+m-1}$, $r_{n+m-2} - p_{n+m-2} = 1$. This completes the proof of statement 1. Statement 2 follows immediately from the recurrence relations for p_{n+j} and q_{n+j} and the initial conditions. Indeed, writing the recurrence relations in matrix form, we have for $j = 0, \ldots, m$,

$$\begin{pmatrix} p_{n+j-1} & q_{n+j-1} \\ p_{n+j} & q_{n+j} \end{pmatrix} = M_{n+j-1} \ldots M_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = M_{n+j-1} \ldots M_n \,, \qquad (4.74)$$

as required. The lemma is proved.

4.2.3 Symmetries of the renormalization strange set

We now consider the symmetries of the renormalization strange sets. The renormalization strange sets are built from copies of the fundamental sets, the symmetries of which, i.e., the number of copies and orientation, are determined by the sign pairs (s_n^u, s_n^t), which in turn depend on the maps the L_b. It transpires that we can analyse the symmetries in terms a group of order 6 determined by the parities of the entries in the p-periodic continued fraction $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$, $a_{k+p} = a_k$. Let us write $\bar{a}$ for $a \mod 2$, and let us consider the reduced continued fraction $\bar{\mathbf{a}} = [\overline{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_p}]$. Let A, B be the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \,. \qquad (4.75)$$

Then over the field of two elements $\mathbb{Z}_2$, i.e., with addition and multiplication modulo 2, A and B satisfy $A^2 = I$, $B^3 = I$, $BA = AB^2$ and thus generate a group of order 6 isomorphic to the symmetric group S_3 and the dihedral group D_3. The elements of this group are I, B, B^2, A, AB, AB^2, with orders given in table 4.1 below.

Element $\bar{M}$	Order	Action on sign-pair (s^u, s^t)	Invariant sign-pair sets other than $\{(-1,1)\}$.	Number of copies of fundamental set
I	1	(s^u, s^t)	{(1,1)}, {(1,-1)} , {(-1,-1)}	1
B	3	$(-s^t, -s^u s^t)$	{(1,1), (1,-1), (-1,-1)}	3
B^2	3	$(s^u s^t, -s^u)$	{(1,1), (1,-1), (-1,-1)}	3
A	2	$(-s^t, -s^u)$	{(1,1), (-1,-1)}, {(1,-1)}	1 or 2
AB	2	$(s^u s^t, s^t)$	{(1,-1), (-1,-1)}, {(1,1)}	1 or 2
AB^2	2	$(s^u, -s^u s^t)$	{(1,1), (1,-1)}, {(-1,-1)}	1 or 2

Tab. 4.1: Action of group elements on sign-pairs.

Now, let us write $\bar{M} = M_{n+p-1}M_{n+p-2}\dots M_n \mod 2$, where

$$M_i = \begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix} . \tag{4.76}$$

From the lemma 6 we obtain immediately that for a block of length p or over,

$$(s^u_{n+p}, s^t_{n+p}) = ((-1)^{r_{n+p-1}}(s^u_n)^{p_{n+m-1}}(s^t_n)^{q_{n+m-1}},\ (-1)^{r_{n+p}}(s^u_n)^{p_{n+m}}(s^t_n)^{q_{n+m}}) \tag{4.77}$$

$$= (-1)^{p_{n+m}}(-(s^u_n)^{p_{n+m-1}}(s^t_n)^{q_{n+m-1}},\ (s^u_n)^{p_{n+m}}(s^t_n)^{q_{n+m}})\,, \tag{4.78}$$

and it is straightforward to obtain the action on a sign-pair over a period as given in the third column of table 4.1.

We are now able to give a criterion for the symmetries of the renormalization strange sets as follows.

Theorem 10. *Let* $(a_k)_{k\in\mathbb{Z}}$*,* $a_{k+p} = a_k$ *have period* $p \geq 1$*, with reduced continued fraction* $\bar{\mathbf{a}} = (\bar{a}_k)_{k\in\mathbb{Z}}$*. Writing* $\bar{M} = M_{n+p-1}M_{n+p-2}\dots M_n \mod 2$*, where*

$$M_i = \begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix}, \tag{4.79}$$

the symmetries of the renormalization strange set are given in terms of $\bar{M}$ *as given by table 4.1. In the case when the number of copies of the fundamental set is one or two, the value of the sign-pair* (s_n^u, s_n^t) *at the start of a block determines the number of copies of the fundamental set.*

To prove the theorem we argue as follows. The sets $\beta_n(\Sigma^{Per})$ have period p, and for $n + kp$ for $k \in \mathbb{Z}$, at the end of a sequence of blocks we will have $M_{n+\ell p-1}M_{n+\ell p-2}\dots M_n \mod 2 = \bar{M}^\ell$ for some $\ell \in \mathbb{N}$. Referring now to the fourth column of table 4.1, we see that the number of copies of the fundamental set is given by the number of elements in an invariant sign-pair set, and therefore the number of copies of the fundamental set is given by fifth column of the table. In the cases when 1 or 2 copies of the fundamental set are possible, the precise number is determined by the sign-pair corresponding to initial condition for the generalized Harper equation.

We illustrate these ideas with the period-1 examples. For $p = 1$ and $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$, $a_k = a$, we have $\bar{M} = A$, if a is even, and B, if a is odd. From the fifth column of table 4.1, we see that there are 1 or 2 copies of the fundamental set if a is even and 3 copies if a is odd. This accords with the conclusions of [46].

4.3 Definition and properties of the renormalization strange set

4.3.1 Definition of the renormalization strange set

We are now in a position to define the renormalization strange set. This we do using the maps β_n and the sign-pair dynamics discussed above.

Let $\mathbf{a} \in \mathbb{N}^{\mathbb{Z}}$ be periodic, with associated bi-infinite sequence $\omega = (\omega_k)_{k\in\mathbb{Z}}$. For a given $\mathbf{c} \in \Sigma^{Per}$, we may choose an initial sign-pair $(s_0^u, s_0^t) \in \{+1, -1\}^2$ and then define, for each $k \in \mathbb{Z}$, a unique sign-pair $(s_k^u, s_k^t) \in \{+1, -1\}^2$ such that for all $L_{b_k}(s_k^u, s_k^t) = (s_{k+1}^u, s_{k+1}^t)$. Here, as before, $\mathbf{b} = \mathbf{c} + \tilde{\mathbf{c}}$. We may now define strange sets for each $n \in \mathbb{Z}$ as follows.

Theorem 11. *There exists a sequence $\mathcal{O} = (\mathcal{O}_k)_{k\in\mathbb{Z}}$, $\mathcal{O}_k \subseteq \mathcal{F}_k$, such that for all $n \in \mathbb{Z}$, $R_n(\mathcal{O}_n) = \mathcal{O}_{n+1}$. The sets $\mathcal{O}_n$ consists of images $(s_n^u, s_n^t)\beta_n(\mathbf{c})$ as $\mathbf{c}$ ranges over Σ^{Per} and $(s_n^u, s_n^t) \in \{+1, -1\}^2$ is a sign-pair depending on $\mathbf{c}$.*

Proof. We may then consider a function-pair $(s_0^u, s_0^t)\beta_0(\mathbf{c})$. In view of the discussion in section (4.2.2), not all such function-pairs are members of the renormalization strange set corresponding to the generalized Harper equation. Indeed, whether or not such a function pair is a member of the renormalization strange set is determined by the sign-pair (s_k^u, s_k^t) at the start of a block for the code $\mathbf{c}$. Let us denote by $B_0(c)$ the set of sign-pairs (s_0^u, s_0^t) such that for each start of a block, $(s_k^u, s_k^t) \in \mathcal{S}_k\}$ where $\mathcal{S}_k$ is set of sign-pairs as given in table 4.1. Note that in the case of 1 or 2 copies of the fundamental set, $\mathcal{S}_k$ is not unique and consists of either one or two sign-pairs, precisely which sign-pairs being determined by the group element at k. (See section 4.2.2.) In view of the analysis in section 4.2.2, if these conditions hold for any one k at the start of a block, then they hold at any other k starting a block. Note that it may be that $k = 0$ is not at the beginning of a block for a given $\mathbf{c}$ so it is not convenient to define these sets in terms of the initial sign pair (s_0^u, s_0^t) alone. We may now define the renormalization strange set $\mathcal{O}_0 = \{(s_0^u, s_0^t)\beta_0(\mathbf{c}) \mid (s_0^u, s_0^t) \in B_0(\mathbf{c}),\ \mathbf{c} \in \Sigma^{Per}\}$. Similarly, we may define for $n \in \mathbb{Z}$, $\mathcal{O}_n = \{(s_n^u, s_n^t)\beta_n(\mathbf{c}) \mid (s_0^u, s_0^t) \in B_0(\mathbf{c}),\ \mathbf{c} \in \Sigma^{Per}\}$ (where $\mathcal{O}_0$ corresponds to the case $n = 0$). Again by the analysis of section 4.2.2, the condition that the initial sign pair $(s_0^u, s_0^t) \in B_0(\mathbf{c})$ means that all sign pairs (s_n^u, s_n^t) at the start of a block satisfy the conditions that $(s_n^u, s_n^t) \in \mathcal{S}_n$. Using the properties of the map β_n it is straightforward to show that $R_n(\mathcal{O}_n) \subseteq \mathcal{O}_{n+1}$. □

4.3.2 Convergence to the scaled orchid

Recall that the initial condition corresponding to the generalized Harper equation is

$$u_1(x) = 1\,, \quad t_1(x) = \frac{1 + \alpha\cos(2\pi(-\omega_0 x + \omega_0/2))}{\alpha(1 - \cos(2\pi(-\omega_0 x + \omega_0)))}\,. \tag{4.80}$$

As in [45] we divide up this initial condition into the numerator and denominator and analyse each separately. Specifically, we write

$$u_1^1(x) = 1 \ , \quad t_1^1(x) = \frac{1 + \alpha \cos\left(2\pi(-\omega_0 x + \omega_0/2)\right)}{\alpha/2} \, , \tag{4.81}$$

$$u_1^2(x) = 1 \quad t_1^2(x) = \frac{1}{2(1 - \cos\left(2\pi(-\omega_0 x + \omega_0)\right)} \, . \tag{4.82}$$

Note that from the multiplicative property of R_n we have that $(u_n, t_n) = (u_n^1, t_n^1)(u_n^2, t_n^2)$ where

$$(u_n^1, t_n^1) = R_{n-1} \ldots R_1(u_1^1, t_1^1) \, , \quad (u_n^2, t_n^2) = R_{n-1} \ldots R_1(u_1^2, t_1^2) \, . \tag{4.83}$$

Our first aim is to establish that the pairs (u_1^1, t_1^1), (u_1^2, t_1^2) are invariant under the projection operator.

Lemma 7.

$$P_1(u_1^1, t_1^1) = (u_1^1, t_1^1), \tag{4.84}$$

$$P_1(u_1^2, t_1^2) = (u_1^2, t_1^2). \tag{4.85}$$

Proof. We prove the lemma by direct calculation using the definition of P_n given in equation (4.22). Let us first show that $\Delta_0^1(\log|(u_1^j, t_1^j)|) = 0$ for $j = 1$, 2. We make use of the integral identity

$$\int_a^{a+2\pi/c} \log|1 + b\cos(-cx + d)| dx = \frac{2\pi}{c} \log(|b|/2) \tag{4.86}$$

valid for a, b, c, $d \in \mathbb{R}$ with $|b| \geq 1$, $c > 0$. Setting $a = 1 - \omega_0^{-1}$, $b = -1$,

$c = 2\pi\omega_0$, $d = 2\pi\omega_0$, we obtain

$$\Delta_0^1(\log|(u_1^2, t_1^2)|) = \int_{1-\omega_0^{-1}}^{1} -\log 2 - \log|1 - \cos(2\pi(-\omega_0 x + \omega_0))|dx$$
$$= -\omega_0^{-1}\log 2 - \omega_0^{-1}\log(1/2) = 0\,, \tag{4.87}$$

and, setting $a = 1 - \omega_0^{-1}$, $b = \alpha$, $c = 2\pi\omega_0$, $d = 2\pi\omega_0/2$, we have

$$\Delta_0^1(\log|(u_1^1, t_1^1)|) = \int_{1-\omega_0^{-1}}^{1} -\log(\alpha/2) + \log|1 + \alpha\cos(2\pi(-\omega_0 x + \omega_0/2))|dx$$
$$= -\omega_0^{-1}\log(\alpha/2) + \omega_0^{-1}\log(\alpha/2) = 0\,. \tag{4.88}$$

We further observe that

$$\Delta_1^1(\log|(u_1^j, t_1^j)|) = \int_{1-\omega_0^{-1}}^{1} \frac{d}{dx}\log|t_1^j(x)|dx = 0 \tag{4.89}$$

for $j = 1, 2$, since t_1^j is periodic with period ω^{-1}. In this calculation it is necessary to take account of the logarithmic singularities both within the interval $(1-\omega_0^{-1}, 1)$ and at the endpoints $1-\omega_0^{-1}$ and 1. This completes the proof of the lemma. □

4.3.3 *Analysis of the zero set of the initial condition* $(u_1^1(x), t_1^1(x))$

In order to show that the renormalization operator converges to a strange set (at least for a countable dense subset of periodic points) we need to analyse the zero set of function pairs $(u_1^1(x), t_1^1(x))$. The function $u_1^1(x) = 1$, so clearly has no zeros, and the relevant zeros of the function $t_1^1(x)$ are the solutions of the equation $1+\alpha\cos(2\pi(-\omega_0 x+\omega_0/2))=0$ with x in the range $1-\omega_0^{-1}\le x\le 1$. As in [45], we write $\alpha^{-1}=\cos 2\pi r$, where r is a parameter with $0<r<1/4$ so that $1<\alpha<\infty$, and we seek solutions of the equation $\cos(2\pi(-\omega_0 x+\omega_0/2))=-\cos 2\pi r$. Thus $x=(1-\omega_0^{-1})/2+(\pm r-k)/\omega_0$, with $k\in\mathbb{Z}$.

It is straightforward to show that for $0<r<\min\{(1-\omega_0)/2,1/4\}$, there are precisely two solutions $x_L=1/2-r/\omega_0$ and $x_U=1/2+r/\omega_0$, in the required range with $x_L+x_U=1-\omega_0^{-1}$. Furthermore, for $(1-\omega_0)/2<r<1/4$ (which can only occur when $\omega_0>1/2$) the solution x_U lies outside the required range, but there is one additional solution $x_{U'}=1/2+(r-1)/\omega_0$ satisfying $x_{U'}+x_L=1$.

For $0<r<1/4$, there is a countable dense subset corresponding to those α for which the solutions x_L,x_U or $x_L,x_{U'}$ have periodic codes (as defined above). In both cases the codes are partners, as a consequence of Lemma 1.

Furthermore, for the case $(1-\omega_0)/2<r<1/4$, x_L and $x_{U'}$ correspond to the start of a block (since their sum is 1), whilst for $r<1(1-\omega_0)/2$, x_L and x_U are not at the start of a block since their sum is $1-\omega_0^{-1}$. The precise position in the block depends in general on the values of r and ω_0 and a complete analysis is not attempted here. The position in the block determines the sign-pair that applies at the start of the next block, which in turn determines the number of copies of the fundamental set according to the theory in section 4.2.2.

We may summarize the position as follows:

Lemma 8. *There is a countable dense subset of* $\alpha > 1$ *for which the initial condition* $(u_1^1, t_1^1) = h^1(\mathbf{c})h^1(\tilde{\mathbf{c}})(\hat{u}_1^1, \hat{t}_1^1)$ *with* $\mathbf{c}, \tilde{\mathbf{c}} \in \Sigma^{Per}$*, where*

$$h^1(\mathbf{c}) = (1, x_L - x)\,, \tag{4.90}$$

$$h^1(\tilde{\mathbf{c}}) = \begin{cases} (1, x_{U'} - x), & 1 < \alpha \quad \textit{if} \quad \omega_0 < 1/2 \\ (1, x_{U'} - x), & 1 < \alpha < (\cos 2\pi(1-\omega_0)/2)^{-1} \quad \textit{and} \quad \omega_0 > 1/2 \\ (1, x_U - x), & (\cos 2\pi(1-\omega_0)/2)^{-1} < \alpha \quad \textit{and} \quad \omega_0 > 1/2 \end{cases} \tag{4.91}$$

and $(\hat{u}_1^1, \hat{t}_1^1) \neq 0$ *on* $\bar{V}_1^n \times \bar{V}_0^n$ *and* $(\hat{u}_1^1, \hat{t}_1^1) > 0$ *on* $\bar{V}_1^n \times \bar{V}_0^n \cap \mathbb{R}^2$*.*

4.3.4 Analysis of the zero set of the initial condition (u_1^2, t_1^2)

We now turn to the denominator of the initial condition (u_1, t_1). We shall see that its function is merely to scale the renormalization strange set given by the numerator rather than to alter its fundamental structure. Indeed, due to the multiplicative structure of the renormalization transformation R_n, we may analyse (u_1^2, t_1^2) separately, and its structure is given by the dynamics of its poles, or rather, the zeros of the denominator in the interval I_0. The pole structure is straightforward to determine. Again $u_1^2 = 1$ is without singularities and $t_1^2(x)$ has poles of order 2 at $x = 1$ and $1 - \omega_0^{-1}$. Applying the renormalization transformation $R_{j-1} \ldots R_1$, gives that, for $j \geq 2$, u_j^2 has a pole of order 2 at $-\omega_j^{-1}$ and t_j^2 has a pole of order 2 at 1. A consequence is that the denominator converges to a periodic function of period p, the period of ω_0. Indeed, we have the following result, the proof of which we omit.

Proposition 4. *There exists a period-p function pair (u^*, t^*) which is periodic of period p under the renormalization transformation R_n, such that, under iteration of the renormalization transformation,*

$$R_{j-1} \dots R_1(u_1^2, t_1^2)/R_{j-1} \dots R_1(u^*, t^*) \tag{4.92}$$

converges to $(1, 1)$.

The function pair (u^*, t^*) corresponds to the strong-coupling fixed point in the golden-mean case, and the proof of the proposition is similar to that contained in [45, 9].

4.3.5 Proof of Theorem 12

For a dense set of initial conditions (corresponding to periodic codes) the dynamics under the renormalization transformation converges to a scaled version of the renormalization strange set. Specifically, we have:

Theorem 12. *Let $\omega = \omega_0$ have periodic continued fraction expansion $[\overline{a_1, \dots a_p}]$ with period $p \geq 1$, and let $(\omega_n)_{n\in\mathbb{Z}}$ be the associated p-periodic sequence $\boldsymbol{\omega} = (\omega_k)_{k\in\mathbb{Z}}$ with periodic continued fraction $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$. Then there exist a dense set of α in the range $\alpha > 1$ such that iteration of the associated renormalization operator R_n, for $n \geq 0$, converges to $\mathcal{O}_n^*$, where $\mathcal{O}_n^*$ is the set $\mathcal{O}_n$ scaled by a function-pair (u_n^*, t_n^*) of period a multiple of p.*

Proof. From Lemma 8 there is a countable dense set of $\alpha > 1$ for which (u_1^1, t_1^1) has the structure given by the lemma, where $\mathbf{c}$, $\tilde{\mathbf{c}}$ are partners in

Σ^{Per}. Because the structure of $h^1(\mathbf{c})$ and $h^1(\tilde{\mathbf{c}})$ correspond to (4.33) with c_0 and $\tilde{c}_0 \neq 0$, which may not be the case, we iterate once, and consider $(u_2, t_2) = R_1(u_1, t_1)$. We have $(u_2, t_2)/((u_2^*, t_2^*)\beta_2(\mathbf{c}))$ satisfies the hypothesis of Lemma 4. Here $(u_2^*, t_2^*) = L_{b_1}(+1, +1)R_1(u^*, t^*)$ which is of period a multiple of p. Theorem 3 follows. □

4.4 Reformulation in terms of sequence spaces

In our analysis we construct several objects indexed by an integer n, which it is convenient to regard as 'time'. In order to simplify our expressions, we may reformulate our results in terms of bi-infinite sequences. Recall that we consider a fixed bi-infinite period-p sequence $\mathbf{a}$ of continued-fraction entries, which gives rise to a periodic sequence $\boldsymbol{\omega}= (\omega_n)_{n\in\mathbb{Z}}$. All our constructions are implicitly dependent on $\mathbf{a}$ and, to simplify notation, we often suppress explicit dependence. First, we adopt the convention that for a object X_n indexed by $n \in \mathbb{Z}$, the corresponding symbol without an index (either in bold type face or otherwise) denotes the associated bi-infinite sequence, so that, for example, $X = (X_n)_{n\in\mathbb{Z}}$. Let us also adopt the general convention that σ denotes the left shift map so that, when considering a bi-infinite sequence of objects $X = (X_n)_{n\in\mathbb{Z}}$, the expression X_σ denotes the bi-infinite sequence $(X_{n+1})_{n\in\mathbb{Z}}$. This convention is needed because the renormalization transformations R_n take a function pair $(u_n, t_n) \in \mathcal{F}_n$ to the function pair $(u_{n+1}, t_{n+1}) \in \mathcal{F}_{n+1}$, so that the induced map R on $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}}$ is defined $R : \mathcal{F} \to \mathcal{F}_\sigma$, and, defining $P = (P_n)_{n\in\mathbb{Z}}$, the equation $R_nP_n = P_{n+1}R$ becomes $RP = P_\sigma R$.

