TOPICS IN SEMIGROUP ALGEBRAS

BY

J. RICHARD WORDINGHAM

A thesis submitted in fulfilment of

the requirements of the degree of

Doctor of Philosophy

Department of Mathematics
University of Stirling

June, 1982

CONTENTS

ACKNOWLEDGEMENTS

I thank my supervisor, Professor J. Duncan, for introducing me to semigroup algebras, and for his advice and constant encouragement. I also thank Professor B.A. Barnes for a suggestion of his.

My studies embodied in this thesis were financed by the University of Stirling.

I thank Mrs. M. Abrahamson for typing this thesis.

ABSTRACT

Much work has been done on the ℓ^1-algebras of groups, but much less on ℓ^1-algebras of semigroups. This thesis studies those of inverse semigroups, also known as generalised groups, with emphasis on the involutive structure. Where results extend to the semigroup ring, I extend them.

I determine the characters of a semilattice in terms of its order structure. The simplest suffice to separate its ℓ^1-algebra. I also determine the algebra's minimal idempotents.

I introduce a generalisation of Banach *-algebras which has good hereditary properties and includes the inverse semigroups rings. These latter have an ultimate identity which can be used to test for representability. Involutive semigroups with s^*s an idempotent yield inverse semigroups when quotiented by the congruence induced by their algebras' *-radical.

The left regular *-representation of inverse semigroups is faithful and acts like that of groups. The corresponding idea of amenability coincides with the traditional one. Brandt semigroups have the weak containment property iff the associated group does. The relationship of ideals to weak containment is studied, and inverse semigroups with well ordered semilattices are shown to have the property if all their subgroups do. The converse is extended for Clifford semigroups.

Symmetry and related ideas are considered, and basic results proved for the above mentioned generalisation, and a better version for a possibly more restricted generalisation. The symmetry of

an ℓ^1-algebra of an E-unitary inverse semigroup is shown to depend on the symmetry of the ℓ^1-algebra of its maximal group homomorphic image if the semilattice has a certain structure or the semigroup is a Clifford semigroup. Inverse semigroups with well ordered semilattices are shown to have symmetric ℓ^1-algebra if all the subgroups do.

Finally, some topologically simple ℓ^1-algebras and simple semigroup rings are constructed, extending results on simple inverse semigroup rings.

INTRODUCTION

In this thesis I study semigroup rings and their ℓ^1-completions, semigroup algebras. Nearly all the semigroups studied are inverse semigroups, that is, semigroups in which each element has a unique (von Neumann) inverse.

First I examine the simplest type of inverse semigroups, the semilattices, that is, commutative semigroups of idempotents. Hewitt and Zuckermann [14] thoroughly examined the ℓ^1-algebras of commutative semigroups, so we already know that $\ell^1(E)$ is semisimple, and has a unit only if E is the union of finitely many principal ideals. I determine the characters in terms of the algebraic structure of the semilattice and also in terms of the simplest characters. We show that these suffice to separate the elements of the ℓ^1-algebra. I then use the algebraic descriptions of the characters to determine the ℓ^1-algebra's minimal idempotents.

Next I sketch the elementary theory of positive functionals and representations on Hilbert space of *-algebras. Then I study the class of *-algebras that have enveloping C*-algebras and all of whose positive functionals are admissible. I show that this class, the uniform admissibility algebras, is closed under most of the methods used to obtain new algebras from old. Then I establish the most comprehensive result I know on extending non-degenerate *-representations from ideal-like *-algebras. I then establish relationships between enveloping C*-algebras and *-algebras manufactured from others.

I then sketch the elementary *-representation theory of inverse semigroups, and develop a test for the representability of positive functions on the semigroup ring using not bounded approximate identities, but ultimate identities.

Then I examine the left regular *-representation of an inverse semigroup introduced by Barnes [1]. I prove that it is faithful, and produce a decomposition in terms of the semilattice. I then show that it provides a satisfying generalisation of the convolution of $\ell^1(G)$ and $\ell^p(G)$ for a group G, and that the corresponding notion of amenability agrees with the traditional one. As Wilde and Argabright (but see Duncan & Namioka [9] for a quicker proof) have already determined when an inverse semigroup is amenable, there is little gain from this fact.

Now the amenability of G is equivalent to two interesting properties. G is amenable if and only if $\ell^1(G)$ is amenable ([4] Proposition 43.3). For S an E-unitary inverse semigroup, Duncan and Namioka determine precisely when $\ell^1(S)$ is amenable. Secondly, G is amenable if and only if G has the weak containment property, i.e. the left regular representation of $\ell^1(G)$ on $\ell^2(G)$ produces the greatest B*-seminorm on $\ell^1(G)$.

Sufficient conditions for weak containment seem easier to establish than necessary ones. I establish that if I is an ideal of S, then I has the weak containment property if S does, and S does if I and S/I do. Paterson [25] proved that for Clifford semigroups (i.e. inverse semigroups whose idempotents are central) these norms coincide if all the subgroups are amenable. The converse has yet to be settled; I establish it in the case where every element of the semilattice has a minimal idempotent associated with it in the semilattice algebra. Next I determine when a Brandt semigroup has the weak containment property, and hence prove that semigroups with well-founded semilattices have the weak containment property if all their subgroups are amenable. On the way I establish that every inverse semigroup with zero has a greatest ideal with the weak containment property.

A group is called Hermitian if its group algebra is symmetric. The investigation of the symmetry of group algebras has been greatly advanced by Leptin. From the algebraic viewpoint a nice study of symmetry is provided by Wichmann [29]. We undertake an investigation of various generalisations of the notion of symmetry for uniform admissibility algebras using Leptin's characterisation of symmetry for Banach *-algebras.