We are now in a position to reformulate our results as follows. We construct a map $\mathcal{E} : \Sigma^{Per} \to \mathcal{F}$, $\mathcal{E} = (\mathcal{E}_n)_{n\in\mathbb{Z}},$, such that for all sequences of sign-pairs $(s^u, s^t) = ((s^u_n, s^t_n)_{n\in\mathbb{Z}})$,

$$R((s^u, s^t)\mathcal{E}) = \kappa(s^u, s^t)\mathcal{E}_\sigma \, , \tag{4.93}$$

where, for a given $\mathbf{c} \in \Sigma^{Per}$, $\kappa : \{-1, +1\}^{\mathbb{Z}} \to \{-1, +1\}^{\mathbb{Z}}$ is given by $\kappa_n(s^u, s^t) = \kappa_{c_n}(s^u_n, s^t_n)$. Noting that the partnering operation extends straightforwardly to bi-infinite sequences, we have a map $\beta : \Sigma^{Per} \to \mathcal{F}$ and a map $L : \{-1, +1\}^{\mathbb{Z}} \to \{-1, +1\}^{\mathbb{Z}}$ such that

$$R((s^u, s^t)\beta) = L(s^u, s^t)\beta_\sigma \, , \tag{4.94}$$

where, for $\mathbf{a} \in \Sigma^{Per}$, $L : \{-1, +1\}^{\mathbb{Z}} \to \{-1, +1\}^{\mathbb{Z}}$ is given by $L(s^u, s^t) = L_{b_n}(s^u_n, s^t_n)$ and $\mathbf{b} = \mathbf{c} + \tilde{\mathbf{c}}$.

Let us call a sequence $(s^u, s^t) \in \{-1, +1\}^{\mathbb{Z}}$ *compatible* with a given sequence $\mathbf{c}$ if $L(s^u, s^t) = (s^u, s^t)_\sigma$. Then, it follows from section 4.2.2 above that, letting $\mathbf{c}$ range over Σ^{Per}, the set of compatible sequences so obtained is partitioned into two disjoint sets if $\mathbf{a}$ contains at least one odd entry or into three disjoint sets if $\mathbf{a}$ contains only even entries. Precisely one of these sets contains the sign pair $(+1, +1)$ at the start of blocks, and we denote by $B(\mathbf{c})$ the set of such (s^u, s^t) for a given $\mathbf{c}$. We may now define invariant strange sets by setting $\mathcal{O} = \{(s^u, s^t)\beta(\mathbf{c})\}$, where (s^u, s^t) ranges over the permitted sign pairs $OS(\mathbf{c})$ and $\mathbf{c}$ ranges over Σ^{Per}. Then $\mathcal{O} = (\mathcal{O}_n)_{n\in\mathbb{Z}}$, where $\mathcal{O}_n$ is the renormalization strange set defined above. The set $\mathcal{O}$ satisfies $R(\mathcal{O}) = \mathcal{O}_\sigma$.

In section 5.4 below we give an alternative formulation in which we conjecture a structure theory for all irrational ω.

4.5 Numerical study

Here we study numerically the evolution of a pair of functions $(u(x), t(x))$ iterating over the renormalization operator (3.83). At each iteration a projection is performed in order to remove unstable eigenvectors to ensure convergence. The projection operator, P_n, is given by (4.4). The resulting renormalization strange set $(u_n(x), t_n(x))$, is evaluated at $(0, 0)$ to produce the two-dimensional projections shown.

Below is a summary of results for periodic continued fractions.

4.5.1 The golden mean case, $\omega = [1, 1, 1, \ldots]$.

- This is the orchid discovered numerically by Ketoja and Satija.
- Mestel and Osbaldestin give a description of the orchid in terms of a code space and sign pairs.
- Found to consist of three copies of a fundamental set figure (4.2) which is the embeding of the model space into function pair space.

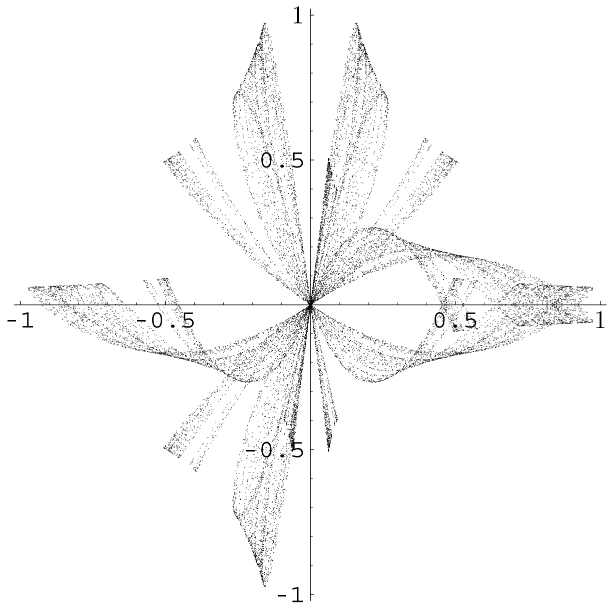


Fig. 4.1: The orchid, $\omega = [1, 1, 1, \ldots]$

4.5.2 Fixed a case $\omega = [a, a, a, \ldots]$.

- For each value of a an orchid like structure is present.
- Sign pairs dictate the transitions between components of these structures.
- It is revealed that for a odd the orchid consists of three copies of a fundamental set. This is shown for the case $a = 1$ where the orchid (figure 4.1) consists of three copies of the fundamental set given in figure 4.2, while for the case $a = 3$ we have figure 4.5 which is made up of three copies of the corresponding fundamental set figure 4.6.
- For a even it is found that the orchid consists of two copies of a fundamental set. This is shown in figure 4.3 which consists of two copies of the fundamental set given in figure 4.4.

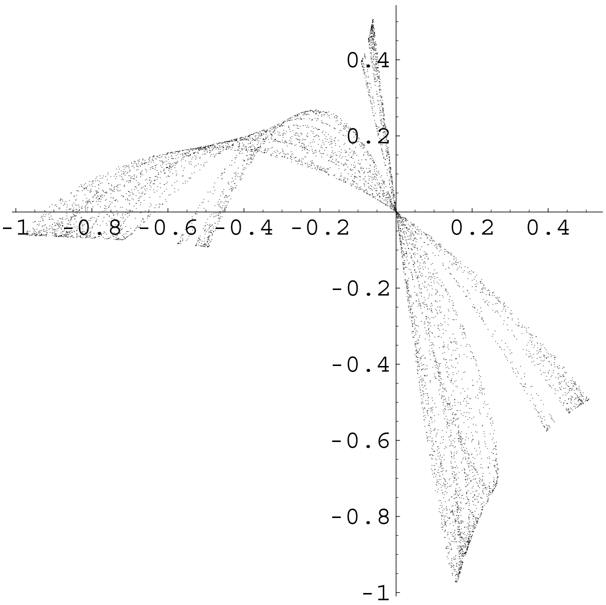


Fig. 4.2: Fundamental set for $\omega = [1, 1, 1, \dots]$

- This example was discussed in section 4.2.2.

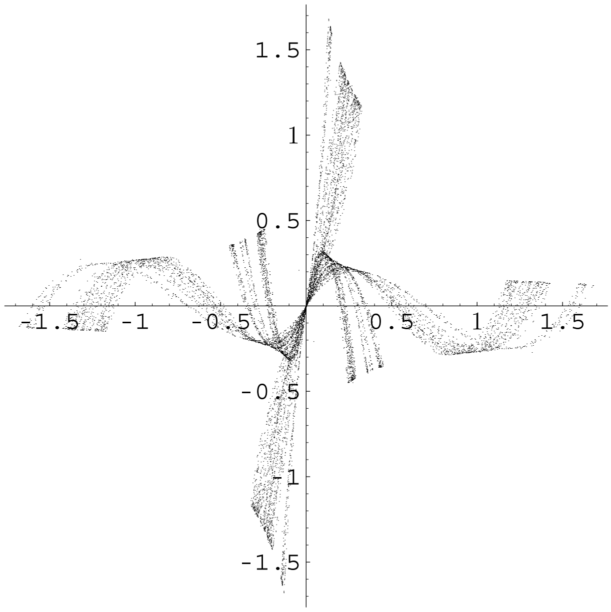


Fig. 4.3: Renormalization strange set for $\omega = [2, 2, 2, \ldots]$.

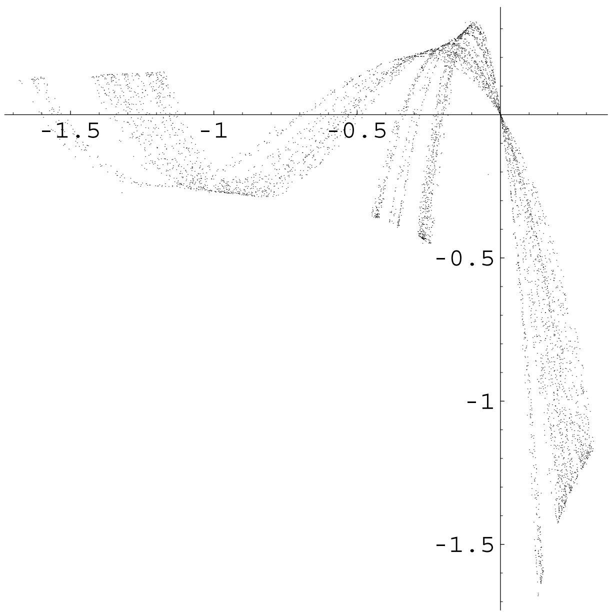


Fig. 4.4: Fundamental set for $\omega = [2, 2, 2, \ldots]$.

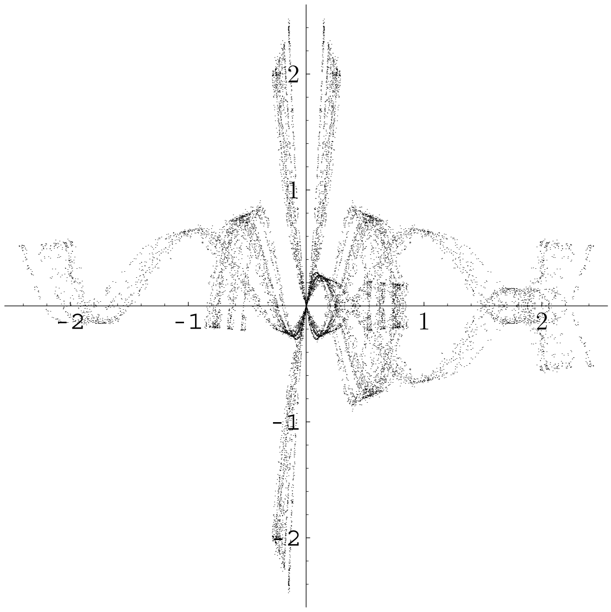


Fig. 4.5: Renormalization strange set for $\omega = [3, 3, 3, \ldots]$.

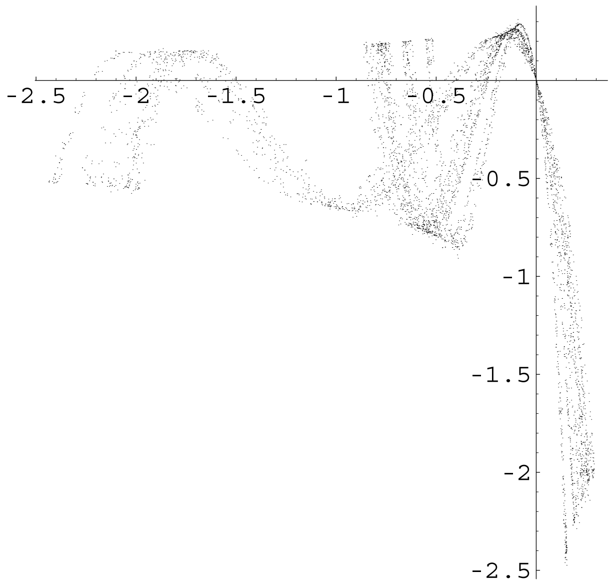


Fig. 4.6: Fundamental set for $\omega = [3, 3, 3, \ldots]$.

4.5.3 Periodic case $\omega = [\overline{a_1, \ldots, a_p}]$, $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$, $a_{k+p} = a_k$

Here we present the numerical results for period 2, 3, 4, and 5 continued fractions consisting of 1's and 2's only. Higher entries in the continued fraction expansion are found to cause greater numerical error.

Taking a closer look at the period 2 case, with continued fraction $\omega = [\overline{1, 2}]$, we find that the period 2 orchid (figure 4.8) consists of two copies of each of the fundamental sets given in figure (4.7) with orientation given by the corresponding sign-pairs. We find the following permitted sign-pairs:

n even	n odd
$(-1, -1)$	$(+1, +1)$
$(+1, -1)$	$(+1, -1)$

Thus for n even we have two copies of the fundamental set for n even corresponding to the sign-pairs $(-1, -1)$ and $(+1, -1)$ and for n odd we have two copies of the fundamental set for n odd corresponding to the sign-pairs $(+1, +1)$ and $(+1, -1)$. These four fundamental sets are combined in figure (4.8).

Recall table 4.1, which, for convenience, we reproduce here.

Now, recalling the theory of section 4.2.2, the following table lists the group element $\bar{M}$ as defined in that section. Note that when $\omega = [\overline{1, 2}]$, we have $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$ with $a_k = 2$, k even, $a_k = 1$ otherwise. Thus

$n \equiv 0$	$n \equiv 1$
$BA = AB^2$	AB

This table is easily seen to correspond to the sign-pairs given above.

Element $\bar{M}$	Order	Action on sign-pair (s^u, s^t)	Invariant sign-pair sets other than $\{(-1,1)\}$.	Number of copies of fundamental set
I	1	(s^u, s^t)	{(1,1)}, {(1,-1)} , {(-1,-1)}	1
B	3	$(-s^t, -s^u s^t)$	{(1,1), (1,-1), (-1,-1)}	3
B^2	3	$(s^u s^t, -s^u)$	{(1,1), (1,-1), (-1,-1)}	3
A	2	$(-s^t, -s^u)$	{(1,1), (-1,-1)}, {(1,-1)}	1 or 2
AB	2	$(s^u s^t, s^t)$	{(1,-1), (-1,-1)}, {(1,1)}	1 or 2
AB^2	2	$(s^u, -s^u s^t)$	{(1,1), (1,-1)}, {(-1,-1)}	1 or 2

Tab. 4.2: Action of group elements on sign-pairs.

The fundamental sets in figure (4.7) were obtained by using the sign-pair $(-1,+1)$ which is invariant under the block to block transitions. The fundamental set in figure (4.7) (left) corresponds to n odd and is given in the period 2 renormalization strange set figure (4.8) with orientation given by the sign-pairs $(+1,+1)$ and $(+1,-1)$. This is easily seen since the transformation $(-1,+1) \to (+1,+1)$ and $(-1,+1) \to (+1,-1)$ are given by a reflection in the x–axis and in both the x and y axes respectively.

Similarly the fundamental set figure (4.7) (right) corresponds to n even and is shown in figure (4.8) with orientations given by sign-pairs $(-1,-1)$ and $(+1,-1)$. As before this is easily seen by noting that the transformations $(-1,+1) \to (-1,-1)$ and $(-1,+1) \to (+1,-1)$ are given by a reflection in the y axis and in both the x and y axes respectively.

We now turn to the case $\omega = [\overline{1,1,2}]$. The renormalization strange set shown in figure (4.9) is a combination of the sets shown in figures (4.10–4.12) which are for $n \equiv 0,\ 1,\ 2 \pmod 3$. Iterating L_{b_n} we find that starting with $n = 0$ with the sign-pair $(+1,+1)$, the following are the permitted sign-pairs:

$n \equiv 0$	$n \equiv 1$	$n \equiv 2$
$(+1,+1)$	$(-1,-1)$	$(+1,-1)$

Note that in this case there is no need for re-orientation of the figures (4.10–4.12) since they were not generated by the invariant sign-pair $(-1,+1)$.

We remark that it is possible to obtain renormalization strange sets with two copies of the fundamental sets by starting the map L_{b_n} at $n = 0$ with the sign-pair $(-1,-1)$ or $(+1,-1)$. In this case the permitted sign-pairs are given by the following table:

$n \equiv 0$	$n \equiv 1$	$n \equiv 2$
$(-1,-1)$	$(+1,+1)$	$(-1,-1)$
$(+1,-1)$	$(1,-1)$	$(+1,+1)$

We note that this accords with the theory of section 4.2.2. For, as above, $\bar{M}$ is given by

$n \equiv 0$	$n \equiv 1$	$n \equiv 2$
$B^2A = AB$	AB^2	$ABA = A$

For $\omega = [\overline{1,1,1,2}]$, we have two copies of the fundamental sets for each of the residual classes $n = 0,\ 1,\ 2,\ 3 \pmod 4$, corresponding to the following permitted sign-pairs:

$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$
$(-1,-1)$	$(+1,+1)$	$(1,-1)$	$(+1,+1)$
$(+1,+1)$	$(-1,-1)$	$(-1,-1)$	$(+1,-1)$

Inspecting the sign-pairs in the above table we have the following symmetries, for $n \equiv 0$ reflection in x and y axes, for $n \equiv 1$ reflection in x and y axes, for $n \equiv 2$ reflection in x axis, and for $n \equiv 3$ reflection in y axis. These symmetries and the order can be clearly seen in figures (4.14–4.17).

This result agrees with the theory in 4.2.2. For we have the following table listing the group element $\bar{M}$ in that section.

$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$
$AB^3 = A$	$B^3A = A$	$B^2AB = AB^2$	$BAB^2 = AB$

which corresponds to the sign-pairs given above.

Similarly for the case $\omega = [\overline{1,1,1,1,2}]$ the period 5 renormalization strange set, figure (4.18), consists of two copies of each of the fundamental sets figures (4.19–4.23) which correspond to each of the residual classes $n \equiv 0,\ 1,\ 2,\ 3,\ 4 \pmod 5$. Iteration of the sign-pair $(+1,-1)$ starting with $n = 0$ gives the permitted sign-pairs in this case to be:

$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$	$n \equiv 4$
$(1,-1)$	$(+1,-1)$	$(+1,+1)$	$(-1,-1)$	$(+1,-1)$
$(+1,+1)$	$(-1,-1)$	$(+1,-1)$	$(+1,+1)$	$(-1,-1)$

with the following symmetries, for $n \equiv 0$ reflection in y axis, for $n \equiv 1$ reflection in x axis , for $n \equiv 2$ reflection in y axis, for $n \equiv 3$ reflection in x and y axes, and for $n \equiv 4$ reflection in the x axis.

Again, this result agrees with the theory in section 4.2.2. For we have the following table listing the group element $\bar{M}$ in that section.

$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$	$n \equiv 4$
$B^4A = AB^2$	$AB^4 = AB$	$BAB^3 = AB^2$	$B^2AB^2 = A$	$B^3AB = AB$

which corresponds to the sign-pairs given above.

Obtaining the renormalization strange sets for higher period becomes difficult numerically due to the build up of error in the projection. Hence the orchid corresponding to period 3, 4, and 5, (figures 4.9, 4.13, and 4.18) become increasingly fuzzy. The reasons for the build up of numerical error are as follows.

- Firstly the evolution of the zeros and the rotation number are governed by the map G_n and the Gauss map γ. These maps are both chaotic which leads to the build up of numerical error.

- The renormalization strange sets are repellors in function-pair space because of the expanding eigenvalues and therefore a projection is needed to prevent divergence. This projection is increasingly hard to calculate as the period of the continued fraction ω increases, this can be seen in appendix F.

- The operator R_n becomes increasingly complex for large a_n which makes the program increasingly inefficient and makes it difficult to take polynomial approximations.

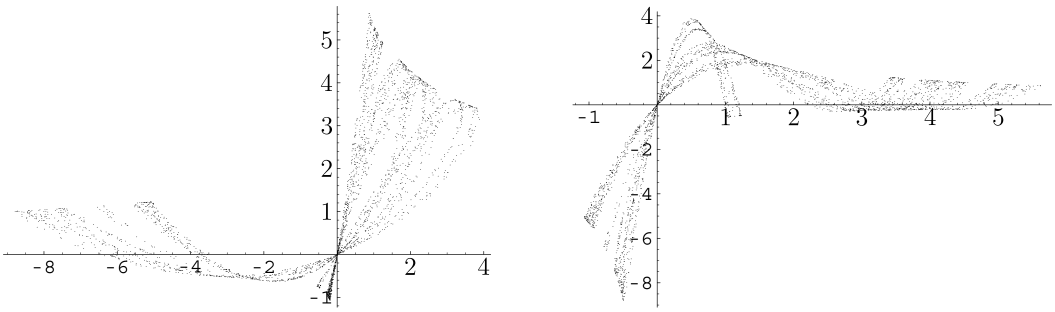


Fig. 4.7: Fundamental sets for $\omega = [\overline{1,2}]$.

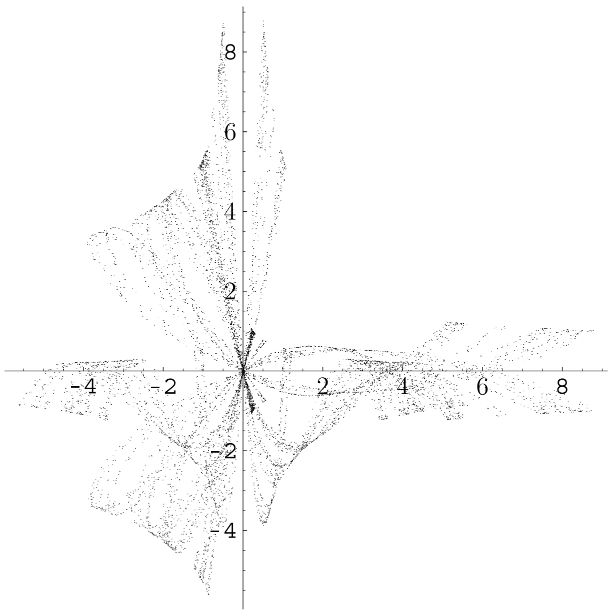


Fig. 4.8: Renormalization strange set for period 2 $\omega = [\overline{1,2}]$.

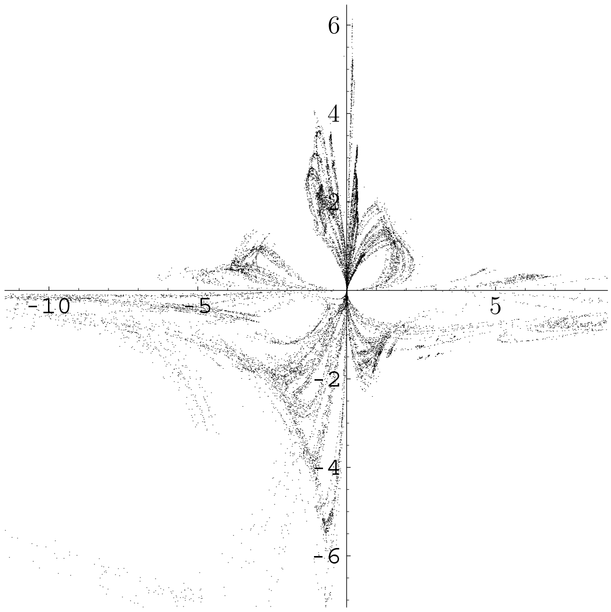


Fig. 4.9: Renormalization strange set for period 3 $\omega = [\overline{1, 1, 2}]$.

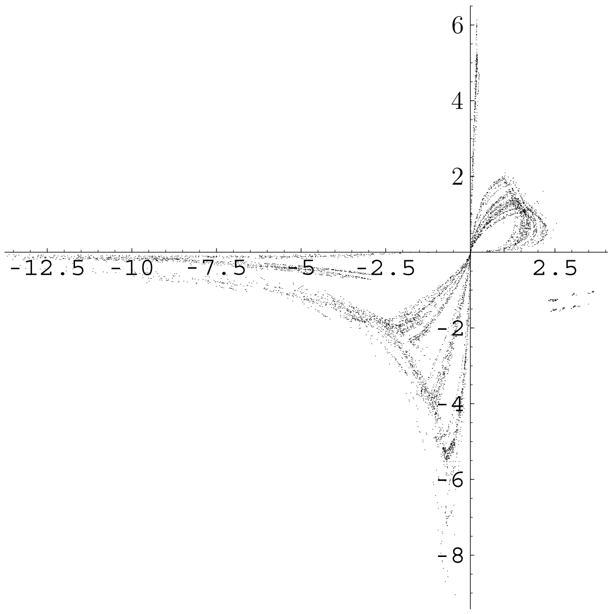


Fig. 4.10: Fundamental set for period 3 $\omega = [\overline{1, 1, 2}]$.

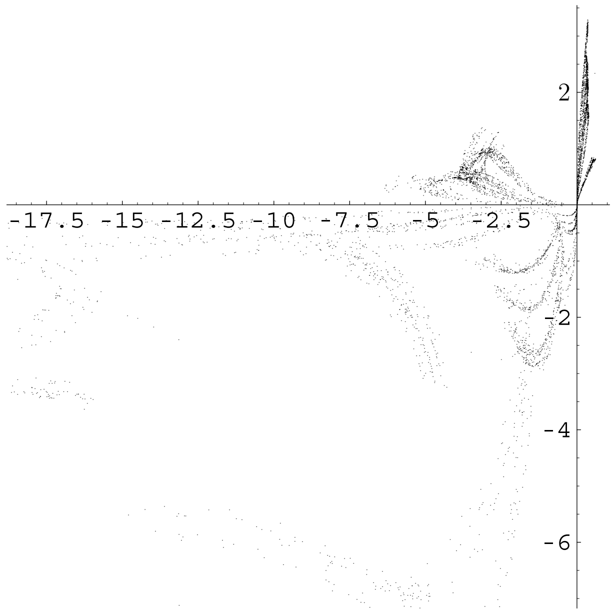


Fig. 4.11: Fundamental set for period 3 $\omega = [\overline{1, 1, 2}]$.

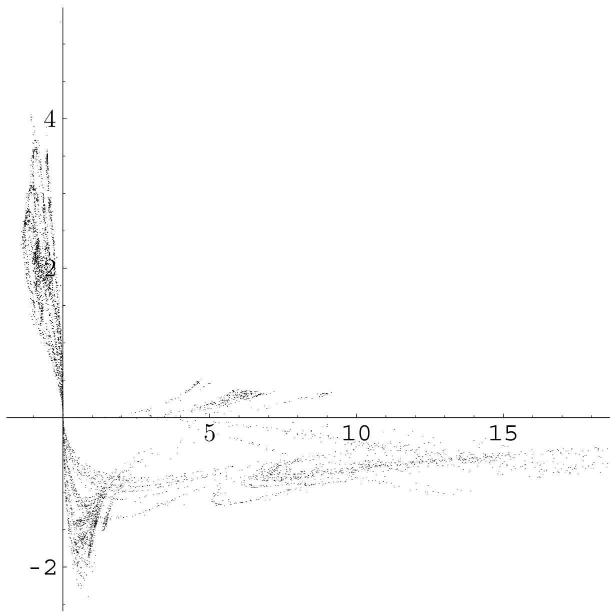


Fig. 4.12: Fundamental set for period 3 $\omega = [\overline{1, 1, 2}]$.

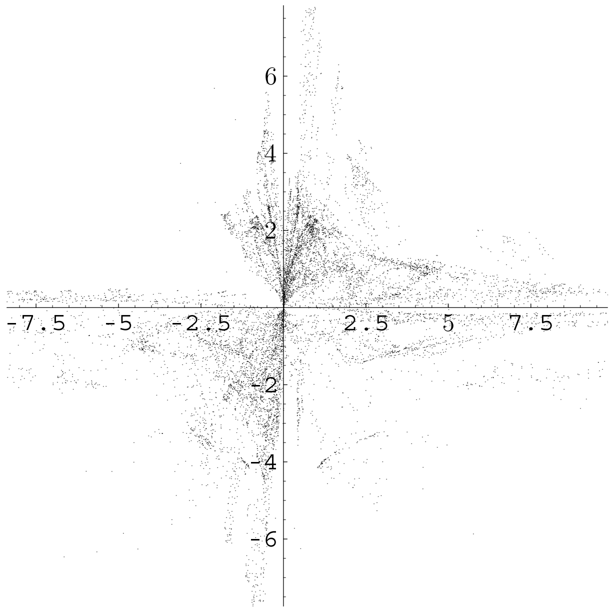


Fig. 4.13: Renormalization strange set for period 4 $\omega = [\overline{1,1,1,2}]$.

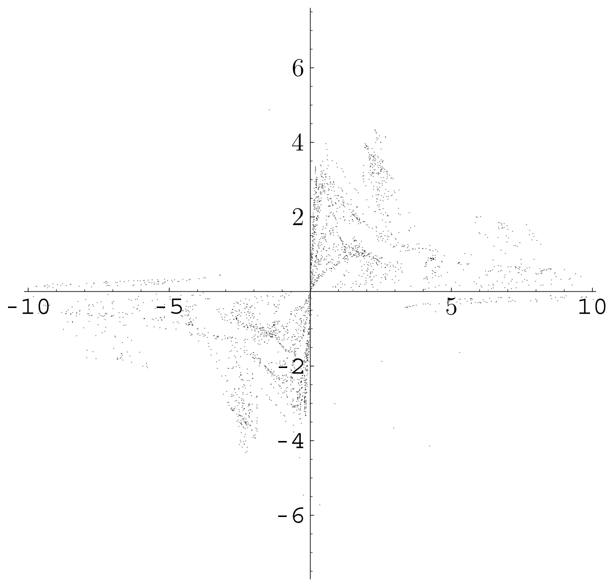


Fig. 4.14: Renormalization strange set, $n \equiv 0$, for period 4 $\omega = [\overline{1,1,1,2}]$.

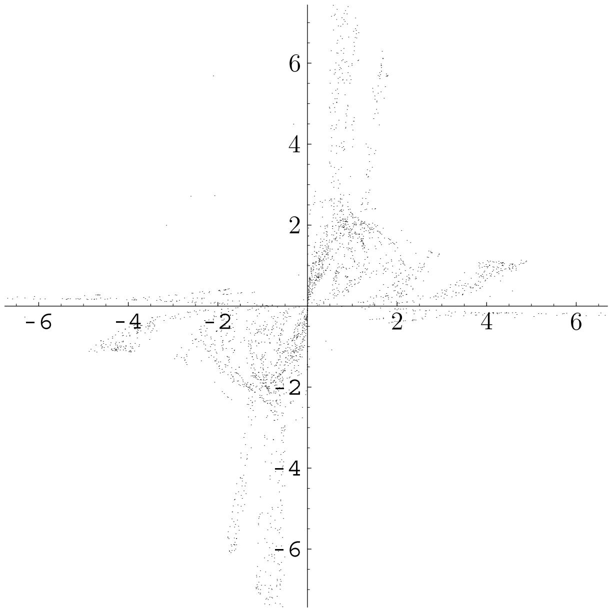


Fig. 4.15: Renormalization strange set, $n \equiv 1$, for period 4 $\omega = [\overline{1, 1, 1, 2}]$.

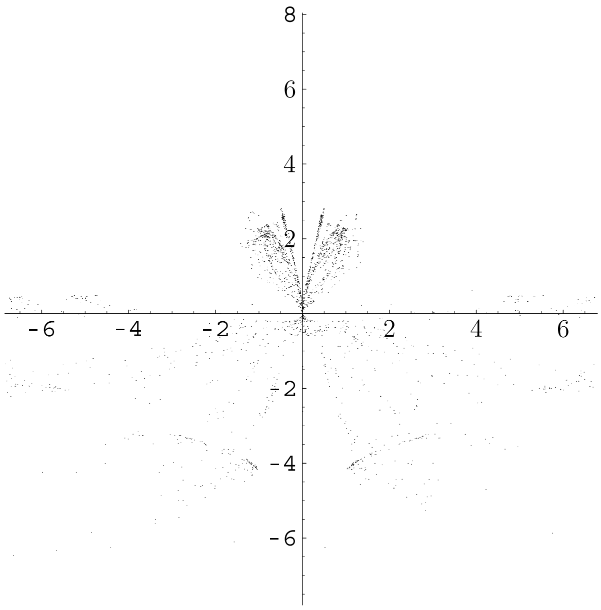


Fig. 4.16: Renormalization strange set, $n \equiv 2$, for period 4 $\omega = [\overline{1, 1, 1, 2}]$.

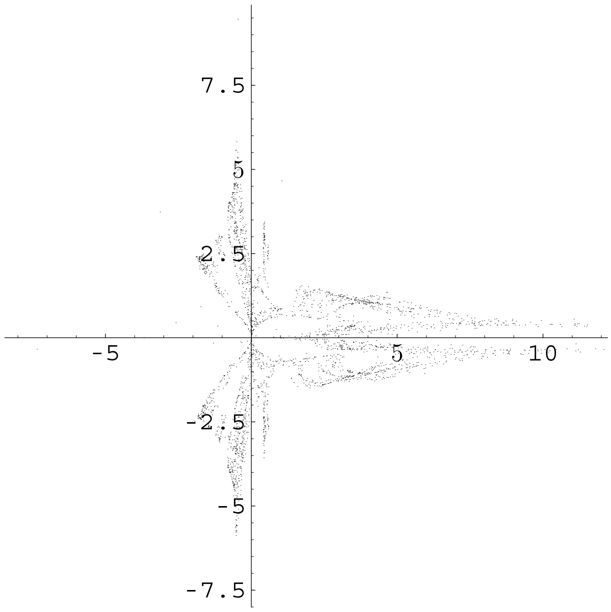


Fig. 4.17: Renormalization strange set, $n \equiv 3$, for period 4 $\omega = [\overline{1,1,1,2}]$.

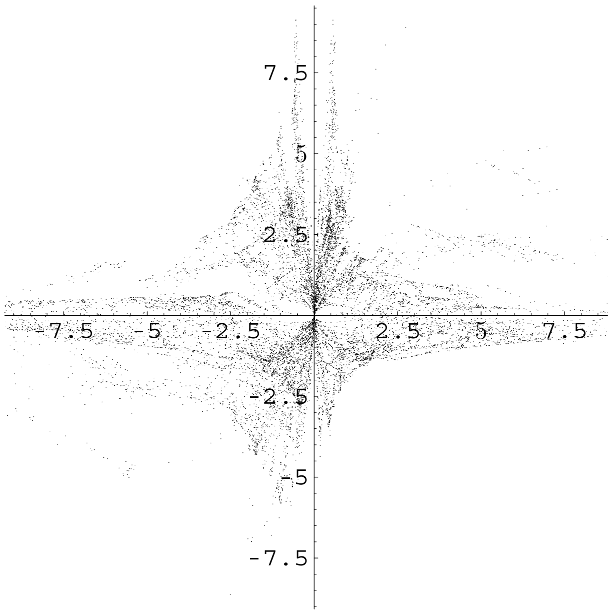


Fig. 4.18: Renormalization strange set for period 5 $\omega = [\overline{1,1,1,1,2}]$.

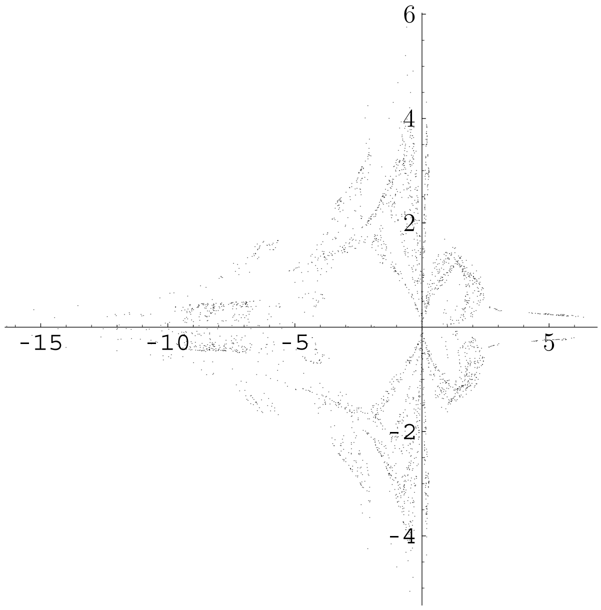


Fig. 4.19: Renormalization strange set, $n \equiv 0$, for period 5 $\omega = [\overline{1, 1, 1, 1, 2}]$.

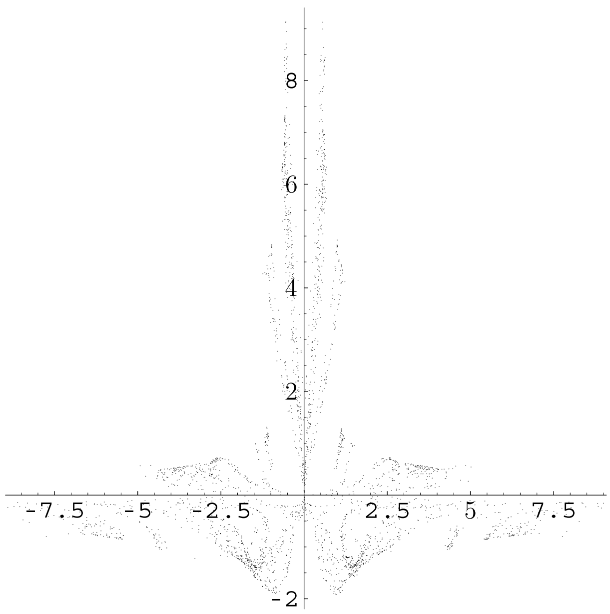


Fig. 4.20: Renormalization strange set, $n \equiv 1$, for period 5 $\omega = [\overline{1, 1, 1, 1, 2}]$.

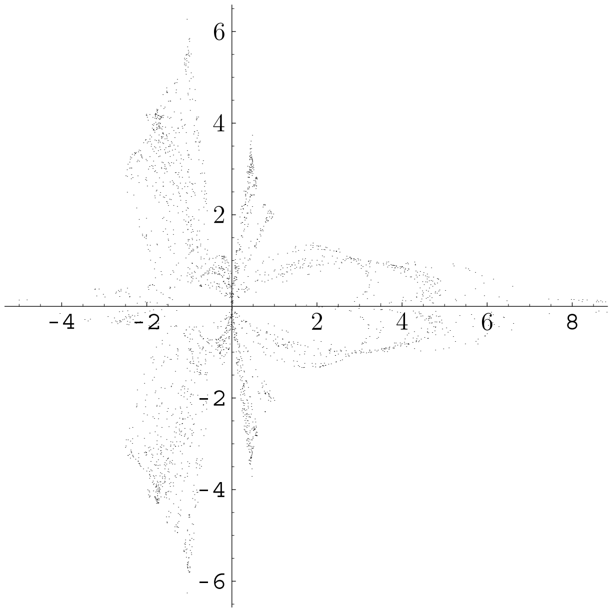


Fig. 4.21: Renormalization strange set, $n \equiv 2$, for period 5 $\omega = [\overline{1, 1, 1, 1, 2}]$.

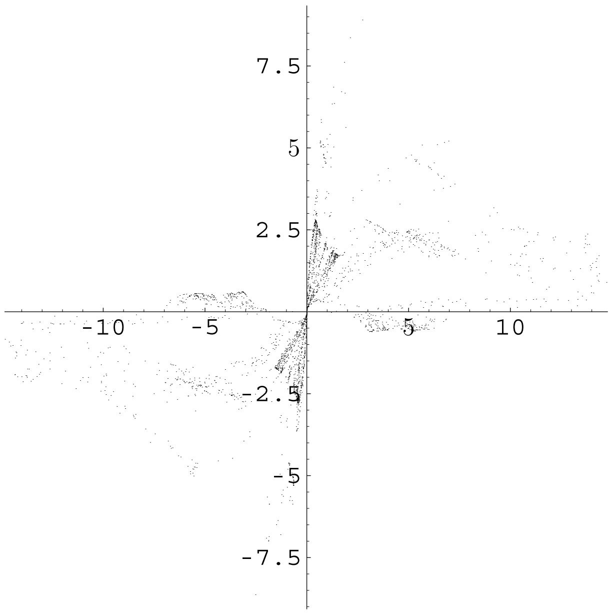


Fig. 4.22: Renormalization strange set, $n \equiv 3$, for period 5 $\omega = [\overline{1, 1, 1, 1, 2}]$.

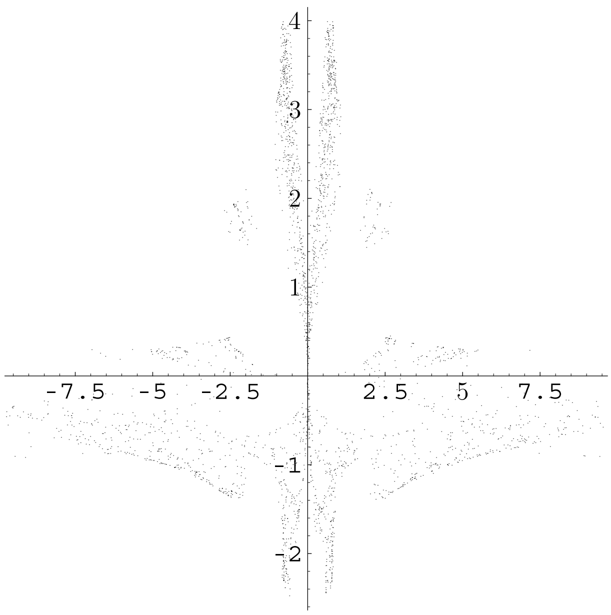


Fig. 4.23: Renormalization strange set, $n \equiv 4$, for period 5 $\omega = [\overline{1, 1, 1, 1, 2}]$.