Leptin has already established that the algebra of a Brandt semigroup is symmetric if and only if its associated group is Hermitian. Calling a semigroup Hermitian if its algebra is symmetric, we establish that every inverse semigroup has a greatest Hermitian ideal, and thus an inverse semigroup with well-founded semilattice is Hermitian if and only if all its subgroups are Hermitian. For an E-unitary inverse semigroup S, I am inspired by the hypothesis that S is Hermitian if G_S is. This is shown to be the case if S is a Clifford semigroup, and also if the idempotent semilattice has a certain structure.

I then push these results through for the complete symmetry of semigroup rings. As the group ring of the integers is not even symmetric, there are very few completely symmetric group rings, and finiteness plays a large role.

Finally sufficient conditions are found for contracted inverse semigroup rings and algebras to be simple and topologically simple respectively. These generalise the earlier results of Munn [23] on inverse semigroup rings. So that many examples may be found, we investigate inverse semigroups constructed from left cancellative semigroups and translate these sufficient conditions into conditions on the left cancellative semigroups.

All the algebras I consider will be associative and, except in Chapter 4 where arbitrary fields are considered, will be over the complex field. For a set of vector spaces or algebras $\{A_\lambda : \lambda \in \Lambda\}$, its direct sum $\sum_{\lambda\in\Lambda} A_\lambda$ will be the set of $\{f \in (\bigcup_\lambda A_\lambda)^\Lambda : f(\lambda) \in A_\lambda$ and $f(\lambda) \neq 0$ for finitely many $\lambda\}$ with pointwise operations. Their ℓ^p-direct sum is the closure in the norm $\|f\| = (\sum_\lambda \|f(\lambda)\|^p)^{1/p}$ if the A_λ are Banach spaces. A directed union of subobjects $\{A_\lambda : \lambda \in \Lambda\}$ where Λ is a directed set and $A_\lambda \subset A_\mu$ if $\lambda < \mu$ is $\bigcup_\lambda A_\lambda$. An algebra is simple if it has no ideals; an algebra with a topology is toplogically simple if it has no closed ideals. An idempotent e of an algebra A over F is called minimal if $eAe = Fe$.

Unless confusion may be caused by taking it out of context, the identity of a semigroup or algebra will be denoted 1, and in an algebra with identity $\lambda 1$ and λ will be used interchangeably for $\lambda \in \mathbb{C}$. For a complex algebra A with identity and $x \in A$, $Sp_A(x) = \{\lambda \in \mathbb{C} : \lambda - x$ is not invertible$\}$. For a complex algebra A without identity we define $Sp_A(x) = Sp_{\tilde{A}}(x)$ where $\tilde{A} = A \oplus \mathbb{C}1$ with multiplication $(x + \lambda)(y + \mu) = (xy + \lambda y + \mu x) + \lambda\mu$. Alternatively, noting that $(1 - x)(1 - y) = 1 - (x + y - xy)$, we define a multiplication $\circ$ on A by $x \circ y = x + y - xy$.

$x \in A$ is said to be left quasiregular if there exists $y \in A$ such that $y \circ x = 0$, and left quasisingular if there is not. $x \in A$ is said to be quasiregular if it is both left and right quasiregular, and quasisingular if it is not. Then $Sp_A(x) = 0 \cup \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}x$ is quasisingular$\}$. Hence if $T : A \to B$ is an algebra morphism, $Sp(Tx) \subset \{0\} \cup Sp(x)$. The spectral radius, $\rho_A(x)$, is $\sup\{|\lambda| : \lambda \in Sp_A(x)\}$. It may be infinite, or if $Sp_A(x) = \emptyset$, undefined, neither of which can happen in a Banach algebra. The subscript will be dropped when it is clear which algebra we are considering. A modular left ideal of an algebra A is a left ideal for which there exists $e \in A$ such that $x - xe \in L$ for all $x \in A$. Such an e is called a right modular unit for L. e is a modular unit for some proper modular left ideal if and only if it is left quasisingular. By a maximal (modular) (left) ideal we mean a maximal proper (modular) (left) ideal.

An involution on object X is a bijection whose square is the identity with, denoting the image of x by x^*, $(xy)^* = y^*x^*$ if X has a multiplication, $(x + y)^* = x^* + y^*$ if X has an addition, and if X is a real or complex vector space, $(\lambda x)^* = \lambda^* x^*$ where λ^* is the complex conjugate of λ. An object with a distinguished involution is called involutive. A *-algebra is an algebra with a distinguished involution; a *-ideal is an ideal closed under the involution. The quotient of a *-algebra by a *-ideal inherits the involution. If A and B have distinguished involutions $*$ and $\dagger$, homomorphism $\phi : A \to B$ is a *-homomorphism if $\phi(x^*) = \phi(x)^\dagger$. An element h is called self-adjoint if $h = h^*$, and the set of self-adjoint elements of A is denoted $\mathrm{sym}(A)$. A Banach *-algebra is a Banach algebra with a distinguished involution.

Following Wichmann [29] and Palmer [24], we refer to hereditary radical properties. A property (P) of rings is said to be a hereditary radical property if

(i) Quotients of rings with property (P) by ideals have property (P) .

(ii) Every ring A has a greatest ideal with property (P) ; we denote it $P\text{-rad}(A)$.

(iii) No non-zero ideal of $A/P\text{-rad}(A)$ has property (P) .

(iv) If I is an ideal of A , then $P\text{-rad}(I) = I \cap P\text{-rad}(A)$.

When we study algebras rather than rings, we use the definition with algebras in place of rings, and ideals remain in it. If *-algebras, *-algebras replace rings and *-ideals replace ideals. If Banach algebras, Banach algebras replace rings and closed ideals replace ideals. If Banach *-algebras, Banach *-algebras replace rings and closed *-ideals replace ideals. A is called P-semisimple if $P\text{-rad}(A) = \{0\}$. Our most important example is the Jacobson radical, for which we use "rad" and "semisimple" unprefixed.