5. GENERAL CONTINUED FRACTIONS

We now consider general continued fractions and give some conjectures based on work of Mestel and Osbaldestin on a the likely structure of the renormalization strange set.

5.1 *Number theoretical preliminaries*

Let $I_* = (0,1)\backslash\mathbb{Q}$, the irrationals in the unit interval, and let $\gamma : I_* \to I_*$ be the Gauss map,

$$\gamma(x) = \left\{\frac{1}{x}\right\} = \frac{1}{x} - \left[\frac{1}{x}\right] . \tag{5.1}$$

For a general continued fraction $\omega \in (0,1)$, we define $\omega_0 = \omega$ and for $n \geq 1$, $\omega_n = \gamma(\omega_{n-1})$. Setting, for $n \geq 1$,

$$a_n = \left[\frac{1}{\omega_{n-1}}\right] , \tag{5.2}$$

then we have

$$\omega_n = [a_{n+1}, a_{n+2}, a_{n+3}, \ldots] \tag{5.3}$$

with convergents p_n/q_n given by the recurrence relations (A.3).

Definition. *Let* $\mathbf{C} : \mathbb{N}^{\mathbb{Z}} \to I_*^{\mathbb{Z}}$ *be given by*

$$\mathbf{C}(\mathbf{a}) = (\omega_n = [a_{n+1}, a_{n+2}, \ldots])_{n\in\mathbb{Z}}\,, \tag{5.4}$$

where $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$.

The map $\mathbf{C}$ is a continuous.

Extending the Gauss map we have

$$\gamma : I_*^{\mathbb{Z}} \to I_*^{\mathbb{Z}}, \quad \gamma((\omega_n)_{n\in\mathbb{Z}}) = (\gamma(\omega)_n)_{n\in\mathbb{Z}}\,, \tag{5.5}$$

where $\mathbf{C} \circ \sigma = \gamma \circ \mathbf{C}$ so that the following diagram commutes

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{Z}} & \xrightarrow{\sigma} & \mathbb{N}^{\mathbb{Z}} \\ {\scriptstyle\mathbf{C}}\downarrow & & \downarrow{\scriptstyle\mathbf{C}} \\ I_*^{\mathbb{Z}} & \xrightarrow{\gamma} & I_*^{\mathbb{Z}} \end{array}$$

5.2 function spaces

Let $\omega = (\omega_n)_{n\in\mathbb{Z}} \in I_*^{\mathbb{Z}}$. For $c \in \mathbb{C}$ and $r > 0$, let $D(c, r)$ denote the disc centered at c with radius r. Let $V_1^n = D(c_1^n, r_1^n)$ and $V_0^n = D(c_0^n, r_0^n)$ be the discs in $\mathbb{C}$ where

$$c_0^n = 1 - \frac{\omega_{n-1}^{-1}}{2}, \quad r_0^n = \frac{\omega_{n-1}^{-1}}{2}, \quad c_1^n = \frac{1}{2} - \omega_{n-1}^{-1}, \quad r_1^n = \frac{1}{2} \tag{5.6}$$

The function spaces $\mathcal{F}_n$ are defined by:

$$\mathcal{F}_n = \{(u,t) : u : V_1^n \to \mathbb{C}, t : V_0^n \to \mathbb{C}, u,t \text{ real analytic and} \\ \|(u,t)\| = \|u\|_1 + \|t\|_1 < \infty\}, \qquad (5.7)$$

so that we have

$$R_n : \mathcal{F}_n \to \mathcal{F}_{n+1}\,. \qquad (5.8)$$

5.3 Extension to sequence space

Let $\mathbf{a} \in \mathbb{N}^{\mathbb{Z}}$ be fixed, and let $\boldsymbol{\omega} = (\omega_n)_{n\in\mathbb{Z}} = \mathbf{C}(\mathbf{a})$. All the following constructions are either explicitly or implicitly dependent on $\mathbf{a}$. Let $\mathcal{F} = \mathcal{F}(\mathbf{a}) = (\mathcal{F}_n)_{n\in\mathbb{Z}}$, where $\mathcal{F}_n$ is as above. Let $\mathcal{F}^a$ be the extended function-pair sequence space

$$\mathcal{F}^a = \{(\mathbf{a}, \mathbf{h}) \mid \mathbf{h} \in \mathcal{F}(\mathbf{a})\}. \qquad (5.9)$$

We may define an extended renormalization map $\mathbf{R}$. Let

$$(\mathbf{a}, \mathbf{h}) = ((a_n)_{n\in\mathbb{Z}}, (h_n)_{n\in\mathbb{Z}}) \in \mathcal{F}^a. \qquad (5.10)$$

Then

$$\mathbf{R}(\mathbf{a}, \mathbf{h}) = (\sigma(\mathbf{a}), \mathbf{R}(\mathbf{h})) \qquad (5.11)$$

where $\mathbf{R}(\mathbf{h}) = (R_n(h_n))_{n\in\mathbb{Z}}$, and R_n is as is usual. Note that $R_n(h_n) \in \mathcal{F}_{n+1}(\mathbf{a}) = \mathcal{F}_n(\sigma(\mathbf{a}))$.

We can reformulate this map in terms of $\boldsymbol{\omega}$. Let $\mathcal{F}^\omega = \{(\boldsymbol{\omega}, \mathbf{h}) \mid \mathbf{h} \in \mathcal{F}(\boldsymbol{\omega})\}$

where $\mathcal{F}(\boldsymbol{\omega}) = (\mathcal{F}_n)_{n\in\mathbb{Z}}$ and $\mathcal{F}_n$ is the function space with parameters ω_{n-1}. We can define $\mathbf{R} : \mathcal{F}^\omega \to \mathcal{F}^\omega$ by $\mathbf{R}(\boldsymbol{\omega}, \mathbf{h}) = (\gamma(\boldsymbol{\omega}), \mathbf{R}(\mathbf{h}))$, and $\mathbf{R}(\mathbf{h}) = (R_n(h_n))_{n\in\mathbb{Z}}$ where

$$R_n(u_n(x), t_n(x)) = (t_n(-\gamma(\omega_n)x), \left(\prod_{i=0}^{a_n-1} t_n(-\gamma(\omega_n)x - i)\right) u_n(-\gamma(\omega_n)x - a_n)) \tag{5.12}$$

and where $a_n = [\omega_n^{-1}]$.

5.4 Conjectures

Let $\mathbf{a} \in \mathbb{N}^{\mathbb{Z}}$ be fixed. All the following constructions are either explicitly or implicitly dependent on $\mathbf{a}$.

Define the code space

$$\Sigma_M(\mathbf{a}) = \{\mathbf{c} = (c_i)_{i\in\mathbb{Z}} : c_i \in \{0, 1, \ldots, a_i\},\ c_i = a_i \implies c_{i-1} = 0\}, \tag{5.13}$$

and

$$\Sigma_E = \{\{\mathbf{a}\} \times \Sigma_M(\mathbf{a})\}. \tag{5.14}$$

We extend the shift map σ in the obvious way.

For each $a \in \mathbb{N}$ and $b \in \mathbb{N}_0$, we define the map $L_{a,b} : \{-1, +1\}^2 \to \{-1, +1\}^2$ by

$$L_{a,b}(s^u, s^t) = (s^t, (-1)^b (s^t)^{a_n} s^u). \tag{5.15}$$

Let us now consider a biinfinite sequence of sign-pairs: $\mathbf{s} = (s_i^u, s_i^t)_{i\in\mathbb{Z}}$, and, in a slight abuse of notation, we may write, $\mathbf{s} = (\mathbf{s}^u, \mathbf{s}^t)$, where $\mathbf{s}^u = (s_i^u)_{i\in\mathbb{Z}}$ and $\mathbf{s}^t = (s_i^t)_{i\in\mathbb{Z}}$. For a given $(\mathbf{a}, \mathbf{c}) \in \Sigma_E$, we define $\mathbf{L}_{\mathbf{a},\mathbf{c}} : \{-1, +1\}^{\mathbb{Z}} \to \{-1, +1\}^{\mathbb{Z}}$ by

$$\mathbf{L}_{\mathbf{a},\mathbf{c}}(\mathbf{s}) = (L_{a_i,b_i}(s_i^u, s_i^t))_{i\in\mathbb{Z}}\,, \tag{5.16}$$

where $b_i = c_i + \tilde{c}_i \mod 2$ and $\mathbf{s} = (s_i^u, s_i^t)_{i\in\mathbb{Z}}$.

We call a sign-pair sequence $\mathbf{s}$ *compatible* with $(\mathbf{a}, \mathbf{c})$ if $\mathbf{L}_{\mathbf{a},\mathbf{c}}(\mathbf{s}) = \sigma(\mathbf{s})$ where σ is the usual left-shift map.

Let us now define an extended model space

$$\Sigma_E^e = \{(\mathbf{a}, \mathbf{c}, \mathbf{s}) \mid \mathbf{s} \text{ is compatible with } (\mathbf{a}, \mathbf{c})\}, \tag{5.17}$$

and let σ denote the usual shift.

Conjecture 1. *There exists an onto continuous map $\beta : \Sigma_E^e \to \mathcal{F}^a$ such that*

$$R(\beta(\mathbf{a}, \mathbf{c}, \mathbf{s})) = \beta(\sigma(\mathbf{a}, \mathbf{c}, \mathbf{s})) \tag{5.18}$$

so that the following commutative diagram holds.

$$\begin{array}{ccc} \Sigma_E^e & \xrightarrow{\sigma} & \Sigma_E^e \\ \beta\downarrow & & \downarrow\beta \\ \mathcal{F}^a & \xrightarrow{\mathbf{R}} & \mathcal{F}^a \end{array}$$

The map is two-to-one with $\beta(\mathbf{a}, \mathbf{c}, \mathbf{s}) = \beta(\mathbf{a}', \mathbf{c}', \mathbf{s}')$ if and only if $\mathbf{a} = \mathbf{a}'$, $\mathbf{s} = \mathbf{s}'$, and $\mathbf{c} = \mathbf{c}'$ or $\tilde{\mathbf{c}}'$.

The model space Σ_E^e *splits into two disjoint pieces that are invariant under* σ*, viz.,* Σ_E^O *and* Σ_E^B*, i.e., satisfying* $\sigma(\Sigma_E^O) = \Sigma_E^O$ *and* $\sigma(\Sigma_E^B) = \Sigma_E^B$*. The space* Σ_E^O *is the model for the orchid-like sets and corresponds to those sequence-triples* $(\mathbf{a}, \mathbf{c}, \mathbf{s})$ *for which* $(s_i^u, s_i^t) \in \{(+1, +1), (-1, -1), (+1, -1)\}$ *for each* i *at the start of a block (as defined for the partnering operation).* Σ_E^O *consists of the remainder of* Σ_E^B*. Likewise the images* $\beta(\Sigma_E^O)$ *and* $\beta(\Sigma_E^B)$ *are disjoint. The first is a renormalization strange set* $\mathcal{O}$ *corresponding to the orchid-flower for the special case of golden-mean* ω*. We have* $\mathbf{R}(\mathcal{O}) = (\mathcal{O})$ *and* $\mathbf{R}$ *is chaotic on* $\mathcal{O}$*.*

In terms of ω the conjecture becomes as follows:

Conjecture 2. *There exists an onto continuous map* $\beta : \Sigma_E^e \to \mathcal{F}^\omega$ *such that*

$$R(\beta(\mathbf{a}, \mathbf{c}, \mathbf{s})) = \beta(\sigma(\mathbf{a}, \mathbf{c}, \mathbf{s})), \tag{5.19}$$

so that the following commutative diagram holds.

$$\begin{array}{ccc} \Sigma_E^e & \xrightarrow{\sigma} & \Sigma_E^e \\ \beta \downarrow & & \downarrow \beta \\ \mathcal{F}^\omega & \xrightarrow{\mathbf{R}} & \mathcal{F}^\omega \end{array}$$

The map is two-to-one with $\beta(\mathbf{a}, \mathbf{c}, \mathbf{s}) = \beta(\mathbf{a}', \mathbf{c}', \mathbf{s}')$ *if and only if* $\mathbf{a} = \mathbf{a}'$*,* $\mathbf{s} = \mathbf{s}'$*, and* $\mathbf{c} = \mathbf{c}'$ *or* $\tilde{\mathbf{c}}'$*.*

The model space Σ_E^e *splits into two disjoint pieces that are invariant under* σ*, viz.,* Σ_E^O *and* Σ_E^B*, i.e., satisfying* $\sigma(\Sigma_E^O) = \Sigma_E^O$ *and* $\sigma(\Sigma_E^B) = \Sigma_E^B$*. The space* Σ_E^O *is the model for the orchid-like sets and corresponds to those sequence-triples*

$(\mathbf{a}, \mathbf{c}, \mathbf{s})$ *for which* $(s_i^u, s_i^t) \in \{(+1, +1), (-1, -1), (+1, -1)\}$ *for each* i *at the start of a block (as defined for the partnering operation).* Σ_E^O *consists of the remainder of* Σ_E^B*. Likewise the images* $\beta(\Sigma_E^O)$ *and* $\beta(\Sigma_E^B)$ *are disjoint. The first is a renormalization strange set* $\mathcal{O}$ *corresponding to the orchid-flower for the special case of golden-mean* ω*. We have* $\mathbf{R}(\mathcal{O}) = (\mathcal{O})$ *and* $\mathbf{R}$ *is chaotic on* $\mathcal{O}$*.*

6. DISCUSSION AND CONCLUSIONS

In this thesis we have used renormalization methods, based on those of Mestel, Osbaldestin and coworkers, to analyse fluctuations for quasiperiodic systems. We have applied the theory to the fluctuations in a generalized Harper equation in the strong-coupling limit. For quadratic-irrational ω, we have constructed a dense set of points in the renormalization strange set in function-pair space (corresponding to periodic points) and have shown that the corresponding parameter values for the generalized Harper equation result in convergence to the strange set under renormalization. We have also extended the theory for irrational ω having non-periodic continued fractions and in section 5.4 we have given some conjectures based on work of Mestel and Osbaldestin on a the likely structure of the renormalization strange set in this case.

Clearly, there are two directions in which to take this research further. First, our construction for periodic continued fractions is only for periodic codes. The next step would be to consider all codes in the periodic continued fraction case, as was done in [45]. Whilst a non-rigorous theory would be relatively straightforward to formulate, there are considerable technical obstacles to overcome to justify the theory rigorously. These obstacles arise from the possibility of small divisors as the zeros of the functions u_n and t_n approach the boundaries of V_1^n and V_0^n respectively. Secondly, there are other physical and mathematical problems involving similar renormalization operators of a similar type, and an investigation of their properties, and, perhaps a general theory of these type of renormalization operators might usefully be explored.

We conclude with some brief remarks concerning the physical application of the theory that has been developed in this thesis. Models such as the Harper and generalized Harper equations provide not only valuable insight into the phenomena that occur in real systems, but, perhaps somewhat surprisingly, often capture the actual behaviour of real systems more accurately that might be reasonably expected. This remark is especially relevant to the renormalization theory, since systems within the same universality class will exhibit the same universal behaviour.

Strictly speaking, the results considered in this thesis apply only in the strong-coupling limit $\lambda \to \infty$. However, it is likely that the theory also succeeds for large and finite λ. This is known from numerical simulations in the case of golden-mean ω. The restriction to the measure-zero set of quadratic irrationals is also less of a problem, as such numbers frequently organize the wider dynamics. A greater concern, perhaps, is the importance of the symmetry of the cosine potential in the results considered here. A potential having period-1, but without the additional symmetry about the midpoint, would not be in the same universality class as the generalized Harper equation and would converge to a different strange set.

It would certainly be interesting to see whether the results presented here could be observed in a real-life system. Such an observation would present quite a challenge, not least because the fluctuations in the exponentially decaying wave function would need to be obtained from statistical data from repeated experiments.

BIBLIOGRAPHY

[1] D. K. Arrowsmith and C. M. Place, *An introduction to Dynamical Systems*, Cambridge, (1990)

[2] D. K. Arrowsmith and C. M. Place, *Dynamical Systems*, Chapman & Hall, (1992)

[3] S. Aubry and G. André, *Analyticity breaking and Anderson localization in incommensurate lattices*, Ann. Israel Phys. Soc. **3** (1980), 133–164.

[4] T. Bedford, and J. Swift, *New Directions in Dynamical Systems*, Cambridge University Press (1987).

[5] A. Bondeson, E. Ott, and T. M. Antonsen, *Quasiperiodically forced damped pendula and Schrödinger equations with quasiperiodic potentials: implications of their equivalence*, Phys. Rev. B **55** (1985), 2103–2106.

[6] Thanks to A. Burbanks for providing this figure.

[7] J. R. Chapman and A. H. Osbaldestin, *Self-similar correlations in a barrier billiard*, Physica D **180** (2003), 71–91.

[8] F. H. Claro and G. H. Wannier, *Magnetic subband structure of electrons in a hexagonal lattice*, Phys. Rev. B **19** (1979), 6068–6074.

[9] J. Dalton and B. D. Mestel, *Renormalization for the Harper equation for quadratic irrationals*, J. Math. Phys. **44** (2003), 4776–4783.

[10] H. Davenport *The Higher Arithmetic*, Cambridge University Press, (1982).

[11] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems.* Benjamin/ Cummings, Menlo Park, (1985).

[12] D. Domínguez, C. Wiecko, and J. V. José, *Wave-function and resistance scaling for quadratic irrationals in Harper's equation*, Phys. Rev. B **45** (1992), 13919–13929.

[13] U. Feudel, A. Pikovsky, and A. Politi, *Renormalization of correlations and spectra of a strange non-chaotic attractor*, J. Phys. A **29** (1996), 5297–5311.

[14] U. Feudel, A.S. Pikovsky, and M. A. Zaks, *Correlation properties of a quasiperiodically forced two-level system*, Phys. Rev. E **51** (1995), 1762–1769.

[15] R. Gilmore, *Baker-Campbell-Hausdorff Formulas*, J. Math. Phys. **15**, 2090-2092, (1974)

[16] J. H. Han, D. J. Thouless, H. Hiramoto, and M. Kohmoto, *Critical and bicritical properties of Harper's equation with next-nearest-neighbor coupling*, Phys. Rev. B **50** (1994), 11365–11380.

[17] G. H. Hardy, E. M. Wright, *An Introduction to The Theory of Numbers*, Oxford University Press, (1979).

[18] P. G. Harper, *Single band motion of conduction electrons in a uniform magnetic field*, Proc. Phys. Soc. London A **68** (1955), 874–892.

[19] Y. Hatsugai and K. Kohmoto, *Energy spectrum and the quantum Hall effect on the square lattice with next-nearest-neighbor hopping*, Phys. Rev. B **42** (1990), 8282–8294.

[20] M. Hénon *A two-dimensional mapping with a strange attractor*, Communications in Mathematical Physics **50** (1976), 69-77.

[21] *Henon map*, retrieved 01/08/2006 from `http://en.wikipedia.org/wiki/Image:HenonMapImage.png`

[22] M. R. Herman, *Sur la Conjugaisons Differentiable des Diffeomorphismes du Cercle à des rotations*, Publ. I.H.E.S. 49 (1979) 5.

[23] D. R. Hofstadter, *Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*, Phys. Rev. B**14**, (1976), 2239.

[24] *Hofstadter Butterfly*, retrieved 02/08/2006 from `http://physics.technion.ac.il/∼odim/hofstadter.html`

[25] R. A. Holmgren, *A first course in discrete dynamical systems*, Springer, (1996)

[26] S. Ya. Jitomirskaya, *Metal-insulator transition for the almost Mathieu operator*, Ann. Math. **150** (1999), 1159–1175.

[27] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag 1995.

[28] J. A. Ketoja and I. I. Satija, *Renormalization approach to quasiperiodic tight binding models*, Phys. Lett. A **194** (1994), 63–70.

[29] J. A. Ketoja and I. I. Satija, *Renormalization approach to quasiperiodic quantum spin chains*, Physica A **219** (1995), 212–233.

[30] J. A. Ketoja and I. I. Satija, *Self-similarity and localization*, Phys. Rev. Lett. **75** (1995), 2762–2765.

[31] J. A. Ketoja and I. I. Satija, *The re-entrant phase diagram of the generalized Harper equation*, J. Phys. C **9** (1997), 1132.

[32] J. A. Ketoja and I. I. Satija, *Harper equation, the dissipative standard map and strange nonchaotic attractors: Relationship between an eigenvalue problem and iterated maps*, Physica D **109** (1997), 70–80.

[33] D. L. Kreider, R. G. Kuller, D. R. Ostberg, F. W. Perkins, *An introduction to linear analysis*, Addison–Wesley, (1966).

[34] M. Kohmoto, *Metal-insulator transition and scaling for incommensurate systems*, Phys. Rev. Lett. **51** (1983), 1198–1201.

[35] M. Kohmoto, L. P. Kadanoff, and C. Tang, *Localization problem in one-dimension: mapping and escape*, Phys. Rev. Lett. **50** (1983), 1870–1872.