Let A and B be linear (sub)spaces with a linear space C such that ab is defined to be an element of C for $a \in A$ and $b \in B$ under some linear composition. For example, A and B might be subalgebras of C , or A might be an algebra of linear operators on vector space $B = C$. Then AB will denote the linear span of $\{ab : a \in A \text{ and } b \in B\}$. Otherwise AB will denote that set itself.

Let π be a representation of an algebra A by bounded operators on a Banach space X . It is called degenerate if $\pi = 0$ or $(\pi(A)X)^-$ is a proper subspace of X . An element ξ of X is called a cyclic vector if $\xi \in (\pi(A)X)^-$ and $(\pi(A)\xi)^- = (\pi(A)X)^-$; then π is called a cyclic representation.

An element x of a semigroup S is called its zero if $xy = yx = x$ for all $y \in S$. We will denote it by θ. We can adjoin an identity to a semigroup S, and we denote the new semigroup S^1. We may adjoin a zero to a semigroup S; we denote the new semigroup S^0. A subset $I \subset S$ is called a left ideal if $sx \in I$ for all $s \in S$ and $x \in I$; it is called an ideal if it is both a left and right ideal. If I is an ideal of S we define its (Rees) quotient S/I to be, assuming for notational reasons that I is not an element of S, $(S\backslash I) \cup \{I\}$ with I the zero and for $s, t \in S\backslash I$, $s \circ t = st$ if $st \notin I$ and I if $st \in I$. A subsemigroup G of S is called a subgroup if G is a group.

An element s of semigroup S is called invertible if S has an identity and there exists $t \in S$ such that $st = ts = 1$. $t \in S$ is called a (von Neumann) inverse of s if $sts = s$ and $tst = t$. A semigroup is called regular if every element has an inverse. It is called an inverse semigroup if every element has a unique inverse. A regular semigroup is an inverse semigroup if and only if its idempotents commute, in which case the idempotents form a subsemigroup. A commutative semigroup of idempotents is called a semilattice; we define an order on it by $e \leq f$ if $e = ef$. For S an inverse semigroup we denote its set of idempotents by E_S, or where no ambiguity may arise, E. For $s \in S$ we denote its inverse by s^*. Then $(st)^* = t^*s^*$. A homomorphic image of an inverse semigroup is an inverse semigroup, and thus a semigroup homomorphism is a $*$-homomorphism. For proofs see [15] §V.1.

A Clifford semigroup S is an inverse semigroup in which the idempotents are central. Then for $e \in E_S$, let $G_e = \{s \in S : s^*s = e\}$. Then each G_e is a group, $S = \bigcup_{e \in E} G_e$ and $G_e G_f \subset G_{ef}$. A

Clifford semigroup is also known as a semilattice of groups. An inverse semigroup has a minimal group homomorphism, which we denote χ_S, and we denote its image by G_S. It is given by $s \sim t$ if there is $e \in E_S$ such that $es = et$. It is called E-unitary if $\chi_S^{-1}(1) = E_S$. Green defined equivalences $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ on an arbitrary semigroup. For an inverse semigroup, $a \mathcal{L} b$ iff $a^*a = b^*b$, $a \mathcal{R} b$ if $aa^* = bb^*$, $a \mathcal{H} b$ if $a \mathcal{L} b$ and $a \mathcal{R} b$, and $a \mathcal{D} b$ if there exists c such that $a^*a = c^*c$ and $cc^* = bb^*$. For details see [15].

An idempotent u of an inverse semigroup is called primitive if the only idempotent it exceeds is the zero element. The Brandt semigroup $\mathcal{M}_0(I, G)$ is $\{(g)_{ij} : g \in G, i, j \in I\} \cup \{\theta\}$ with θ its zero and

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ \theta & \text{otherwise} \end{cases}$$

where G is a group. I could not find an explicit proof of our first theorem. It is well known.

<u>Theorem 0.1</u>

Let u be a primitive idempotent of inverse semigroup S. Then SuS is a group or Brandt semigroup.

<u>Proof</u>

Let u be a primitive idempotent of S. If u is the only idempotent of SuS then SuS is a group. Suppose u is not its only idempotent.

Let $v \in (SuS \cap E_S)\backslash\{\theta\}$. Then $v = xuy$ for some x and $y \in S$. Then $v = v^*v = y^*ux^*xuy = y^*uy$ as $v \neq \theta$, $ux^*xu \leq u$ and u is primitive. Then as $yvy^* \neq \theta$, $u = yvy^*$. Then if $e \in E_S$, $ev = ev(y^*y) = y^*(yey^*)(yvy^*)y = y^*((yey^*)u)y$, so $ev = \theta$ or $ev = y^*uy = v$, so v is also primitive.

Let $I = E_{SuS}\backslash\{\theta\}$, and G be the subgroup of SuS containing u. For $e \in I$ pick $x_e \in S$ such that $e = x^*_e u x_e$. Let

$$f : \mathcal{M}_0(I, G) \to SuS \quad \text{and} \quad \phi : SuS \to \mathcal{M}_0(I, G)$$

$$(g)_{ij} \to x^*_i g x_j \qquad z \to (ux_i z x^*_j u)_{ij} \quad \text{where } x^*_i u x_i = zz^*$$

$$\text{and} \quad x^*_j u x_j = z^* z \,,$$

and $f(\theta) = \theta$, $\phi(\theta) = \theta$. Then f and ϕ are mutually inverse, and $(x^*_i g x_j)(x^*_k h x_l) = (x^*_i g x_j)(x^*_j u x_j)(x^*_k u x_k)(x^*_k h x_l)$ as $(x^*_e u x_e)^2 \neq \theta$ and hence $u x_e x^*_e = u, = \theta$ if $j \neq k$ and $(x^*_i g x_j)(x^*_j u x_k)(x^*_k h x_l) = x^*_i g h x_l$ if $j = k$. $\square$