[36] S. P. Kuznetsov, A. S. Pikovsky, and U. Feudel, *Birth of a strange nonchaotic attractor: A renormalization group analysis*, Phys. Rev. E **51** (1995), R1629–R1632.

[37] A. Lesne, *Renormalization Methods: Critical Phenomena, Chaos, Fractal Structures*, Wiley, (1998)

[38] E. N. Lorenz, *Deterministic Nonperiodic Flow*, J. Atmos. Sci. **20**, 130–141, (1963)

[39] *Lorenz attractor*, retrieved 01/08/2006 from `http://en.wikipedia.org/wiki/Image:Lorenz_attractor_yb.svg`

[40] W. D. McComb, *Renormalizaton methods* , Oxford, (2004)

[41] B. D. Mestel, *A computer-assisted proof of universality for cubic critical maps of the circle with golden mean rotation numbers.* PhD thesis, Warwick University, (1985)

[42] B. D. Mestel, A. H. Osbaldestin, and B. Winn, *Golden mean renormalisation for the Harper equation: the strong coupling fixed point*, J. Math. Phys. **41** (2000), 8304–8330.

[43] B. D. Mestel and A. H. Osbaldestin, *Periodic orbits of renormalisation for the correlations of strange nonchaotic attractors*, Math. Phys. EJ **6** (2000), no. 5, 27 pp.

[44] B. D. Mestel and A. H. Osbaldestin, *Renormalization analysis of correlation properties in a quasiperiodically forced two-level system*, J. Math. Phys. **43** no. 7 (2002), 3458–3483.

[45] B. D. Mestel and A. H. Osbaldestin, *Golden mean renormalization for a generalized Harper equation: The Ketoja-Satija orchid*, J. Math. Phys. **45**, no. 12 (2004), 5042–5075.

[46] B. D. Mestel and A. H. Osbaldestin, *A garden of orchids: a generalised*

Harper equation at quadratic irrational frequencies, J. Phys. A: Math. Gen. , **37** (2004), 9071-9086.

[47] B. D. Mestel and A. H. Osbaldestin, *Renormalization of correlations in a quasiperiodically forced two-level system: quadratic irrationals*, J. Phys. A: Math. Gen. **37** (2004) 8969-8989.

[48] C. D. Olds, *Continued Fractions*, Random House, (1963).

[49] A. H. Osbaldestin and B. D. Mestel, *Renormalization in quasiperiodically forced systems*, (in Proceedings of Synchro-2002 – Synchronization of chaotic and stochastic oscillations: applications in physics, chemistry, biology and medicine, Saratov, Russia, Sept. 2002), Fluctuation and Noise Letters, **3**, no. 2 (2003) L251–L258.

[50] Thanks to A. H. Osbaldestin for providing this figure.

[51] S. Ostlund, D. A. Rand, J. Sethna, and E. Sigga, *Universal properties of the transition from quasi-periodicity to chaos in dissipative systems*, Physica D8 (1983) 303–342

[52] S. Ostlund and R. Pandit, *Renormalization-group analysis of the discrete quasiperiodic Schrödinger equation*, Phys. Rev. B **29** (1984), 1394–1414.

[53] S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, and E. D. Siggia, *One-dimensional Schrödinger equation with an almost periodic potential*, Phys. Rev. Lett. **50** (1983), 1873–1876.

[54] H. O. Peitgen, H. Jürgens, D. Saupe, *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag, (1992).

[55] A. M. Rockett, P. Szüsz, *Continued fractions*, World Scientific, (1992).

[56] H. G. Schuster, *Deterministic Chaos*, VCH (1988)

[57] S. J. Shenker, *Scaling behaviour in a map of the circle onto itself: empirical results*, Physica 5D (1982) 405–411

[58] B. Simon, *Schrödinger operators in the twentieth century*, J. Math. Phys. **41** (2000), 3523–3555.

[59] B. Simon, *Schrödinger operators in the twenty-first century.* in Mathematical Physics 2000, A. Fokas, A. Grigoryan, T. Kibble, B. Zegarlinski (eds.), Imperial College Press, London, 2000, 283 – 288.

[60] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817.

[61] J. B. Sokoloff, *Unusual band structure, wave functions and electrical sonductance in crystals with incommensurate periodic potentials*, Physics Reports **126** (1985), 189–244.

[62] H. J. Stöckmann, *Quantum Chaos: An introduction*, Cambridge University Press (1999)

[63] J. H. Stover, *A rate of convergence for a particular estimate of a noise-contaminated chaotic time series*, retrieved 19/10/2006, `http://cybertiggyr.com/gene/ctfs`

[64] C. Tang and M. Kohmoto, *Global scaling properties of the spectrum for a quasiperiodic Schrödinger equation*, Phys. Rev. B **34** (1986), 2041–2044.

[65] D. J. Thouless, *Bandwidths for a quasiperiodic tight-binding model*, Phys. Rev. B **28** (1983), 4272–4276.

[66] Weisstein, Eric W. *Devil's Staircase*, retrieved 15/10/2005 from MathWorld–A Wolfram Web Resource, `http://mathworld.wolfram.com/DevilsStaircase.html`

[67] Weisstein, Eric W. *Lyapunov Characteristic Exponent*, retrieved 16/10/2006 from MathWorld–A Wolfram Web Resource, `http://mathworld.wolfram.com/LyapunovCharacteristicExponent.html`

[68] M. Yampolsky, *Hyperbolicity of renormalization of critical circle maps*, Publ. Math. Inst. Hautes Etudes Sci. No. **96** (2002), 1–41.

APPENDIX

A. CONTINUED FRACTIONS AND NUMBER THEORY

A.1 Introduction to continued fractions

Continued fractions are a way of expressing any real number, $x \in \mathbb{R}$, as a fraction:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \tag{A.1}$$

usually abbreviated by $x = [a_0, a_1, a_2, \dots]$, where $a_0 \in \mathbb{Z}$ and $a_1, a_2, \dots \in \mathbb{N}$. The terms $a_0, a_1, a_2, \dots$ are known as the *partial quotients* and may be finite or infinite in number corresponding to rational or irrational x.

Theorem 13. *x is rational if and only if it can be expressed as a finite continued fraction. x is irrational if and only if its continued fraction expansion is infinite [55].*

A.2 Convergents

The continued fraction expansion $x_0 = [a_1, a_2, a_3, \ldots]$ of a real number has partial quotients a_n and partial remainders x_n calculated by the recursion

$$x_n = a_n + \frac{1}{x_{n+1}} \tag{A.2}$$

where $a_n = \lfloor x_n \rfloor$ is the largest integer less than x_n. The process stops if some $x_n = a_n$ as occurs when x is rational. The convergents $c_n = p_n/q_n = [a_1, a_2, \ldots, a_n]$ are given by

$$p_{n+1} = a_n p_n + p_{n-1} \quad \text{and} \quad q_{n+1} = a_n q_n + q_{n-1}, \tag{A.3}$$

with $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, and $q_0 = 1$.

They are exactly the set of best rational approximations to x, i.e. p/q is a convergent to x if and only if

$$0 \leq |qx - p| < |q'x - p'| \tag{A.4}$$

holds for all integers p, q, p', q' with $0 < q' < q$. The partial remainders x_n satisfy

$$x_{n+1} = \gamma(x_n) \tag{A.5}$$

where $\gamma : (0,1) \mapsto (0,1)$ is the *Gauss map* given by

$$\gamma(x) = \begin{cases} 0 & \text{if } x = 0 \\ \{1/x\} & \text{otherwise} \end{cases} \tag{A.6}$$

where $\{\}$ denotes the fractional part.

Theorem 14. *Any two consecutive convergents satisfy the relation*

$$p_i q_{i-1} - q_i p_{i-1} = (-1)^i, \tag{A.7}$$

where p_i and q_i are defined by the recurrence relations in (A.3).

For example, if we take $i = 2$ then we know from above that $p_1 = a_1,\ q_1 = 1,\ p_2 = a_2 a_1 + 1$, and $q_2 = a_2$, so

$$p_2 q_1 - q_2 p_1 = (a_2 a_1 + 1) - a_2 a_1 = 1 = (-1)^2. \tag{A.8}$$

Theorem 15. *The convergents, c_k, $k < n$, of the continued fraction $x = [a_1, a_2, \ldots]$ are alternately less than and greater than x.*

Proof. Writing (A.7) in the form

$$\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} = \frac{(-1)^i}{q_{i-1} q_i}, \tag{A.9}$$

we see that the right hand side is positive if i is even and negative if i is odd. Since the value of $q_1, q_2, q_3, \ldots$ increases, the difference in (A.9) decreases as i increases. Thus, p_2/q_2 is greater than p_1/q_1, and p_3/q_3 is less than p_2/q_2

but greater than p_1/q_1, etc. Since $p_n/q_n \to x$ as $n \to \infty$ it follows that all the even convergents are greater than x and all the odd convergents are less than x. □

A.3 Quadratic irrationals

The continued fraction, $x = [a_1, a_2, \dots]$, is said to be *eventually periodic* if there exists positive integers N and k such that for all $n \geq N$, $a_n = a_{n+k}$. This can be written

$$x = [a_1, a_2, \dots, a_N, a_{N+1}, \dots, a_{N+k-1}, a_N, a_{N+1}, \dots] \tag{A.10}$$

or abbreviated by

$$x = [a_1, a_2, \dots, \overline{a_N, a_{N+1}, \dots, a_{N+k-1}}]. \tag{A.11}$$

Theorem 16. *(Euler) An irrational solution of a quadratic equation with integer coefficients has an eventually periodic continued fraction.*

Conversely,

Theorem 17. *(Lagrange) Any x with eventually periodic continued fraction expansion satisfies a quadratic equation with integer coefficients.*

Example 1. *The simplest example of this is the continued fraction for the (positive) root of the quadratic equation $x^2 - x - 1 = 0$. Rearranging this gives $x = 1 + \frac{1}{x}$, repeatedly replacing the x on the right-hand-side by its equal* *gives the continued fraction*

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}} \tag{A.12}$$

A.3.1 Reverse periodic continued fractions

Theorem 18. *(Galois) If a quadratic irrational x_1 has the continued fraction expansion $x_1 = [\overline{a_1, \ldots, a_k}]$ then*

$$-\frac{1}{\overline{x}_1} = [\overline{a_k, \ldots, a_1}] \tag{A.13}$$

where $\overline{x}_1$ is the quadratic conjugate to x_1, i.e. the other root of the quadratic equation satisfied by x_1.

A.4 Euclidean algorithm

The Euclidean algorithm is a method for finding the continued fraction expansion, $[a_1, a_2, a_3 \ldots]$, of a number x. Initially a_1 is set equal to $\lfloor x \rfloor$, the integer part of x, this may be positive, negative, or zero, and x_1 is set equal to $\{x\}$, the fractional part of x. Thus,

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \tag{A.14}$$

$$= \frac{1}{a_1 + x_1}, \tag{A.15}$$

so that $x_1 = [a_2, a_3, a_4, \ldots]$. In the next step a_2 is set equal to $\lfloor x_1^{-1} \rfloor$, the integer part of the reciprocal of x_1, and x_2 equal to the fractional part, $\{x_1^{-1}\}$. This process is repeated so that

$$a_{i+1} = \lfloor x_i^{-1} \rfloor \quad \text{and} \quad x_{i+1} = \{x_i^{-1}\}, \quad i = 0, 1, 2, \ldots . \tag{A.16}$$

Note that a_k is positive for $k = 1, 2, \ldots$, and that since x_k is the fractional part, each x_k is in the interval $(0, 1)$. This process gives a unique continued fraction for each real number x, and the process terminates if and only if x is rational.

This algorithm is related to the Euclidean algorithm for finding the greatest common divisor of two integers m and n [48]. If this method is used to find the continued fraction of m/n, then the quotients from the Euclidean algorithm are the integers, $a_0, a_1, a_2, \ldots$, that make up the continued fraction, and the last nonzero remainder from the Euclidean algorithm is the greatest common divisor of m and n.

As an illustration we calculate the first entries of the continued fraction for π, for which there is no known general formula. We have, $x_0 = \pi = 3.14159\ldots$,

$$x_0^{-1} = 0.31830\ldots \qquad a_1 = 0 \qquad x_1 = 0.31830\ldots \tag{A.17}$$

$$x_1^{-1} = 3.14159\ldots \qquad a_2 = 3 \qquad x_2 = 0.14159\ldots \tag{A.18}$$

$$x_2^{-1} = 7.06251\ldots \qquad a_3 = 7 \qquad x_3 = 0.06251\ldots \tag{A.19}$$

$$x_3^{-1} = 15.99659\ldots \qquad a_4 = 15 \qquad x_4 = 0.99659\ldots \tag{A.20}$$

$$x_4^{-1} = 1.00341\ldots \qquad a_5 = 1 \qquad x_5 = 0.00341\ldots \tag{A.21}$$

$$x_5^{-1} = 292.63459\ldots \qquad a_6 = 292 \qquad x_6 = 0.63459\ldots \tag{A.22}$$

so the first entries of the continued fraction for π are given by $\pi = [0; 3, 7, 15, 1, 292, \ldots]$.

A.5 The Gauss map

The Gauss map $\gamma : (0,1) \to (0,1)$, defined in equation (A.6), is an example of a chaotic discrete dynamical system. In terms of the Gauss map γ, the above algorithm (A.16) becomes

$$x_{i+1} = \{x_i^{-1}\} = \gamma(x_i) \tag{A.23}$$

$$a_{i+1} = \lfloor x_i^{-1} \rfloor, \qquad i = 0, 1, 2, \ldots. \tag{A.24}$$

Thus the continued fraction $x_k = [x_{k+1}, x_{k+2}, x_{k+3}, \ldots]$ is generated as a byproduct of the iteration of the Gauss map.

Taking the continued fraction, $x_0 = [a_1, a_2, a_3, \ldots]$, we see that $\gamma(x_0) = x_1 = [a_2, a_3, a_4, \ldots]$, $\gamma(x_1) = x_2 = [a_3, a_4, a_5, \ldots]$, $\gamma(x_2) = x_3 = [a_4, a_5, a_6, \ldots]$, and so on. This relates the Gauss map to the shift map discussed in appendix C.

B. DYNAMICAL SYSTEMS

B.1 Continuous dynamical systems

A continuous dynamical system is a set of first order differential equations

$$\frac{dx}{dt} = F(x,t), \quad x \in X \tag{B.1}$$

the system is said to be *autonomous* if it does not vary with time, i.e. $F_n = F_n(x)$.

The set of curves given by the solutions $x_n(t)$ $n = 1, 2, \ldots$ are known as *trajectories* or *orbit*.

B.2 Discrete dynamical systems

A discrete dynamical system is described by a function $f : X \to X$ and its iterates. The dynamics of the system is given by the behaviour of the points $x \in X$ under iteration, i.e. the properties of the sequence $x, f(x), f^2(x) = f(f(x)), f^3(x) = f(f(f(x))) \ldots$ and the limit $\lim_{n\to\infty} |f^n(x)|$.

B.3 Invariant sets

If the orbit of a point x_t remains within a particular region of phase space for all $t \in \mathbb{R}$ then it constitutes an *invariant set*. The following definition is given in [1].

Definition. *A set $\Lambda \subseteq M$ is said to be invariant under f if $f^m(x) \in \Lambda$ for each $x \in \Lambda$ and all $m \in M$.*

B.3.1 Fixed points

A fixed point of a function f is one which remains under iteration of f.

Definition. *If f is a function and $f(c) = c$ then c is a fixed point of f.*

B.3.2 Periodic points

Definition. *If f is a function and $f^k(x) = x$, and $f^n(x) \neq x$ for $0 < n < k$, then x is a periodic point of f with period k.*

Definition. *If x is a periodic point of f with period k then the iterates $x, f(x), f^2(x), \ldots, f^{k-1}(x)$ are called a periodic orbit.*

Definition. *The point x is an eventually periodic point of f with period k if there exists N such that $f^{n+k}(x) = f^n(x)$ whenever $n \geq N$.*

Example 2. *Let $f(x) = |x - 1|$ then f has a periodic orbit $\{0, 1\}$ since $f(0) = 1$ and $f^2(0) = f(1) = 0$. For any integer starting value, this map is eventually periodic ending in the periodic orbit $\{0, 1\}$.*

Fixed points and periodic orbits are examples of invariant sets which are periodic. A more complex example of an invariant set is Smale's horseshoe [60]

B.3.3 Horseshoe map

Smale's horseshoe is a diffeomorphism of the plane $h : S \to S$ where S is a square capped by two semi discs (figure B.1). The action of the map is to contract the square vertically and elongate it horizontally to make a thin strip which is then bent into a horseshoe shape and placed back onto S (figure B.2). The curved part of the horseshoe overlaps the square, and under iteration the points in this region have orbits which are attracted to a fixed point outside the square. Most orbits leave the square under iteration, the points which remain form a fractal invariant set.

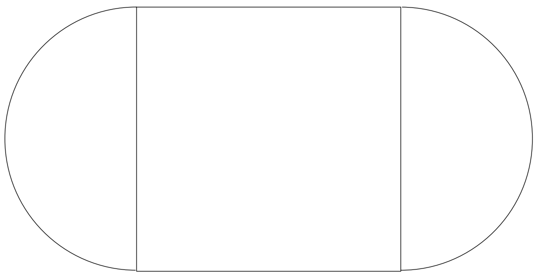


Fig. B.1: Original square S_0 capped by two semi discs.

Concentrating on the square, under forward iteration the original square is mapped to two horizontal strips as in figure B.3. Let $H_n = h^n(S_0) \cap S_0$ be the two horizontal strips created at time n where S_0 is the original square. Under backward iteration it is revealed that these strips H_n come from vertical strips V_n of the original square where $V_n = h^{-n}(H_n)$. For a point to remain in the

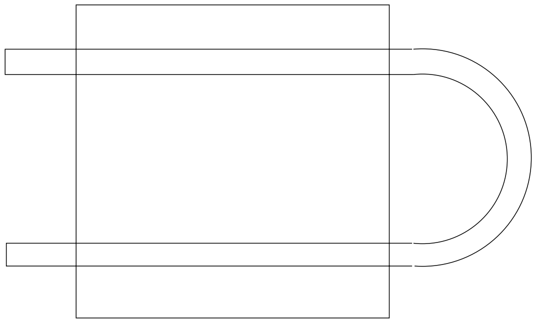


Fig. B.2: Smale's horseshoe.

square indefinitely it must belong to a set Λ which maps to itself. The squares where the horizontal and vertical stripes intersect $H_n \cap V_n$ converge to this invariant set,

$$\lim_{n\to\infty} (H_n \cap V_n) = \Lambda. \tag{B.2}$$

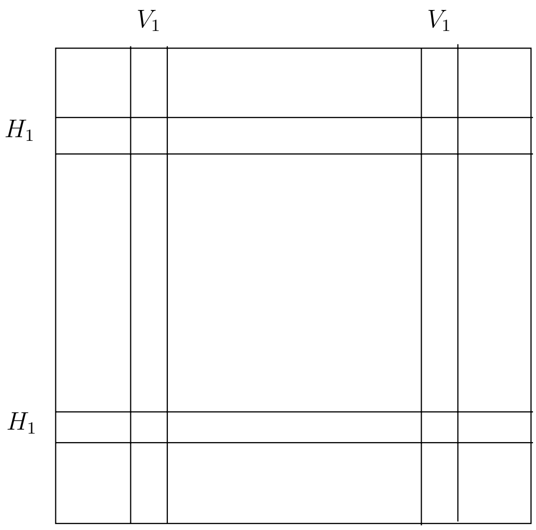


Fig. B.3: Illustration of $H_1 \cap V_1$.

B.3.4 Solenoid attractor

Another example of an invariant set is the solenoid attractor. Consider the map on the solid torus given by

$$F_\beta(\theta, x) = (2\theta, \beta x + \frac{1}{2}e^{i\theta}) \tag{B.3}$$

The map describes a torus which is cut to get long cylinder. The cylinder is stretched to twice its length while contracting its width by β. The resulting long, thin cylinder is wrapped around itself twice, the ends are rejoined and it is replaced inside the original space.

Iterating the solenoid map n times results in a very long, thin tube that winds around the inside of the torus 2^n times (as shown in figure B.4). Notice $F^n(T)$ is a closed set contained completely inside the interior of $F^{n-1}(T)$. Thus $\cap_{n\geq 0} F^n(T) = \Lambda$. It can be shown that F is a homeomorphism on Λ. Since $\lim_{n\to\infty} F^n(x) \in \Lambda$ for any $x \in T$, Λ is the attractor for F [63].

B.4 Chaos and strange attractors

A dynamical system is said to be chaotic if it has the following properties [11]:

1. it displays sensitive dependence on initial conditions,
2. it is transitive
3. its periodic orbits are dense.