For S a semigroup we define a multiplication on $\ell^1(S)$ by $fg(s) = \sum\{f(t)g(u) : tu = s\}$. This makes $\ell^1(S)$ a Banach algebra, the semigroup algebra. We imbed S in $\ell^1(S)$ as the co-ordinate vectors. The semigroup ring $k(S) = \{f \in \ell^1(S) : f(s) = 0$ except for finitely many $s\}$ inherits this multiplication, and for an arbitrary field F, we define FS to be $\{f \in F^S : f(s) = 0$ except for finitely many $s\}$ and define multiplication as before. Now if S has a zero θ, θ is an ideal of $\ell^1(S)$ and $k(S)$. We regard the quotients $\ell^1_\theta(S)$ and $k_\theta(S)$ as functions with domain $S\backslash\{\theta\}$ rather than as cosets. The same multiplication formula holds, so we may write $\ell^1(S\backslash\{\theta\})$ or $k(S\backslash\{\theta\})$ rather than $\ell^1_\theta(S)$ or $k_\theta(S)$. Similarly we define $F_\theta S$ to be functions on $S\backslash\{\theta\}$, and it is isomorphic to $FS/F\theta$. If I is an ideal of S, $\ell^1(S)/\ell^1(I) \cong \ell^1_\theta(S/I)$ and similarly for $k(S)$ and FS. If S has an involution we extend it to $\ell^1(S)$ etcetera by $f^*(s) = f(s^*)^*$. This involution is isometric on $\ell^1(S)$.

CHAPTER 1

SEMILATTICES

Here we establish some basic properties of semilattices, which we shall use later.

Definition 1.1

A subset J of E is called a filter if:

(i) when $e \geq f$ and $f \in J$ then $e \in J$;

(ii) when e and $f \in J$ then $ef \in J$; and

(iii) $J \neq \emptyset$.

Proposition 1.2

There is a one-one correspondence between the characters on E and its filters, given by

$$\phi \leftrightarrow \{e \in E : \phi(e) = 1\} .$$

Proof

If ϕ is a character on E , $\phi : E \to \{0, 1\}$.

Let $J_\phi = \{e \in E : \phi(e) = 1\}$. Then J_ϕ is a filter, and

$$\phi(e) = \begin{cases} 1 & \text{if } e \in J_\phi \\ 0 & \text{if } e \notin J_\phi \end{cases}$$

Let J be a filter. Let

$$\psi_J(e) = \begin{cases} 1 & \text{if } e \in J \\ 0 & \text{if } e \notin J \end{cases} .$$

If $ef \in J$ then $e, f \in J$ and so $\psi_J(ef) = \psi_J(e)\psi_J(f)$.
If $ef \notin J$ then $e \notin J$ or $f \notin J$, and then $\psi_J(ef) = \psi_J(e)\psi_J(f)$.
Thus ψ_J is a character. $\square$

For $e, f \in E$ let $\psi_e(f) = \begin{cases} 1 & \text{if } f \geq e \\ 0 & \text{otherwise.} \end{cases}$

<u>Proposition 1.3</u>

Let J be a filter. Direct J by $\geq$. Then for all $e \in E$,

$\psi_J(e) = \lim_{f \in J} \psi_f(e)$.

<u>Proof</u>

Let J be a filter. If $f, g \in J$ then $fg \in J$ and $f, g \geq fg$. Therefore $(J, \geq)$ is a directed set, and so for each e, $f \to \psi_f(e)$ is a net. If $e \in J$, $\psi_e(e) = 1$ and $\psi_f(e) = 1$ whenever $f \leq e$, so $\lim_{f \in J} \psi_f(e) = 1$. If $e \notin J$, then $\psi_f(e) = 0$ for all $f \in J$. $\square$

The following lemma is but a watered down version of Theorem 3.4 of [14] but the proof is simpler.

<u>Lemma 1.4</u> (Wordingham [30])

$\{\psi_e : e \in E\}$ separates $\ell^1(E)$.

<u>Proof</u>

Without loss of generality, E is infinite.

Let $x \in \ell^1(E)$ with $\psi_e(x) = 0$ for all $e \in E$, yet $x \neq 0$.

Let $\phi = \ell^\infty(E)$ with $\phi(e) = \prod_{i=1}^{n} \psi_{u_i}(e)$ for all $e \in E$. Such a product will be called a product of ψ_e's. Let

$F = \{e \in E : e \geq u_r \text{ for } 1 \leq r \leq n\}$. Then F is a filter or empty. If empty, $\phi(e) = 0$ for all $e \in E$, so $\phi(x) = 0$. If F is a filter, then $\phi = \psi_F$. Then for all $e \in E$, $\phi(e) = \lim_{f \in F} \psi_f(e)$. The ψ_f are uniformly bounded and converge pointwise on E, so they converge weak*ly. So $\phi(x) = \lim_{f \in F} \psi_f(x) = 0$.

Let $x = \sum_{r=1}^{\infty} a_r e_r$ with $a_r \in \mathbb{C}$, $a_1 \neq 0$ and the e_r distinct. Then $\sum_{r=1}^{\infty} a_r = \psi_E(x) = \lim_{e \in E} \psi_e(x) = 0$. For $r \geq 2$ there exists f_r such that $\psi_{f_r}(e_1) \neq \psi_{f_r}(e_r)$. Define $\phi_n \in \ell^\infty(E)$ by

$$\phi_n(e) = \prod_{r=2}^{n} \frac{\psi_{f_r}(e) - \psi_{f_r}(e_r)}{\psi_{f_r}(e_1) - \psi_{f_r}(e_r)}$$

for $e \in E$.

Then $\phi_n(e_1) = 1$, $\phi_n(e_r) = 0$ for $2 \le r \le n$ and $||\phi_n||_\infty = 1$.

Then $\phi_n(x) = 0$ as ϕ_n is a sum of multiples of ψ_e's and ψ_E , so $|a_1| \le \sum_{r=n+1}^{\infty} |a_r|$. Thus $a_1 = 0$, which is a contradiction. □

$x \in l^1(E)$ is a minimal idempotent if $x^2 = x$ and, since $xex \in \mathbb{C}x$, $xe = x$ or $xe = 0$ for all $e \in E$. Now if $u, v \in E$ and $u > v$, then $u - v$ is an idempotent. Then if $w \in E$ and $wu \le v$, $w(u - v) = 0$, and if $w \ge u$, $w(u - v) = u - v$.

Proposition 1.5

Let $u \in E$. If $Eu\backslash\{u\}$ is the union of finitely many principal ideals Ev_i , $1 \le i \le n$, then $\prod_{i=1}^{n} (u - v_i)$ is a minimal idempotent. The value of this expression depends only on u and not on the choice of principal ideals. All the minimal idempotents of $\ell^1(E)$ but the zero of E (if it exists) are of this form.

Proof

Let $u \in E$ and $Eu\backslash\{u\} = \bigcup_{i=1}^{n} Ev_i$. Then $v_i < u$, so $\prod_{i=1}^{n} (u - v_i)$ is an idempotent. Let $w \in E$. If $wu = u$, then $w\prod_{i=1}^{n} (u - v_i) = \prod_{i=1}^{n}(u - v_i)$. If $wu < u$, then $wu \in Ev_r$ for some r . Then $w(u - v_r) = wu - wv_r = wuv_r - wuv_r = 0$. Thus $\prod_{i=1}^{n}(u - v_i)$ is a minimal idempotent.

Now if $x_i < u$, $u \in \mathrm{Supp}(\prod_{i=1}^{p} (u - x_i))$. So if $Eu\backslash\{u\} = \bigcup_{i=1}^{m} Ew_i$, $(\prod_{i=1}^{n}(u - v_i))(\prod_{j=1}^{m} (u - w_j)) \neq 0$, so $\prod_{i=1}^{n} (u - v_i) = \prod_{j=1}^{m}(u - w_j)$.

Let $A \Delta B$ denote $(A\backslash B) \cup (B\backslash A)$. We shall now determine the minimal idempotents of $\ell^1(E)$. Let Ω be the character space of $\ell^1(E)$. For $\phi \in \Omega$ let $J_\phi = \{e \in E : \phi(e) = 1\}$, and for filter S let

$$\psi_S(e) = \begin{cases} 1 & \text{if } e \in J \\ 0 & \text{if } e \notin J \end{cases}$$

for $e \in E$ and extend to $\ell^1(E)$. For $e \in E$ let $J_e = \{f \in E : f \geq e\}$.

Let x be a minimal idempotent of $\ell^1(E)$ and let $\hat{x}$ be its Gelfand transform. Then $\hat{x}(\Omega) = \{0, 1\}$. Suppose $\phi_1(x) = \phi_2(x) = 1,\ \phi_1(y) \neq \phi_2(y)$ for some $y \in \ell^1(E)$. Then $\phi_1(xy) \neq \phi_2(xy)$ although $xy \in \mathbb{C}x$. Therefore $\hat{x}^{-1}(1) = \{\psi\}$ is an open singleton. Therefore there exists $\varepsilon \in (0, 1)$ and finite non-empty subset U of E such that $\{\psi\} = \{\phi \in \Omega : |\phi(u) - \psi(u)| < \varepsilon$ for all $u \in U\} = \{\phi \in \Omega : \phi(u) = \psi(u)$ for $u \in U\}$. So for all $\phi \in \Omega\backslash\{\psi\}$, $(J_\phi \Delta J_\psi) \cap U \neq \emptyset$.

The proof splits into two cases.

Suppose J_ψ is a singleton, say $\{u\}$. Then u is a maximal element of E. Whenever $g < u,\ J_g \Delta J_\psi = \{e \in E : e \neq u$ and $e \geq g\}$, so there exists $e \in U$ such that $e \neq u$ and $e \geq g$. Let $B = \{eu : e \in U,\ e \neq u\}$. Then B is finite and non-empty.

Suppose J_ψ is not a singleton. Suppose $U \cap J_\psi = \emptyset$. Now there exists $f \in J_\psi$ such that $J_\psi \neq J_f$. Then $U \cap (J_f \Delta J_\psi) \subset U \cap J_\psi = \emptyset$, which is impossible. Let $u = \prod\{e : e \in U \cap J_\psi\}$. Then $u \in J_\psi$, and so $J_u \subset J_\psi$. $U \cap (J_u \Delta J_\psi) = U \cap (J_\psi\backslash J_u) = \emptyset$. Therefore $J_u = J_\psi$.

Suppose u is not the minimal element of E, for if it is then $x = u$. Then there exists $f < u$, and so $J_f \neq J_u$. But

$\emptyset \neq (J_u \ \Delta \ J_f) \cap U = U \cap \{e \in E : e \geq f, e \geq u\} = (U \backslash J_\psi) \cap \{e \in E : e \geq f\}$.
Therefore there exists $v \in U$ such that $v \ngeq u$ and $v \geq f$, indeed $vu \geq f$. Let $B = \{vu : v \in (U \backslash J_\psi)\}$.

In either case, suppose u is not the minimal element of E . Let J be a filter distinct from $J_u = J_\psi$. If $u \notin J$, $u \in J \ \Delta \ J_u$. If $u \in J$ there exists $w \in J$ such that $w \ngeq u$, i.e. $wu \neq u$. But there exists $v \in B$ such that $v \geq wu \in J$. Then $v \in J$. But $v \ngeq u$, so $v \in J \ \Delta \ J_u$. Thus in either case, $(\{u\} \cup B) \cap (J \ \Delta \ J_u) \neq \emptyset$.