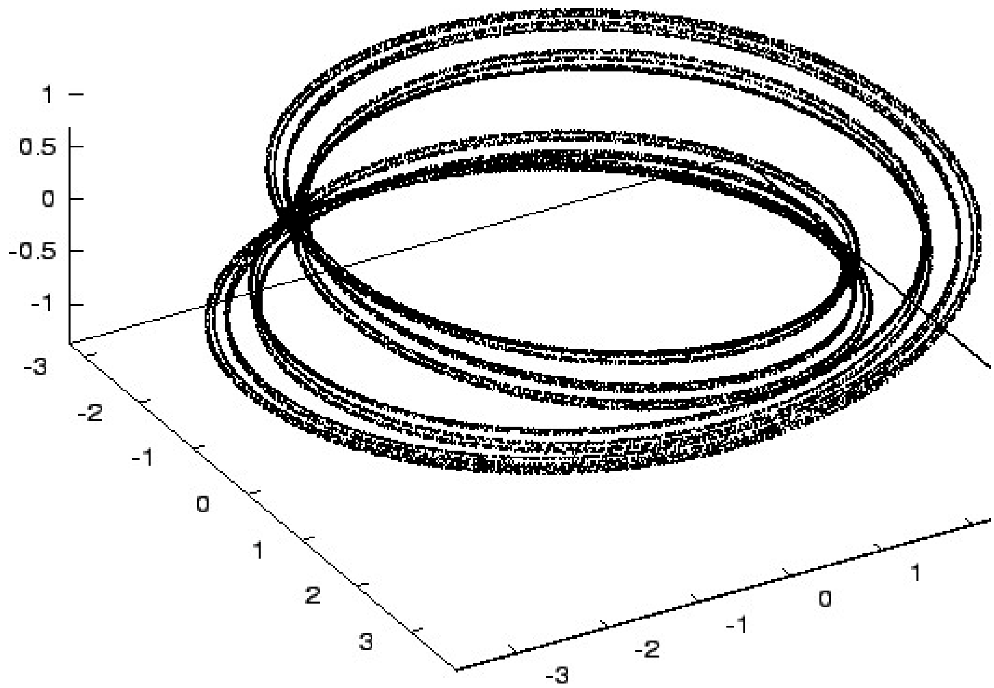


Fig. B.4: Smale's solenoid map [63].

B.4.1 Sensitivity to initial conditions

A map f displays sensitive dependence on initial conditions if points arbitrarily close become separated under iteration.

Definition. *Let X be the phase space for the map f, then f displays sensitive dependence to initial conditions if there is a $\delta > 0$ such that for every point $x \in X$ and any neighborhood N containing x there exist a point y from that neighborhood N and a time τ such that the distance*

$$d(f^{\tau}(x), f^{\tau}(y)) > \delta. \tag{B.4}$$

This means that for each x there are points arbitrarily close to x whose orbits eventually move far away from the orbit of x.

B.4.2 Transitive orbits

Definition. *A dynamical system is* transitive *if for any pair of points* x, y *and any neighbourhood* $U \ni x$ *and* $V \ni y$*, there exists a third point* $z \in U$ *with an orbit* $f^n(z) \in V$ *for some* $n \geq 0$.

If a dynamical system has a dense orbit then it is transitive since this orbit comes arbitrarily close to all points. There is also the following theorem given in [1]

Theorem 19. *A dynamical system is transitive if and only if it has a dense orbit.*

B.4.3 Density of periodic orbits

For a map $f : X \to X$ an orbit is said to be *dense* if the set $\{f^n(x) : n = 1, 2, \ldots\}$ is dense in X.

Definition. *Let* A *be a subset of space* X*.* A *is said to be* dense *in* X *if for any point* $x \in X$*, any neighborhood of* x *contains at least one point from* A*.*

If X *is a metric space then* A *is dense in* X *if every* $x \in X$ *is a limit of a sequence of elements in* A*.*

B.4.4 Strange attractors

The four main properties of a strange attractor according to [54] are:

- There is a trapping region within which all initial points have orbits leading to the attractor,
- the orbits display sensitive dependence on initial conditions,
- it contains a dense orbit, its periodic orbits are dense,
- the attractor has a fractal structure.

A *trapping region*, R, is a region of phase space from which no orbit can escape, each orbit started in R remains in R for all iterations. The *basin of attraction* is the set of all points that have orbits that are eventually caught by this trapping region.

Strange attractors occur in both continuous dynamical systems such as the Lorenz system [38] (see figure B.5) and in discrete systems such as the Hénon map [20] (see figure B.6) .

B.5 Lyapunov exponents

The Lyapunov exponent is used to study the stability of dynamical systems. It gives the rate of exponential divergence from a perturbed initial condition. The larger the Lyapunov exponent, the greater the rate of exponential divergence.

There is whole spectrum of Lyapunov exponents, the number of them is equal to the number of dimensions of the phase space. For an n-dimensional system there are n Lyapunov exponents $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ dominated by the largest one which determines the stability of the system.

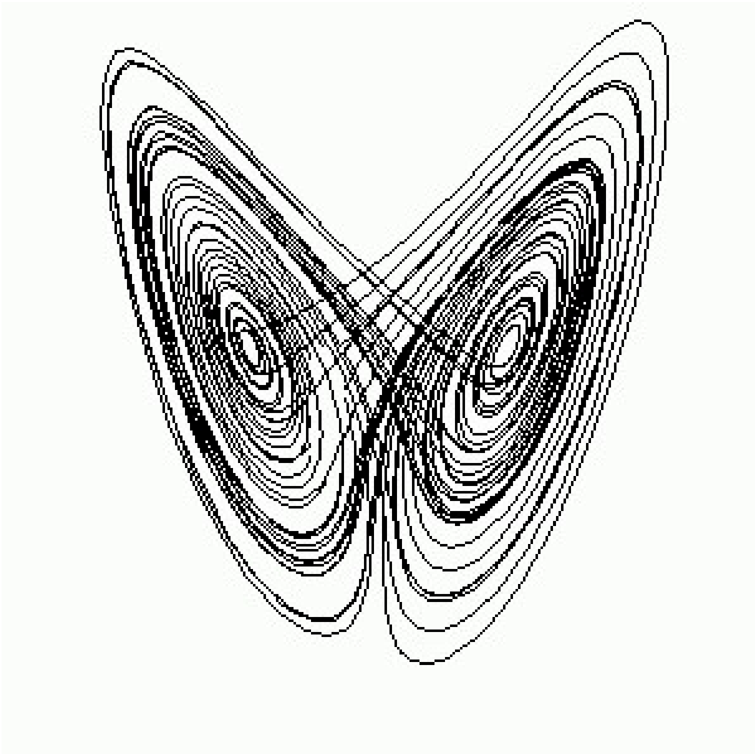


Fig. B.5: Lorenz attractor [39].

Consider the orbit around a point $x^*(t)$ with average perturbation $u(t)$ at time t and let $x(t) = x^*(t) + u(t)$ then for an n-dimensional mapping the Lypanunov exponent is given by [67]

$$\sigma_i = \lim_{N \to \infty} \ln |\lambda_i(N)|, \tag{B.5}$$

for $i = 1, \ldots, n$, where $\lambda_i = \varepsilon^{\sigma_i}$ is the Lyapunov number. For an n-dimensional linear map, $x_{n+1} = Mx_n$, the Lyapunov numbers $\lambda_1, \ldots, \lambda_n$ are given by the eigenvalues of M.

Fig. B.6: Hénon attractor [21].

C. SHIFT SPACES

Shift spaces are used to model dynamical systems, in particular they are the objects of study in symbolic dynamics.

C.1 The shift operator

The shift operator σ maps a sequence $(\ldots, s_n, s_{n+1}, s_{n+2}, \ldots)$ to another sequence, $(\ldots, s_{n+1}, s_{n+2}, s_{n+3}, \ldots)$, by shifting all symbols to the left, i.e. $\sigma(s_n) = s_{n+1}$. If the sequence is infinite in one direction, it is called a one-sided shift and its action is to cut off the first symbol so that $\sigma(s_0, s_1, \ldots) = (0, s_1, s_2, \ldots)$. If σ acts on a bi-infinite sequence, it is called a two-sided shift and $\sigma(\ldots, s_{-1}, s_0, s_1, \ldots) = (\ldots, s_0, s_1, s_2, \ldots)$. Only the two-sided shift is invertible.

C.2 Full shifts

Let $\mathcal{A} = \{0, 1, \ldots, N-1\}$ denote an ordered set of symbols. The phase space Σ_N of this system is the space of all biinfinite sequences of elements from the set of N symbols given by:

$$\Sigma_N = \{s = (s_n)_{n\in\mathbb{Z}} | s_n \in \mathcal{A}\}. \tag{C.1}$$

The *full N-shift* is given by the pair (Σ_N, σ) where $\sigma : \Sigma_N \mapsto \Sigma_N$ is the shift transformation defined above.

The distance between two distinct sequences s and t is given by

$$d(s,t) = \frac{1}{|n|+1} \tag{C.2}$$

where n is the coordinate of smallest absolute value where they differ. Thus if $d(s,t) < 1/n$ for $n > 0$, then $s_k = t_k$ for $-n < k < n$.

C.3 Subshifts

Restricting the shift transformation of a full shift Σ_N to a closed shift-invariant subspace Σ, gives the *subshift* (Σ, σ). A subshift can be any subspace of the full shift that is invariant under the action of the shift operator.

An example of a subshift is the one-sided version of the full N-shift,

$$\Sigma_N^+ = \{s = (s_0, s_1, \dots) | s_n \in \mathcal{A}, n = 0, 1, 2, \dots\}. \tag{2.3.3}$$

On this space the shift transformation σ is similarly defined by $\sigma(s_n) = s_{n+1}$ but now only for non-negative n.

Some subshifts can be characterized by a transition matrix which determines the allowed sequences, these are called *subshifts of finite type*, or *topological Markov shifts.*

C.3.1 Topological Markov shifts

Let $\mathcal{A} = \{0, 1, \ldots, N-1\}$ be a finite set of symbols and let M be an $N \times N$ matrix with entries in $\{0, 1\}$ which determines the allowed transitions. If σ is the shift operator acting on the set X of all allowed sequences then the *subshift of finite type* is defined to be the pair (X, σ).

For a one-side sequence the shift space is given by:

$$\Sigma_M^+ = \{(s_0, s_1, \ldots)|s_n \in A, M_{s_n s_{n+1}} = 1, n \in \mathbb{N}\}. \tag{C.3}$$

This is the space of all allowed sequences, it says that the symbol p can be followed by the symbol q if and only if the $(p,q)^{th}$ entry of the matrix M is 1.

Similarly, for a two-sided shift of finite type the shift space is given by:

$$\Sigma_M = \{(\ldots, s_{-1}, s_0, s_1, \ldots)|s_n \in A, M_{s_n s_{n+1}} = 1, n \in \mathbb{Z}\}. \tag{C.4}$$

Example 3. *The* Fibonacci *shift, so-called because the number of allowed blocks of length* n *are the Fibonacci numbers, has two 1-blocks, three 2-blocks, five 3-blocks,* $\ldots$.

Here the alphabet $A = \{0, 1\}$ *consists of* 0*'s and* 1*'s. The space* Σ_M *is given* *by (C.4) where the allowed transitions are determined by the matrix*

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{C.5}$$

This says that only sequences of 0's and 1's with 1's separated by 0's are allowed.

C.4 Symbolic dynamics

Symbolic dynamics provides a way of obtaining a deeper insight into the nature of chaotic orbits. The dynamics are coded in terms of sequences of symbols. The basic idea is to divide up the set of possible states into a finite number of intervals, and keep track of which interval the state of the system lies in at each time step. Each interval is associated with a symbol, and in this way the evolution of the system is described by an infinite sequence of symbols called a *symbolic trajectory* that reflects the properties of the original dynamical trajectory.

D. SPECTRAL THEORY OF COMPACT LINEAR OPERATORS ON BANACH SPACES

In this section we give a brief synopsis of the Banach space theory that we use in the thesis. Further details may be found in [27], on which this appendix is based.

D.1 Banach spaces

Definition. Normed vector space. *For a real or complex vector space X, a* norm *is any function $\|x\|$, defined for all $x \in X$, which satisfies the following conditions:*

- $\|x\| \geq 0$, $\|x\| = 0$ *if and only if* $x = 0$,
- $\|\alpha x\| = |\alpha|\|x\|$ *(homogeneity),* α *scalar.*
- $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality).*

A norm gives rise to a topology on the vector space X as follows, in a normed vector space X the convergence $x_k \to x$ is defined by $\|x_k - x\| \to 0$ this implies the Cauchy condition $\|x_n - x_m\| \to 0$ is satisfied and $\{x_k\}$ is a Cauchy sequence.

Definition. Cauchy convergence. *A sequence* $\{x_k\}$ *is said to be a Cauchy sequence if for each* $\epsilon > 0$ *there exists an integer* K*, depending on* ϵ*, such that*

$$|x_m - x_n| < \epsilon \tag{D.1}$$

for all $m, n \geq K$ *[33].*

If X is a *finite dimensional* vector space then the Cauchy condition is sufficient for the existence of a limit $x \in X$. For an *infinite dimensional* space X a Cauchy sequence need not have a limit in X. Those spaces X for which every Cauchy sequence has a limit in X are called *complete*.

Definition. *A normed vector space* X *is said to be* complete *if every Cauchy sequence contained in* X *has a limit* $x \in X$*. A complete normed vector space is called a* Banach space.

D.1.1 L^p Spaces

We now give some important examples of Banach spaces which we use in the thesis.

Let $\tilde{X}$ be a Banach space with norm $\|.\|$ and let X be a space of sequences, finite, infinite, or biinfinite, of elements of $\tilde{X}$. For a vector $x \in X$, $x = x_j$, $x_j \in \tilde{X}$, the p–norm is defined for $p \geq 1$ to be

$$\|x\|_p = \left(\sum_j |x_j|^p \right)^{1/p} \tag{D.2}$$

.

The space L^p is defined to be the set of all infinite sequences such that the p–norm is finite i.e. the series $\{|x_j|^p\}$ converges.

The p–norm is defined on the L^p space for $1 \leq p \leq \infty$, where the ∞–norm is given by $\|x\|_\infty = \sup |x_j|$ and

$$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p. \tag{D.3}$$

The L^p spaces are examples of *Banach spaces*.

Especially relevant to this thesis is the case $p = 1$. An example of this are the analytical function spaces defined on discs in $\mathbb{C}$ that we use in equation (3.73).

D.2 Bounded linear operators

A bounded linear operator is a linear transformation L between Banach spaces X and Y for which the ratio of the norm of $L(x)$ to that of x is bounded by the same number, over all non-zero vectors $x \in X$. In other words, there exists some $M > 0$ such that for all $x \in X$,

$$\|L(x)\|_Y \leq M \|x\|_X. \tag{D.4}$$

The smallest such M is called the operator norm of L.

On a Banach space a linear operator is bounded if and only if it is continuous.

D.3 Compact operators

A special class of linear operator is the compact operator.

Definition. *A compact operator is a linear operator L from a Banach space X to another Banach space Y, such that the image under L of any bounded subset of X is a relatively compact subset of Y. Such an operator is necessarily a bounded operator, and so continuous.*

Definition. *A subset $S \subset X$ is said to be* compact *if any sequence of elements of S has a subsequence converging to an element of S [27].*

Let T and A be operators with the same domain space X. Assume that $D(T) \subset D(A)$ and , for any sequence $u_n \in D(T)$ with both u_n and Tu_n bounded, Au_n contains a convergent subsequence. Then A is said to be *relatively compact with respect to T.*

D.4 Spectral values

Let X be a Banach space, then the set of all bounded linear operators on X forms a Banach algebra, $B(X)$. Let $T \in B(X)$ be a bounded linear operator, then the spectrum of T, denoted by $\sigma(T)$, consists of those λ for which $\lambda I - T$ is not invertible in $B(X)$.

If T is a compact operator, then it can be shown that any nonzero λ in the spectrum is an eigenvalue. Therefore $\sigma(T)$ is bounded and consists of discrete eigenvalues plus possibly 0.

Theorem 20. *Let $T \in B(X)$ be compact. $\sigma(T)$ is a countable set with no accumulation point different from zero. Each nonzero $\lambda \in \sigma(T)$ is an eigenvalue of T with finite multiplicity.*

D.4.1 Properties of the spectrum

Definition. *An operator is said to have* discrete spectrum *if it has a finite set or a countable set of eigenvalues. An operator with non-discrete eigenvalues has a* continuous spectrum*.*

The spectrum $\sigma(T)$ of an operator T consists of those λ for which $(T - \lambda I)^{-1}$ does not exist.

$\sigma(T)$ has the following properties:

- $\sigma(T)$ is always compact and non-empty.
- $\sigma(T)$ is bounded and closed, $\sigma(T) \leq B(0, \|T\|)$.
- For $\lambda \in \sigma(T)$, λ is an eigenvalue if there exists a vector $\mathbf{x} \neq 0$ such that $T\mathbf{x} = \lambda\mathbf{x}$, however, not all spectral values are eigenvalues.

Example 4. *Let $\mathbf{v} = \sum_{i=0}^{\infty} f_i x^i = f(x)$, $Tf(x) = 2f(x/2)$, then $\mathbf{v}_n = x^n$ is an eigenvalue with eigenvector $\lambda_n = 2^{-(n+1)} \to 0$ as $n \to \infty$. Therefore $\lambda = 0$ is an example of a spectral value that is not an eigenvalue.*

E. PROOF OF LEMMA 2

Here we give the proof of Lemma 2.

Proof. We have that $R^p_{0,n}$ is analyticity improving on the domains $V^n_1 \times V^n_0$. It is then a standard result that such operators are compact. Indeed the arguments in, for example, [42] may be readily adapted to the case considered here. Let us now give an outline of the calculation of the spectrum. The arguments given here are also analogous to those found in [42, 9].

Recall that

$$R_{0,n}(U(x), T(x)) = (T(\theta^n_0(x)), \sum_{i=0}^{a_n-1} T(\theta^n_i(x)) + U(\theta^n_{a_n}(x))) \,, \qquad \text{(E.1)}$$

and $R^\ell_{0,n} = R_{0,n+\ell-1} R_{0,n+\ell-2} \ldots R_{0,n}$.

We may obtain the spectrum directly by noting that the eigenfunction pairs consist of polynomials. The arguments of [42] may be readily adapted in this case. Letting $U(x) = U_m x^m + U_{m-1} x^{m-1} + \ldots + U_0$ and $T(x) = T_m x^m + T_{m-1} x^{m-1} + \ldots + T_0$ for $m \geq 0$ then

$$R_{0,n}(U,T) = \begin{pmatrix} T_m(-\omega_n x)^m + T_{m-1}(-\omega_n x)^{m-1} + \ldots + T_0 \\ \\ \left(\sum_{i=0}^{a_n-1} T_m(-\omega_n x - i)^m + \ldots + T_0\right) + U_m(-\omega_n x - a_n)^m + \ldots + U_0 \end{pmatrix}. \tag{E.2}$$

Considering the coefficients of the x^m-terms we have

$$R_{0,n} \begin{pmatrix} U_m \\ T_m \end{pmatrix} = M_n \begin{pmatrix} U_m \\ T_m \end{pmatrix}, \tag{E.3}$$

where

$$M_n = (-\omega_n)^m \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}, \tag{E.4}$$

so that

$$R_{0,n}^p \begin{pmatrix} U_m \\ T_m \end{pmatrix} = M_{n+p-1} \begin{pmatrix} U_m \\ T_m \end{pmatrix}, \tag{E.5}$$

where

$$M_{n+p-1} = (-1)^{pm}(\omega_0\omega_1 \ldots \omega_{p-1})^m \begin{pmatrix} 0 & 1 \\ 1 & a_{n+p-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n+p-2} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}. \tag{E.6}$$

Now for $k \in \mathbb{Z}$,

$$\begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix} \begin{pmatrix} 1 \\ -\omega_{k-1} \end{pmatrix} = -\omega_{k-1} \begin{pmatrix} 1 \\ -\omega_k \end{pmatrix}, \tag{E.7}$$

so we see that

$$\begin{pmatrix} 0 & 1 \\ 1 & a_{n+p-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 1 \\ -\omega_{n-1} \end{pmatrix} = (-1)^p \omega_0 \dots \omega_{p-1} \begin{pmatrix} 1 \\ -\omega_{n+p-1} \end{pmatrix} \tag{E.8}$$

gives the eigenvalue $(-1)^p \gamma_0^p$. Similarly,

$$\begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{n+p-1} \end{pmatrix} \begin{pmatrix} \omega_{n+p-1} \\ 1 \end{pmatrix} = \omega_0^{-1} \dots \omega_{p-1}^{-1} \begin{pmatrix} \omega_n \\ 1 \end{pmatrix} \tag{E.9}$$

gives the second eigenvalue $(\gamma_0^p)^{-1}$ since, for all j,

$$\begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix}^t = \begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix}. \tag{E.10}$$

Thus (E.9) also gives the eigenvalues for the product

$$\begin{pmatrix} 0 & 1 \\ 1 & a_{n+p-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}. \tag{E.11}$$

Now from (E.6) we see the eigenvalues of $R^p_{0,n}$ are given by

$$\{(-1)^{p(m+1)}(\gamma_0^p)^{m+1}, (-1)^{pm}(\gamma_0^p)^{m-1}, m = 0, 1, 2, \dots\}. \tag{E.12}$$

Setting $m = 0$ gives the two eigenvalues of $R^p_{0,n}$ outside of the unit disc in $\mathbb{C}$. $\square$

F. CONSTRUCTION OF THE PROJECTION OPERATOR

In this appendix we prove the existence of the projection $P_{0,n}$ given in section 4.1.2 by constructing $\mathbf{v}_0^n$ and $\mathbf{v}_1^n$ explicitly so that they satisfy equations (4.6) and (4.7).