Now x is the unique solution to $\psi_u(x) = 1$, $\psi_J(x) = 0$ if $J \neq J_u$. If $u \notin J$, then $v \notin J$ for all $v \in B$, so $\psi_J(\prod_{v \in B} (u - v)) = 0$. If $u \in J$ and there exists $v \in B \cap J$, then $\psi_J(\prod_{v \in B} (v - u)) = 0$. If $u \in J$ and $B \cap J = \emptyset$ then $(\{u\} \cup B) \cap (J \ \Delta \ J_u) = \emptyset$, so $J = J_u$. But $\psi_u(\prod_{v \in B} (u - v)) = 1$,

so $x = \prod_{v \in B} (u - v)$.

But if $f < u$, $f \in BE$, so $Eu \backslash \{u\} = \bigcup_{v \in B} Ev$, so all minimal idempotents are as described. $\square$

CHAPTER 2

REPRESENTATIONS AND POSITIVE FUNCTIONALS

§1 Elementary Theory

First I give an account of the elementary theory of *-representations and positive functionals.

Definition 2.1.1

Let A be a *-algebra. Then a Hilbert A-module is a Hilbert space H with a module action such that $\langle a\xi, \eta\rangle = \langle \xi, a^*\eta\rangle$ and $\{\|a\xi\| : \|\xi\| \le 1\}$ is bounded for each $a \in A$.

Definition 2.1.2

A *-representation of a *-algebra A is a *-homomorphism π from A to the bounded linear operators on some Hilbert space H. A *-representation π of A on H will be called irreducible if $\pi \neq 0$ and the only closed subspaces K of H such that $\pi(A)K \subset K$ are 0 and H.

Given a *-representation π of A on H, we may equivalently view H as a Hilbert A-module by $\alpha\xi = \pi(a)\xi$, and vice versa.

Definition 2.1.3

A positive functional f on *-algebra A is a linear functional on A such that $f(x^*x) \ge 0$ for all $x \in A$. Let f be a positive functional on a *-algebra A. Then f is said to be Hermitian if $f(x^*) = f(x)^*$ for all $x \in A$, and admissible if for all $y \in A$ there exists $K_y \ge 0$ such that $f(x^*y^*yx) \le K_y^2 f(x^*x)$ for all $x \in A$.

We shall now see the significance of the constant K_y above.

Let f be a positive functional on A. Then we can define an inner product on A by $\langle x, y\rangle_f = f(y^*x)$, and let $L_f = \{x \in A : f(x^*x) = 0\}$. Let $X_f = A/L_f$. Then X_f is a pre-Hilbert space inheriting the above inner product. Let $\|\ \|_f$ be the associated norm. Then we define an A-module structure on X_f by $a(x + L_f) = ax + L_f$. Then $\langle a(x + L_f), y + L_f\rangle = f(y^*ax) = \langle x + L_f, a^*(y + L_f)\rangle$. But $\|ax + L_f\|^2_f = f(x^*a^*ax)$; so A acts as bounded operators on H if and only if f is admissible. Suppose f is admissible. Then let H_f be the completion of X_f. Then the action of A extends to make H_f a Hilbert A-module.

Now every non-degenerate Hilbert module can be decomposed into an ℓ^2-sum of cyclic Hilbert modules [25] Theorem 4.48. Let ξ generate cyclic Hilbert module H. Then define f on A by $f(a) = \langle a\xi, \xi\rangle$. Then if π is the corresponding *-representation,

$$\|\pi(a^*a)\| = \sup\{\|a\eta\|^2 : \|\eta\|^2 \le 1\} = \sup\{\|ax\xi\|^2 : \|x\xi\|^2 \le 1\}$$
$$= \sup\{f(x^*a^*ax) : f(x^*x) \le 1\}.$$

Definition 2.1.4

A positive function f is representable if there exists cyclic Hilbert module H with cyclic vector ξ such that $f(x) = \langle x\xi, \xi\rangle$.

Theorem 2.1.5

Let f and g be representable positive functions on *-algebra A. Then if $f(xy) = g(xy)$ for all $x, y \in A$, then $f = g$.

Proof

[26] lemma 4.5.10.

The next result is well known.

Lemma 2.1.6

Let f be a positive functional on A. Then f extends to a positive functional on $\tilde{A}$ if and only if it is self-adjoint and there exists $\kappa \geq 0$ such that $|f(x)|^2 \leq \kappa f(x^*x)$ for all $x \in A$.
In the least such extension, $f(1)$ is the least κ for which the inequality holds.

Proof

Let g extend f to $\tilde{A}$. Then f is self-adjoint, and

$g((\lambda 1 + x)^*(\lambda 1 + x)) = |\lambda|^2 g(1) + \lambda^* f(x) + \lambda f(x^*) + f(x^*x) \geq 0$

for all $\lambda \in \mathbb{C}$. Therefore $|f(x)|^2 \leq g(1) f(x^*x)$.

Conversely, let $\tilde{f}(1) = \kappa$. Then

$$\begin{aligned} f((\lambda 1 + x)^*(\lambda 1 + x)) &= |\lambda|^2\kappa + \lambda^* f(x) + \lambda f(x)^* + f(x^*x) \\ &\geq |\lambda|^2\kappa - 2|\lambda||(f(x))| + f(x^*x) \\ &\geq |\lambda|^2\kappa - 2|\lambda|\kappa^{\frac{1}{2}} f(x^*x)^{\frac{1}{2}} + f(x^*x) \\ &= (|\lambda|\kappa^{\frac{1}{2}} - f(x^*x)^{\frac{1}{2}})^2 \geq 0 . \quad \square \end{aligned}$$

Definition 2.1.7

If f can be so extended, the least such κ is called its "essential norm" and denoted $\|f\|$.