Let $\mathbf{v}_0^n = (A_n, C_n)$, so that,

$$R_{0,n-1}(\mathbf{v}_0^n) = (C_{n-1}, a_{n-1}C_{n-1} + A_{n-1}) \quad \text{(F.1)}$$

$$= \omega_{n-1}^{-1}(A_n, C_n), \quad \text{(F.2)}$$

which implies that

$$A_n = \omega_{n-1}C_{n-1}, \quad \text{(F.3)}$$

and

$$a_{n-1}C_{n-1} + A_{n-1} = \omega_{n-1}^{-1}C_n. \quad \text{(F.4)}$$

Now, the condition (4.8) together with (F.3 - F.4) gives the recursion:

$$\omega_{n-1}^{-1}C_n + \omega_{n-1}C_{n-1} = 1. \quad \text{(F.5)}$$

Similarly, we let $V_1^n = (A_n x + B_n, C_n x + D_n)$, then

$$R_{0,n-1}(\mathbf{v}_1^n) = \begin{pmatrix} C_{n-1}(-\omega_{n-1}x) + D_{n-1} \\ \\ \sum_{j=0}^{a_{n-1}-1} C_{n-1}(-\omega_{n-1}x - j) + D_{n-1} \\ +A_{n-1}(-\omega_{n-1}x - a_{n-1}) + B_{n-1} \end{pmatrix}$$

$$= -(A_n x + B_n, C_n x + D_n) \quad \text{(F.6)}$$

implies that

$$B_n = -D_{n-1}, \quad \text{(F.7)}$$

and

$$D_n = \frac{1}{2}(a_{n-1} - 1)a_{n-1}C_{n-1} + a_{n-1}A_{n-1} - a_{n-1}D_{n-1} + D_{n-2} \quad \text{(F.8)}$$

$$= \Lambda_{n-1} - a_{n-1}D_{n-1} + D_{n-2}, \quad \text{(F.9)}$$

where $\Lambda_n = \frac{1}{2}(a_n - 1)a_n C_n + a_n A_n$.

F.1 Period 1

If ω is a period 1 continued fraction then a is fixed and there is no time dependence, thus we have:

$$A = \omega C, \tag{F.10}$$

$$B = -D, \tag{F.11}$$

$$C = \frac{1}{\omega + \omega^{-1}}, \tag{F.12}$$

and
$$D = \frac{\Lambda}{a} = \frac{1}{2}(a-1)C + A, \tag{F.13}$$

giving,

$$\mathbf{v}_0 = \frac{1}{\omega + \omega^{-1}}(\omega, 1), \tag{F.14}$$

and

$$\mathbf{v}_1 = \frac{1}{\omega + \omega^{-1}} \left(\omega x - \omega - \frac{(a-1)}{2}, x + \omega + \frac{(a-1)}{2} \right), \tag{F.15}$$

as given in [46].

F.2 Period 2

Now for a period 2 continued fraction $a_n = a_{n-2}$, and we obtain

$$A_n = \omega_{n-1} C_{n-1}, \tag{F.16}$$

$$B_n = -D_{n-1}, \tag{F.17}$$

$$C_n = \omega_{n-1} - \omega_{n-1}^2 C_{n-1} = \frac{\omega_{n-1}}{1 + \omega_{n-1}\omega_n}, \tag{F.18}$$

$$D_n = \Lambda_{n-1} - a_{n-1} D_{n-1} + D_n \tag{F.19}$$

$$= \Lambda_n / a_n \tag{F.20}$$

$$= (\frac{1}{2}(a_n - 1) + \omega_n)\frac{\omega_{n-1}}{1 + \omega_{n-1}\omega_n}. \tag{F.21}$$

Thus,

$$\mathbf{v}_0^n = \frac{1}{1 + \omega_{n-1}\omega_n}(\omega_{n-1}\omega_n, \omega_{n-1}), \tag{F.22}$$

and

$$\mathbf{v}_1^n = \frac{1}{1 + \omega_{n-1}\omega_n} \left(\omega_{n-1}\omega_n x - \omega_n \frac{(a_{n-1} - 1)}{2} - \omega_{n-1}\omega_n, \omega_{n-1} x + \omega_{n-1} \frac{(a_n - 1)}{2} - \omega_{n-1}\omega_n \right). \tag{F.23}$$

F.3 Period-p

For a general period-p continued fraction we have that $a_n = a_{n-p}$, this leads us to the following: again $A_n = \omega_{n-1} C_{n-1}$ and $B_n = -D_{n-1}$, but now we have,

$$\begin{aligned} C_n &= \omega_{n-1} - \omega_{n-2}\omega_{n-1}^2 + \ldots + (-1)^{p-1}\omega_{n-p}\omega_{n-1}^2 \ldots \omega_{n-p+1}^2 \\ &\quad +(-1)^p \omega_{n-1}^2 \ldots \omega_{n-p}^2 C_{n-p} \qquad \text{(F.24)} \\ &= \frac{\sum_{j=0}^{p-1} \omega_{n-1-j} \prod_{i=1}^{j}(-\omega_{n-i}^2)}{1 - \prod_{k=1}^{p}(-\omega_{n-k}^2)}. \qquad \text{(F.25)} \end{aligned}$$

.

For D_n we write the recurrence

$$\begin{pmatrix} D_{n-1} \\ D_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -a_{n-1} \end{pmatrix} \begin{pmatrix} D_{n-2} \\ D_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \Lambda_{n-1} \end{pmatrix} \qquad \text{(F.26)}$$

as

$$x_n = M_{n-1} x_{n-1} + b_n \qquad \text{(F.27)}$$

where

$$M_n = \begin{pmatrix} 0 & 1 \\ 1 & -a_n \end{pmatrix} \quad \text{and} \quad M_n^{-1} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \qquad \text{(F.28)}$$

Using the convention $\omega_k = [a_{k+1}, a_{k+2}, \ldots]$ and $\hat{\omega}_k = [a_k, a_{k-1}, \ldots]$ with

$$\hat{\gamma}_s^t = \begin{cases} \hat{\omega}_s \ldots \hat{\omega}_t, & t \geq s; \\ 1, & t < s, \end{cases} \qquad \text{(F.29)}$$

we see that

$$M_n \begin{pmatrix} 1 \\ \omega_{n-1} \end{pmatrix} = \begin{pmatrix} \omega_{n-1} \\ 1 - a_n \omega_{n-1} \end{pmatrix} = \omega_{n-1} \begin{pmatrix} 1 \\ \omega_n \end{pmatrix} \qquad \text{(F.30)}$$

and

$$M_n^{-1}\begin{pmatrix} -\hat{\omega}_n \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - a_n\hat{\omega}_n \\ -\hat{\omega}_n \end{pmatrix} = -\hat{\omega}_n \begin{pmatrix} -\hat{\omega}_{n-1} \\ 1 \end{pmatrix}. \tag{F.31}$$

We use

$$\mathbf{v}_n^1 = \begin{pmatrix} 1 \\ \omega_{n-1} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_n^2 = \begin{pmatrix} -\hat{\omega}_{n-1} \\ 1 \end{pmatrix} \tag{F.32}$$

as a basis at n. Writing

$$x_n = x_n^1\mathbf{v}_n^1 + x_n^2\mathbf{v}_n^2 \tag{F.33}$$

and

$$b_n = b_n^1\mathbf{v}_n^1 + b_n^2\mathbf{v}_n^2, \tag{F.34}$$

equation (F.27) becomes:

$$\begin{aligned} x_n^1\mathbf{v}_n^1 + x_n^2\mathbf{v}_n^2 &= M_{n-1}(x_{n-1}^1\mathbf{v}_{n-1}^1 + x_{n-1}^2\mathbf{v}_{n-1}^2) + b_n^1\mathbf{v}_n^1 + b_n^2\mathbf{v}_n^2 && \text{(F.35)} \\ &= \omega_{n-2}\mathbf{v}_n^1 x_{n-1}^1 - \hat{\omega}_{n-1}^{-1}\mathbf{v}_n^2 x_{n-1}^2 + b_n^1\mathbf{v}_n^1 + b_n^2\mathbf{v}_n^2, && \text{(F.36)} \end{aligned}$$

which gives the two first order recurrences:

$$x_n^1 = \omega_{n-2}x_{n-1}^1 + b_n^1 \tag{F.37}$$

$$x_n^2 = -\hat{\omega}_{n-1}^{-1}x_{n-1}^2 + b_n^2 \tag{F.38}$$

where b_n is given by

$$\begin{pmatrix} 0 \\ \\ \Lambda_{n-1} \end{pmatrix} = b_n^1 \mathbf{v}_n^1 + b_n^2 \mathbf{v}_n^2 = \begin{pmatrix} 1 \\ \\ \omega_{n-1} \end{pmatrix} b_n^1 + \begin{pmatrix} -\hat{\omega}_{n-1} \\ \\ 1 \end{pmatrix} b_n^2, \qquad \text{(F.39)}$$

so that

$$b_n^1 = \frac{\Lambda_{n-1}}{\omega_{n-1} + \hat{\omega}_{n-1}^{-1}}, \quad \text{and} \quad b_n^2 = \frac{\Lambda_{n-1}}{\omega_{n-1}\hat{\omega}_{n-1} + 1}. \qquad \text{(F.40)}$$

The solution of (F.37) is given by:

$$\begin{aligned} x_n^1 &= b_n^1 + \omega_{n-2} b_{n-1}^1 + \omega_{n-2}\omega_{n-3} b_{n-2}^1 + \ldots + \omega_{n-2} \ldots \omega_{n-p-1} b_{n-p}^1 \\ &\quad + \omega_{n-2} \ldots \omega_{n-p} (\omega_{n-p-1} x_{n-p}^1 + b_{n-p+1}^1) && \text{(F.41)} \\ &= \frac{b_n^1 + \ldots + \omega_{n-2} \ldots \omega_{n-p} b_{n-p+1}^1}{1 - \omega_{n-2} \ldots \omega_{n-p-1}} && \text{(F.42)} \\ &= \frac{1}{1 - \gamma_0^p} \sum_{s=0}^{p-1} b_{n-s}^1 \gamma_{n-s-1}^{n-1} && \text{(F.43)} \end{aligned}$$

while the solution of (F.38) is given by:

$$\begin{aligned} x_n^2 &= -\hat{\omega}_n x_{n+1}^2 + \hat{\omega}_n b_{n+1}^2 && \text{(F.44)} \\ &= \hat{\omega}_n b_{n+1}^2 - \hat{\omega}_n \hat{\omega}_{n+1} b_{n+2}^2 + \ldots + (-1)^{p-1} \hat{\omega}_n \ldots \hat{\omega}_{n+p-1} b_{n+p}^2 \\ &\quad + (-1)^p \hat{\omega}_n \ldots \hat{\omega}_{n+p-1} x_{n+p}^2 && \text{(F.45)} \\ &= \frac{\sum_{s=0}^{p-1} (-1)^s b_{n+s+1}^2 \hat{\omega}_n \ldots \hat{\omega}_{n+s}}{1 - (-1)^p \hat{\omega}_n \ldots \hat{\omega}_{n+p-1}} && \text{(F.46)} \\ &= \frac{1}{1 - (-1)^p \hat{\gamma}_1^p} \sum_{s=0}^{p-1} (-1)^s b_{n+s+1}^2 \hat{\gamma}_n^{n+s}. && \text{(F.47)} \end{aligned}$$

Now equation (F.33) becomes

$$\begin{pmatrix} D_{n-1} \\ D_n \end{pmatrix} = x_n^1 \begin{pmatrix} 1 \\ \omega_{n-1} \end{pmatrix} + x_n^2 \begin{pmatrix} -\hat{\omega}_{n-1} \\ 1 \end{pmatrix} \tag{F.48}$$

then

$$D_n = \frac{\omega_{n-1}}{1-\gamma_0^p} \sum_{s=0}^{p-1} \frac{\gamma_{n-s-1}^{n-1} \Lambda_{n-s-1}}{\omega_{n-s-1} + \hat{\omega}_{n-s-1}^{-1}} + \frac{1}{1-(-1)^p \hat{\gamma}_1^p} \sum_{s=0}^{p-1} \frac{(-1)^s \hat{\gamma}_n^{n+s} \Lambda_{n+s}}{\omega_{n+s} \hat{\omega}_{n+s} + 1}. \tag{F.49}$$

Then $\mathbf{v}_0^n = (A_n, C_n)$ and $\mathbf{v}_1^n = (A_n x + B_n, C_n x + D_n)$ are determined for any period p.

We conclude that the spectral radius of $R_{0,n}^p P_n$ is $(\omega_0 \dots \omega_{p-1})$, and further deduce that there exist constants $K > 0$ and $0 < \rho < 1$ such that

$$\|R_{0,n}^\ell\| \le K \rho^\ell \,, \quad \text{for} \ \ \ell \ge 0. \tag{F.50}$$

F.4 General ω

For non-periodic ω equations (F.43) and (F.47) become:

$$x_n^1 = \sum_{s=0}^{\infty} \gamma_{n-s-1}^{n-1} b_{n-s}^1 \tag{F.51}$$

and

$$x_n^2 = \sum_{s=0}^{\infty} (-1)^s \hat{\gamma}_n^{n+s} b_{n+1+s}^2. \tag{F.52}$$

Thus (F.49) becomes:

$$D_n = \omega_{n-1} \sum_{s=0}^{\infty} \frac{\gamma_{n-s-1}^{n-1} \Lambda_{n-s-1}}{\omega_{n-s-1} + \hat{\omega}_{n-s-1}^{-1}} + \sum_{s=0}^{\infty} \frac{(-1)^s \hat{\gamma}_n^{n+s} \Lambda_{n+s}}{\omega_{n+s} \hat{\omega}_{n+s} + 1}. \tag{F.53}$$

For C_n equation (F.25) becomes:

$$C_n = \sum_{j=0}^{\infty} \omega_{n-1-j} \prod_{i=1}^{j} (-\omega_{n-i}^2) \tag{F.54}$$

We also have $A_n = \omega_{n-1} C_{n-1}$ and $B_n = -D_{n-1}$ as before to give $\mathbf{v}_0^n$ and $\mathbf{v}_1^n$ for any ω .

G. PROOF OF LEMMA 5

Here we give the proof of Lemma 5, which states that there exist $\delta > 0$ and $L > 0$ (depending on $\mathbf{a}$ and $\mathbf{c}$) such that the function $H^{n,1} = (u^1_{n+1}, t^1_{n+1}) \in \mathcal{F}^{\delta}_{n+1}$ satisfies

1. $u^1_{n+1}(x) > 0$ for $x \in \overline{V}^{n+1}_1 \cap \mathbb{R}$, and $t^1_{n+1}(x) > 0$ for $x \in \overline{V}^{n+1}_0 \cap \mathbb{R}$;
2. $\|H^{n,1}\| \leq L$.

Proof. If we let $H^{n,1} = \log(u^1_{n+1}, t^1_{n+1})$, then comparing with equations (4.45–4.51) we show that $u^1_{n+1}(x) > 0$ for $x \in \overline{V}^{n+1}_1 \cap \mathbb{R}$, and $t^1_{n+1}(x) > 0$ for $x \in \overline{V}^{n+1}_0 \cap \mathbb{R}$ in the following way: For case (i) and (ii) $u^1_{n+1}(x) = \omega_n > 0$ for all x and it remains to show that $y_n - \theta^n_i(x) > 0$ for $x \in \overline{V}^{n+1}_0 \cap \mathbb{R}$ where (i)$i = 1, \ldots, a_n$, and (ii)$i = 1, \ldots, a_n - 1$. For $x \in \overline{V}^{n+1}_0 \cap \mathbb{R}$ we have

$$\theta^n_i(x) \in [-\omega_n - \omega_n\delta - i, 1 - \omega_n + \omega_n\delta - i], \tag{G.1}$$

so that for (i)

$$\theta^n_i(x) \in [-\omega_n - a_n - \omega_n\delta, -\omega_n + \omega_n\delta], \tag{G.2}$$

and for (ii)

$$\theta_i^n(x) \in [-\omega_n - (a_n - 1) - \omega_n\delta, -\omega_n + \omega_n\delta], \tag{G.3}$$

then $y_n - \theta_i^n(x) > -\omega_n\delta$ in each case since $y_n \in (-\omega_n, 1)$.

Since $(a_k)_{k\in\mathbb{Z}}$ and $\mathbf{c} = (c_k)_{k\in\mathbb{Z}}$ are periodic, we can choose $\delta > 0$ sufficiently small such that $y_n - \theta_i^n(x) > 0$, i.e. $t_{n+1}^1 > 0$ as required.

Now for cases (iii) and (iv), $u_{n+1}^1 = \theta_0^n(x) - y_n$. We need to show that this is greater than zero for $x \in \overline{V}_1^{n+1} \cap \mathbb{R}$. Here we have

$$\theta_0^n(x) = [1 - \omega_n - \omega_n\delta, 1 + \omega_n\delta] \tag{G.4}$$

and $y_n \in (-\omega_n - c_n, -\omega_n - (c_n - 1))$, so we see that $y_n < \theta_0^n(x)$ so that $u_{n+1}^1 > 0$.

Finally,

$$t_{n+1}^1(x) = \omega_n(\prod_{i=0}^{c_n-1} \theta_i^n(x) - y_n)(\prod_{j=c_n+1}^{k} y_n - \theta_j^n(x)), \tag{G.5}$$

where for case (iii) $k = a_n$, so that

$$\theta_j^n(x) \in [-\omega_n - a_n - \omega_n\delta, -\omega_n + \omega_n\delta - c_n], \tag{G.6}$$

and for (iv) $k = a_n - 1$, and

$$\theta_j^n(x) \in [-\omega_n - (a_n - 1) - \omega_n\delta, -\omega_n + \omega_n\delta - c_n], \tag{G.7}$$

for $x \in \overline{V}_0^{n+1} \cap \mathbb{R}$, in either case:

$$y_n - \theta_j^n(x) > -\omega_n \delta, \tag{G.8}$$

then as before we may choose δ such that $y_n - \theta_j^n(x) > 0$. It can also be shown that $\prod_{i=0}^{c_n-1}(\theta_i^n(x) - y_n) > 0$ since

$$\theta_i^n(x) \in [-\omega_n - (c_n - 1) - \omega_n \delta, 1 - \omega_n + \omega_n \delta], \tag{G.9}$$

for $x \in \overline{V}_0^{n+1} \cap \mathbb{R}$, so that $\theta_i^n(x) - y_n > \omega_n \delta$, if we choose δ sufficiently small such that $\theta_i^n(x) - y_n > 0$, thus in case (iii) and (iv) $t_{n+1}^1 > 0$ as required. □

Part 2 of the lemma is proved separately. See Lemma 9, Appendix I.

H. DETAILS OF THE CONSTRUCTION OF $\mathcal{E}$

In this section we give the details of the construction of $\mathcal{E}_n$. Recalling the definitions given in section 4.1.3, we define $\mathcal{E}_n = h^n \exp(K^n)$ where $K^n = K^n(\mathbf{c}) \in \mathcal{F}_n^\delta$, and satisfies the equation $H^{n,1} + R_{0,n}K^n = K^{n+1}$ in $\mathcal{F}_{n+1}^\delta$. This equation is derived as follows. From (4.36) we have

$$R_n\mathcal{E}_n = \kappa_{c_n}(+1,+1)\mathcal{E}_{n+1} \tag{H.1}$$

$$= \kappa_{c_n}(+1,+1)h^{n+1}\exp(K^{n+1}), \tag{H.2}$$

and, from (4.41), we obtain

$$R_n\mathcal{E}_n = R_n(h^n \exp(K^n)) \tag{H.3}$$

$$= \kappa_{c_n}(+1,+1)h^{n+1}(\mathbf{c})\exp(H^{n,1}(\mathbf{c}))\exp(R_{0,n}K^n) \tag{H.4}$$

$$= \kappa_{c_n}(+1,+1)h^{n+1}\exp(H^{n,1} + R_{0,n}K^n). \tag{H.5}$$

Comparing (H.5) with (H.2) we see that we require

$$K^{n+1} = H^{n,1}(\mathbf{c}) + R_{0,n}K^n\,. \tag{H.6}$$

We now construct K^n. Defining $H^{n,0} = \log|h^n|$ we confirm that

$$R_{0,n}H^{n,0} = H^{n+1,0} + H^{n,1} \tag{H.7}$$

where $H^{n,1} \in \mathcal{F}^{\delta}_{n+1}$ is as before.