Theorem 2.1.8

f is representable if and only if it can be extended to $\tilde{A}$ and is admissible.

Proof

Necessity is clear. Let f be the least extension to $\tilde{A}$. Then the construction after definition 2.1.3 provides the representation. $\square$

Definition 2.1.9

A linear seminorm $|\ |$ on a *-algebra is a B*-seminorm if $|a^*a| = |a|^2$ for all $a \in A$. A norm $|\ |$ on a *-algebra is a C*-norm if it is a B*-seminorm.

Theorem 2.1.10 (Sebestyén [27]

Every B*-seminorm $|\ |$ on a *-algebra satisfies $|ab| \le |a||b|$.

§2 Uniform Admissibility Algebras

For a *-algebra an important consequence of having a complete algebra norm is that every positive function is admissible and the corresponding constants are independent of the function. This follows from Ford's square root lemma, [4] proposition 12.11. We examine the class of algebras with this property, and sidestep the problems of completing and then examining the result.

Definition 2.2.1

A *-algebra A is a uniform admissibility algebra if for all $y \in A$ there exists $K_y \ge 0$ such that $f(x^*y^*yx) \le K_y^2 f(x^*x)$ for all $x \in A$ whenever f is a positive functional on A.

Thus all positive functionals on a uniform admissibility algebra are admissible and, by the argument after definition 2.1.3, there is a greatest B*-seminorm, namely $|y|$ is the least K_y satisfying the above definition. Examples are Banach *-algebras ([4] lemma 37.6), Husain and Warsi's BP*-algebras [15], Palmer's U*-algebras [24], and inverse semigroup rings over $\mathbb{C}$, as we shall see below.

Theorem 2.2.2

Let G generate *-algebra A. Then if for all $g \in G$ there exists $K_g \geq 0$ such that $f(x^*g^*gx) \leq K_g^2 f(x^*x)$ for all $x \in A$ and positive functionals f on A, A is a uniform admissibility algebra.

Proof

Let $B = \{z \in A :$ there exists $M_z > 0$ such that $f(x^*z^*zx) \leq M_z f(x^*x)$ for all $x \in A$ and positive functionals f on $A\}$. B is closed under scalar multiplication. If f is positive, then

$f(x^*(g-h)^*(g-h)x) = f(x^*g^*gx) + f(x^*h^*hx) - f(x^*(g^*h + h^*g)x) \geq 0$.

So if $g, h \in B$ then

$$\begin{aligned} f(x^*(g+h)^*(g+h)x) &= f(x^*g^*gx) + f(x^*h^*hx) + f(x^*(g^*h + h^*g)x) \\ &\leq 2f(x^*g^*gx) + 2f(x^*h^*hx) \\ &\leq 2(M_g + M_h) f(x^*x) \end{aligned}$$

whenever $x \in A$ and f is a positive functional on A, so B is closed under addition. If $g, h \in B$ then

$$f(x^*g^*h^*hgx) \leq M_h f(x^*g^*gx) \leq M_h M_g f(x^*x) ,$$

so B is closed under multiplication.

$M_g f(x^*x) - f(x^*gg^*x) = M_g^{-1} f(x^*((M_g - gg^*)^2 + g(M_g - g^*g)g^*)x) \geq 0$

if $g \in B$, so B is closed under involution. Thus $B = A$. □

It is immediate that unitisations, direct sums (by decomposing the positive functionals onto the summands), directed unions (because the bounding constant is given by the greatest B*-seminorm and every B*-seminorm restricts to a B*-seminorm on each *-subalgebra) and images, because positive functionals induce positive functionals on the original algebra, are all uniform admissibility algebras. Subalgebras need not be, for let S be the free semigroup in one indeterminate. Then $k(S) \subset l^1(S)$, but the former has inadmissible positive functionals and no greatest B*-seminorm.

Proposition 2.2.3

Let A and B be uniform admissibility algebras. Then $A \otimes B$ is a uniform admissibility algebra.

Proof

Let A and B be uniform admissibility algebras and f be a positive functional on $A \otimes B$, and let $x = \sum_i x_i \otimes y_i \in A \otimes B$. Then for $u \in A$ and $v \in B$,

$$f(x^*(u \otimes v)^*(u \otimes v)x) = \sum_{i,j} f(x_i^* u^* u x_j \otimes y_i{}^* v^* v y_j) .$$

Now $z \to \sum_{i,j} f(x_i^* z x_j \otimes y_i^* v^* v y_j)$ is a positive functional on $\tilde{A}$, and thus

$$\sum_{i,j} f(x_i^* u^* u x_j \otimes y_i^* v^* v y_j) \leq |u^*u| \sum_{i,j} f(x_i^* x_j \otimes y_i^* v^* v y_j)$$

where $|\ |$ is the greatest B*-seminorm on A . Similarly

$$\sum_{i,j} f(x_i^* x_j \otimes y_i^* v^* v y_j) \leq |v^*v| \sum_{i,j} f(x^*_i x_j \otimes y_i^* y_j)$$

where $|\ |$ is the greatest B*-seminorm on B . Thus $f(x^*(u \otimes v)^*(u \otimes v)x) \leq |u|^2|v|^2 f(x^*x)$. But the $u \otimes v$ span $A \otimes B$, so the positive functionals on $A \otimes B$ are uniformly admissible. □

Proposition 2.2.4

Let I be a *-ideal of a uniform admissibility algebra. Then I is a uniform admissibility algebra.

Proof

Let f be a positive functional on I . For $x \in I$, $y \in \tilde{A}$, let $f_x(y) = f(x^*yx)$. Then f_x is a positive functional on $\tilde{A}$. Let $|\ |$ be the greatest B*-seminorm on $\tilde{A}$. Then as f_x is representable, $f_x(y^*y) \leq |y|^2 f_x(1) = |y|^2 f(x^*x)$.