Making use of the relation $R_n \exp(U,T) = \exp R_{0,n}(U,T)$ and (4.41) we have:

$$\begin{aligned} R_{0,n}H^{n,0} &= \log|R_n \exp H^{n,0}| && \text{(H.8)} \\ &= \log|R_n h^n| && \text{(H.9)} \\ &= \log|\kappa_{c_n} h^{n+1} \exp(H^{n,1})| && \text{(H.10)} \\ &= \log|h^{n+1}| + H^{n,1} && \text{(H.11)} \\ &= H^{n+1,0} + H^{n,1}. && \text{(H.12)} \end{aligned}$$

Using the notation $R^{\ell}_n = R_{n+\ell-1}R_{n+\ell-2}\ldots R_n$ and similarly for $R^{\ell}_{0,n}$, we define:

$$H^{n,\ell} = R^{\ell-1}_{0,n+1}H^{n,1} \in \mathcal{F}^{\delta}_{n+\ell}, \tag{H.13}$$

and

$$\hat{H}^{n,\ell} = R^{\ell-1}_{0,n+1}P_{0,n+1}H^{n,1}. \tag{H.14}$$

From (H.14) we have that

$$\hat{H}^{n-k,k} = R^{k-1}_{0,n-k+1}P_{0,n-k+1}H^{n-k,1}. \tag{H.15}$$

Now, if we let

$$\hat{G}^n = \sum_{k=1}^{\infty} \hat{H}^{n-k,k}, \tag{H.16}$$

then $\hat{G}^n$ converges and $R_{0,n}\hat{G}^n + \hat{H}^{n,1} = \hat{G}^{n+1}$ since:

$$\begin{aligned} R_{0,n}\hat{G}^n &= \sum_{k=1}^{\infty} R_{0,n}\hat{H}^{n-k,k} && \text{(H.17)} \\ &= \sum_{k=1}^{\infty} \hat{H}^{n-k,k+1} && \text{(H.18)} \\ &= -\hat{H}^{n,1} + \sum_{k=0}^{\infty} \hat{H}^{n-k,k+1}, && \text{(H.19)} \end{aligned}$$

then replacing k by $k' = k+1$ we have,

$$\begin{aligned} R_{0,n}\hat{G}^n + \hat{H}^{n,1} &= \sum_{k'=1}^{\infty} \hat{H}^{n+1-k',k'} && \text{(H.20)} \\ &= \hat{G}^{n+1}, && \text{(H.21)} \end{aligned}$$

as required.

Finally, defining $K^n = \hat{G}^n - (I - P_{0,n})H^{n,0} \in \mathcal{F}_n^{\delta}$, we can show that $H^{n,1} +$

$R_{0,n}K^n = K^{n+1}$ as follows:

$$\begin{aligned}
R_{0,n}K^n &= R_{0,n}\hat{G}^n - R_{0,n}(I - P_{0,n})H^{n,0} && \text{(H.22)}\\
&= \hat{G}^{n+1} - \hat{H}^{n,1} - (I - P_{0,n+1})R_{0,n}H^{n,0} && \text{(H.23)}\\
&= \hat{G}^{n+1} - \hat{H}^{n,1} - (I - P_{0,n+1})(H^{n+1,0} + H^{n,1}) && \text{(H.24)}\\
&= \hat{G}^{n+1} - (I - P_{0,n+1})H^{n+1,0} - \hat{H}^{n,1} - (I - P_{0,n+1})H^{n,1} && \text{(H.25)}\\
&= \hat{G}^{n+1} - (I - P_{0,n+1})H^{n+1,0} - P_{0,n+1}H^{n,1} - H^{n,1} + P_{0,n+1}H^{n,1} && \text{(H.26)}\\
&= K^{n+1} - H^{n,1}, && \text{(H.27)}
\end{aligned}$$

where we have made use of equations (H.21), (4.52), and (H.14) and the fact that $\hat{H}^{n,1} = P_{0,n+1}H^{n,1}$. From (H.27) we see that

$$R_{0,n}K^n + H^{n,1} = K^{n+1}, \tag{H.28}$$

as required.

I. PROOF OF LEMMA 9

Here we prove lemma 9.

Lemma 9. *There exists a constant $L > 0$ (independent of k and n) such that $\|H^{n-k,1}\| \leq L$, provided $\delta > 0$ is chosen sufficiently small.*

Proof. Let $H^{n-k,1} = (U, T)$ where $H^{n-k,1} \in \mathcal{F}^{\delta}_{n-k+1}$. Then

$$\|H^{n-k,1}\| = \|U\|_1 + \|T\|_1, \tag{I.1}$$

where U and T are given by equations (4.45), (4.47), (4.49) and (4.51).

We make use of the identities (3.73) and letting $n' = n - k$ we take first the case $c_{n'} = 0$ then $\|U\|_1 = |\log \omega_{n'}|$. Since the sequence ω_j is periodic with period p we may choose

$$\frac{L}{2} \geq \max_{1 \leq j \leq p} |\log \omega_j|, \tag{I.2}$$

then clearly $\|U\|_1 \leq L/2$ independently of n' and k.

If $c_{n'} \neq 0$ then $\|U\|_1 = \|\log(\theta_0^{n'}(x) - y_{n'})\|_1$. Now,

$$\begin{aligned} \log(\theta_0^{n'}(x) - y_{n'}) &= \log(\theta_0^{n'}(x) - \theta_0^{n'}(\theta_0^{n'})^{-1}(y_{n'})) & \text{(I.3)} \\ &= \log \omega_{n'} + \log((\theta_0^{n'})^{-1}(y_{n'}) - x), & \text{(I.4)} \end{aligned}$$

thus, $\| \log(\theta_0^{n'}(x) - y_{n'})\|_1 \leq |\log \omega_{n'}| + \| \log((\theta_0^{n'})^{-1}(y_{n'}) - x)\|_1$.

Now since $y_{n'} \in (-\omega_{n'} - c_{n'}, -\omega_{n'} - c_{n'} + 1)$, then $(\theta_0^{n'})^{-1}(y_{n'}) \in (1 - \omega_{n'}^{-1} + c_{n'}\omega_{n'}^{-1}, 1 + c_{n'}\omega_{n'}^{-1})$, and in this case $c_{n'} \neq 0$ so $(\theta_0^{n'})^{-1}(y_{n'}) \in (1, 1 + a_{n'}\omega_{n'}^{-1})$ and then $|(\theta_0^{n'})^{-1}(y_{n'}) - c_1^{n'+1}| > 1$. Recalling that U is defined on $V_1^{n'+1}$, we now make use of the following result from [45]:

Lemma 10. *Let $c, r \in \mathbb{R}$, $r > 0$ and let $y \in \mathbb{R}$, $y \notin \overline{D(c,r)}$. Let*

$$f(x) = \begin{cases} \log(y - x), & y > c + r; \\ \log(x - y), & y < c - r. \end{cases} \tag{I.5}$$

Then as a function of $x \in D(c, r)$,

$$\|f\|_1 = |\log |y - c|| + \log |y - c| - \log(|y - c| - r) \tag{I.6}$$

Hence, we see that

$$\| \log((\theta_0^{n'})^{-1}(y_{n'}) - x)\|_1 = \log \frac{|(\theta_0^{n'})^{-1}(y_{n'}) - c_1^{n'+1}|^2}{|(\theta_0^{n'})^{-1}(y_{n'}) - c_1^{n'+1}| - r_1^{n'+1}}, \tag{I.7}$$

which assumes that $(\theta_0^{n'})^{-1}(y_{n'}) > -\omega_{n'+1} - a_{n'+1} + 1 + \delta$, i.e. $c_{n'}\omega_{n'}^{-1} > \delta$. This is true since $c_{n'} \neq 0$ in this case, provided $\delta < 1$ which we assume from now on.

So now,

$$\|U\|_1 = |\log \omega_{n'}| + \log \frac{|(\theta_0^{n'})^{-1}(y_{n'}) - c_1^{n'+1}|^2}{|(\theta_0^{n'})^{-1}(y_{n'}) - c_1^{n'+1}| - r_1^{n'+1}}. \tag{I.8}$$

For $\delta < 1/2$, there clearly exists $\delta_1 > 0$ such that $|(\theta_0^{n'})^{-1}(y_{n'}) - c_1^{n'+1}| - r_1^{n'+1} > \delta_1$ since $|(\theta_0^{n'})^{-1}(y_{n'}) - c_1^{n'+1}| > 1$ and $r_1^{n'} = \frac{1}{2} + \delta$. We also see that $|(\theta_0^{n'})^{-1}(y_{n'}) - c_1^{n'+1}| < \frac{1}{2} + (c_{n'} + 1)\omega_{n'}^{-1} < K$ for constant K, independently of n', since $c_n' \leq \max_{1\leq j\leq p} a_j$ and $\omega_{n'}^{-1} \leq (\min_{1\leq j\leq p} \omega_j)^{-1}$. Then,

$$\begin{aligned} \|U\|_1 &\leq \max_{1\leq j\leq p} |\log \omega_j| + \log\left(\frac{K^2}{\delta_1}\right) && \text{(I.9)} \\ &\leq L/2, && \text{(I.10)} \end{aligned}$$

for some choice of L.

Firstly, for $c_{n'} = 0$:

$$T(x) = \log(\omega_{n'} \prod_{i=1}^{\ell} (y_{n'} - \theta_i^{n'}(x))), \tag{I.11}$$

where $\ell = a_{n'}$ for $c_{n'-1} = 0$ and $\ell = a_{n'} - 1$ for $c_{n'-1} \neq 0$. We can write this as

$$T(x) = \log \omega_{n'} + \sum_{i=1}^{\ell} \log(y_{n'} - \theta_i^{n'}(x)). \tag{I.12}$$

Then

$$\|T\|_1 \leq |\log \omega_{n'}| + \sum_{i=1}^{\ell} \|\log(y_{n'} - \theta_i^{n'}(x))\|_1. \tag{I.13}$$

Now,

$$\begin{aligned} \log(y_{n'} - \theta_i^{n'}(x)) &= \log(\theta_i^{n'}(\theta_i^{n'})^{-1}(y_{n'}) - \theta_i^{n'}(x)) && \text{(I.14)} \\ &= \log \omega_{n'} + \log(x - (\theta_i^{n'})^{-1}(y_{n'})), && \text{(I.15)} \end{aligned}$$

where $1 \le i \le \ell$. Thus,

$$\|\log(y_{n'} - \theta_i^{n'}(x))\|_1 \le |\log \omega_{n'}| + \|\log(x - (\theta_i^{n'})^{-1}(y_{n'}))\|_1. \tag{I.16}$$

Since $c_{n'} = 0$ we have $y_{n'} \in (-\omega_{n'}, 1)$ and $(\theta_i^{n'})^{-1}(y_{n'}) \in (-\omega_{n'}^{-1} - i\omega_{n'}^{-1}, 1 - i\omega_{n'}^{-1})$, and since T is defined on $V_0^{n'+1}$ we require that $(\theta_i^{n'})^{-1}(y_{n'}) < 1 - \frac{1}{2}\omega_{n'}^{-1} - \frac{1}{2}\omega_{n'}^{-1} - \delta$ i.e. $1 - i\omega_{n'}^{-1} < 1 - \omega_{n'}^{-1} - \delta$ which is true for $i > 1$ provided $\delta > 0$ is sufficiently small. Since we have $i \ge 1$ then in principle $(\theta_i^{n'})^{-1}(y_{n'})$ could get arbitrarily close to $V_0^{n'+1}$. However, assuming a periodic code $(c_k)_{k\in\mathbb{Z}}$ as well as a periodic continued fraction, we can say that there exists $\delta_2 > 0$ such that

$$\min_{n'} |(-\omega_{n'+1} - a_{n'+1} + 1) - (\theta_i^{n'})^{-1}(y_{n'})| - r_0^{n'+1} > \delta_2, \tag{I.17}$$

again provided δ is taken small enough. Now making use of lemma 10,

$$\begin{aligned}\|\log(x - (\theta_i^{n'})^{-1}(y_{n'})\|_1 &= |\log|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}|| \\ &\quad + \log|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| \\ &\quad - \log(|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| - \frac{1}{2}\omega_{n'}^{-1} - \delta).\end{aligned} \tag{I.18}$$

Now if $|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| > 1$ then

$$\|\log(x - (\theta_i^{n'})^{-1}(y_{n'}))\|_1 = \log\left(\frac{|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}|^2}{|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| - \frac{1}{2}\omega_{n'}^{-1} - \delta}\right), \tag{I.19}$$

where $|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| < |(\frac{1}{2} - i)|\omega_{n'}^{-1} < K$ for some constant K then

$$\begin{aligned}
\|T\|_1 &\le \max_{1\le j\le p} |\log \omega_j| + \left(\sum_{i=1}^{\ell}\left(\max_{1\le j\le p}|\log\omega_j|\right) + \log\left(\frac{K^2}{\delta_2}\right)\right) && \text{(I.20)}\\
&\le (1+\ell)\max_{1\le j\le p}|\log\omega_j| + \ell\log\left(\frac{K^2}{\delta_2}\right) && \text{(I.21)}\\
&\le \frac{L}{2}, && \text{(I.22)}
\end{aligned}$$

for some L independent of n' and k. If $|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| \le 1$, then

$$\begin{aligned}
\|T\|_1 &\le (1+\ell)\max_{1\le j\le p}|\log\omega_j| + \ell\log\left(\frac{1}{\delta_2}\right) && \text{(I.23)}\\
&\le \frac{L}{2}, && \text{(I.24)}
\end{aligned}$$

again for some L independent of n' and k.

Finally, for $c_{n'} \neq 0$

$$T(x) = \log\left(\omega_{n'}\prod_{i=0}^{c_{n'}-1}(\theta_i^{n'}(x) - y_{n'})\prod_{j=c_{n'}+1}^{\ell}(y_{n'} - \theta_j^{n'}(x))\right), \tag{I.25}$$

where, as before, $\ell = a_{n'}$ or $a_{n'} - 1$. Then

$$\|T\|_1 \le |\log\omega_{n'}| + \sum_{i=1}^{c_{n'}-1}\|\log(\theta_i^{n'}(x) - y_{n'})\|_1 + \sum_{j=c_{n'}+1}^{\ell}\|\log(y_{n'} - \theta_j^{n'}(x))\|_1. \tag{I.26}$$

Now for $0 \le i \le c_{n'} - 1$, $\log(\theta_i^{n'}(x) - y_{n'}) = \log\omega_{n'} + \log((\theta_i^{n'})^{-1}(y_{n'}) - x)$, thus,

$$\|\log(\theta_i^{n'}(x) - y_{n'})\|_1 \le |\log\omega_{n'}| + \|\log((\theta_i^{n'})^{-1}(y_{n'}) - x)\|_1. \tag{I.27}$$

Now since $c_{n'} \neq 0$ then $y_{n'} \in (-\omega_{n'} - c_{n'}, -\omega_{n'} - c_{n'} + 1)$ and $(\theta_i^{n'})^{-1}(y_{n'}) \in (1 - \omega_{n'}^{-1}(1 - c_{n'} + i), 1 + \omega_{n'}^{-1}(c_{n'} - i))$. Using lemma 10 and the fact that T is defined on $V_0^{n'+1}$ we have:

$$\begin{aligned} \|\log((\theta_i^{n'})^{-1}(y_{n'}) - x)\|_1 &= |\log|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}|| \\ &\quad + \log|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| \\ &\quad - \log(|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| - \frac{1}{2}\omega_{n'}^{-1} - \delta). \end{aligned} \tag{I.28}$$

We have that $(\theta_i^{n'})^{-1}(y_{n'}) > 1$, since $i \leq c_{n'} - 1$ then $(\theta_i^{n'})^{-1}(y_{n'}) > 1$ but $(\theta_i^{n'})^{-1}(y_{n'})$ could be arbitrarily close to $V_0^{n'+1}$. However, following the above argument, if we assume a periodic code $\mathbf{c}$ then we can say there exists $\delta_3 > 0$ such that $\min_{n'} |(\theta_i^{n'})^{-1}(y_{n'}) - 1 - \delta| > \delta_3$, provided δ is chosen sufficiently small.

Now since $|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| < \omega_{n'}^{-1}(c_{n'} - \frac{1}{2}) < K$ for some constant $K > 1$, we have that

$$\begin{aligned} \|\log((\theta_i^{n'})^{-1}(y_{n'}) - x)\|_1 &= \log\left(\frac{\max\{1, |(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}|^2\}}{|(\theta_i^{n'})^{-1}(y_{n'}) - 1 + \frac{1}{2}\omega_{n'}^{-1}| - \frac{1}{2}\omega_{n'}^{-1}}\right) \\ &\leq \log\left(\frac{K^2}{\delta_3}\right). \end{aligned} \tag{I.29}$$

So now we have

$$\|T\|_1 = |\log\omega_{n'}| + \sum_{i=0}^{c_{n'}-1}\left(|\log\omega_{n'}| + \log\frac{K^2}{\delta_3}\right) + \sum_{j=c_{n'}+1}^{\ell} \|\log(y_{n'} - \theta_j^{n'}(x))\|_1. \tag{I.30}$$

Writing for $c_{n'}+1 \leq j \leq \ell$, $\log(y_{n'}-\theta_j^{n'}(x)) = \log\omega_{n'} + \log(x-(\theta_j^{n'})^{-1}(y_{n'}))$ then $\|\log(y_{n'}-\theta_j^{n'}(x))\|_1 \leq |\log\omega_{n'}| + \|\log(x-(\theta_j^{n'})^{-1}(y_{n'}))\|_1$, where again $y_{n'} \in (-\omega_{n'}-c_{n'}, -\omega_{n'}-c_{n'}+1)$ and $(\theta_j^{n'})^{-1}(y_{n'}) \in (1-\omega_{n'}^{-1}(1-c_{n'}+j), 1+\omega_{n'}^{-1}(c_{n'}-j))$. Using lemma 10 we have:

$$\begin{aligned}\|\log(x-(\theta_j^{n'})^{-1}(y_{n'})\|_1 &= |\log|(\theta_j^{n'})^{-1}(y_{n'})-1+\frac{1}{2}\omega_{n'}^{-1}|| \\ &\quad +\log|(\theta_j^{n'})^{-1}(y_{n'})-1+\frac{1}{2}\omega_{n'}^{-1}| \\ &\quad -\log(|(\theta_j^{n'})^{-1}(y_{n'})-1+\frac{1}{2}\omega_{n'}^{-1}|-\frac{1}{2}\omega_{n'}^{-1}-\delta),\end{aligned} \tag{I.31}$$

and since T is defined on $V_0^{n'+1}$ we require that $(\theta_j^{n'})^{-1}(y_{n'}) < 1-\omega_{n'}^{-1}-\delta$. Given that $j \geq c_{n'}+1$, we have $(\theta_j^{n'})^{-1}(y_{n'}) \leq 1-\omega_{n'}^{-1}$ i.e. $(\theta_j^{n'})^{-1}(y_{n'})$ could be arbitrarily close to $V_0^{n'+1}$. However, again, using the periodicity of $\mathbf{c}$, we can say that there exists $\delta_4 > 0$ such that $\min_{n'} |1-\omega_{n'}^{-1}-\delta-(\theta_j^{n'})^{-1}(y_{n'})| > \delta_4$, provided, as before, that $\delta > 0$ is taken sufficiently small.

Finally $(\theta_j^{n'})^{-1}(y_{n'})-1+\frac{1}{2}\omega_{n'}^{-1} > (\frac{1}{2}-\ell)\omega_{n'}^{-1}$, hence $|(\theta_j^{n'})^{-1}(y_{n'})-1+\frac{1}{2}\omega_{n'}^{-1}| < (\ell-\frac{1}{2})\omega_{n'}^{-1} < K$, for some constant $K > 1$, larger than the K above. We then have, for $c_{n'} \neq 0$:

$$\begin{aligned} \|T\|_1 &\leq \max_{1\leq m\leq p} |\log \omega_m| + \sum_{i=0}^{c_{n'}-1} \left(\max_{1\leq m\leq p} |\log \omega_m| + \log \frac{K^2}{\delta_3} \right) \\ &\quad + \sum_{j=c_{n'}+1}^{\ell} \left(\max_{1\leq m\leq p} |\log \omega_m| + \log \frac{K^2}{\delta_4} \right) \qquad \text{(I.32)} \\ &\leq (1+\ell) \max_{1\leq m\leq p} |\log \omega_m| + c_{n'} \log \frac{K^2}{\delta_3} + (\ell - c_{n'}) \left(|\log K| + \log \frac{K}{\delta_4} \right) \qquad \text{(I.33)} \\ &\leq L/2, \qquad \text{(I.34)} \end{aligned}$$

for some L using the periodicity of $\omega_{n'}$ to ensure that $\log K$ is bounded, as is $\log \omega_m$.

For all cases then we see that

$$\begin{aligned} \|H^{n-k,1}\| &= \|U\|_1 + \|T\|_1 \qquad \text{(I.35)} \\ &\leq L/2 + L/2 \qquad \text{(I.36)} \\ &\leq L, \qquad \text{(I.37)} \end{aligned}$$

for some $L > 0$ independent of k and n.

□