Therefore, for all positive functionals f on I and elements, y, x of I, $f(x^*y^*yx) \leq |y|^2 f(x^*x)$. $\square$

Definition 2.2.5

Let A be a *-algebra with a greatest B*-seminorm $|\ |$. Then the enveloping C*-algebra of A is the completion of $(A/I, |\ |)$ where $I = \{x \in A : |x| = 0\}$, and is denoted $C^*(A)$. If $I = 0$, A is called *-semisimple and will often be regarded as a subalgebra of $C^*(A)$. I is known as the *-radical, and is a hereditary radical.

Then every *-representation π of A extends to a unique *-representation $\tilde{\pi}$ of $C^*(A)$, and every *-representation π of $C^*(A)$ induces a *-representation of A, and π is irreducible if and only if $\tilde{\pi}$ is. Then for $x \in A$, $|x| = \sup\{\|\pi(x)\| : \pi$ is a *-representation$\} = \sup\{\|\pi(x)\| : \pi$ is an irreducible *-representation:$\}$, where $\sup \emptyset$ is defined to be 0.

Closely related to the idea of the proof of proposition 2.2.4 is the problem of extending *-representations from ideals to algebras. Results using approximate identities can be found in [7] and [17]. For arbitrary Banach *-algebras the result may be found in Leptin [18]. Sebestyén [27] determines when a particular representation may be extended.

A linear operator S on algebra A is a left multiplier if $S(xy) = (Sx)y$ for all $x, y \in A$ and similarly a linear operator T on algebra A is a right multiplier if $T(xy) = x(Ty)$ for all $x, y \in A$.

The double centraliser [17] of A is the algebra of pairs (S, T) of linear operators on A such that S is a left and T a right multiplier and $x(Sy) = (Tx)y$ with

$$\lambda(S, T) = (\lambda S, \lambda T) \quad \text{for} \quad \lambda \in \mathbb{C}$$

$$(S, T) + (U, V) = (S + U, T + V)$$

$$(S, T)(U, V) = (SU, VT)$$

For $x = (S, T)$ and $a, b \in A$, let $xa = Sa$, $ax = Ta$, and $axb = a(Sb) = (Ta)b$. Then any formal product of at least one element of A and elements of the double centraliser is well defined and independent of the bracketing.

Any involution on A can be lifted to the double centraliser by $(S, T)^* = (T^*, S^*)$ where $V^*(x) = (V(x^*))^*$.

<u>Theorem 2.2.6</u>

Let A be a uniform admissibility *-subalgebra of the double centraliser of *-algebra B. Then any non-degenerate *-representation π of B on H determines a unique *-representation π^b of A on H such that $\pi(ab) = \pi^b(a)\pi(b)$.

<u>Proof</u>

I use the method of [28]4.1.

Let A, B, H and π be as above. Without loss of generality, A has a unit. Let $|\ |$ be the greatest B*-seminorm on A. For $\xi = \sum_i \pi(b_i)\xi_i$ where $b_i \in B$, $\xi_i \in H$ define f_ξ on A by $f_\xi(x) = \sum_i \langle \pi(xb_i)\xi_i, \xi\rangle$. Now for $b \in B$, $\eta \in H$,

$$\langle \sum_i \pi(xb_i)\xi_i, \pi(b)\eta\rangle = \sum_i \langle \pi(b^*xb_i)\xi_i, \eta\rangle = \sum_i \langle \pi(b_i)\xi_i, \pi(x^*b)\eta\rangle,$$

so f_ξ is well defined. Now

$$f_\xi(x^*x) = \sum_{i,j} \langle \pi(x^*xb_i)\xi_i,\ \pi(b_j)\ \xi_j \rangle = \sum_{i,j} \langle \pi(b_j^*x^*xb_i)\xi_i,\ \xi_j \rangle$$

$$= \|\sum_i \pi(xb_i)\xi_i\|^2 \geq 0 \ ,$$

so f_ξ is positive. By hypothesis f_ξ is admissible, and hence $f_\xi(x^*x) \leq |x|^2 f_\xi(1) = |x|^2\|\xi\|^2$. Therefore

$\pi^b(x)(\sum_i \pi(b_i)\xi_i) = \sum_i \pi(xb_i)\xi_i$ defines a bounded operator on $\pi(B)H$, so can be extended to $H = \overline{\pi(B)H}$, so π^b is a *-representation of A on H.

Suppose $\tau(a)$ were another such representation. Then $\tau(a)\xi = \pi^b(a)\xi$ for all $\xi \in \pi(B)H$, which is dense in H, so $\tau = \pi^b$. □

Corollary 2.2.7

Let I be a *-ideal of uniform admissibility algebra A. Then every non-degenerate *-representation π of I on H extends to a unique *-representation of A on H.

Proof

Let A, I, π and H be as above. We shall produce a *-homomorphism from A to the double centraliser of I, and thus extend π.

For $a \in A$ define linear operators L_a and R_a on I by $L_a x = ax$ and $R_a x = xa$. Then $x(L_a y) = x(ay) = (xa)y = (R_a x)y$ for $x, y \in I$. $L_a^* x = (L_a x^*)^* = (ax^*)^* = xa^* = R_{a^*}x$, so $(L_a, R_a)^* = (R_a^*, L_a^*) = (L_{a^*}, R_{a^*})$. Now $(L_{ab}, R_{ab}) = (L_a L_b, R_b R_a) = (L_a, R_a)(L_b, R_b)$, so $T : a \to (L_a, R_a)$ is a *-homomorphism from A to the double centraliser of I.

Let π^{b} be the $*$-representation of $T(A)$ on H induced by π. Then define $\tilde{\pi}$ on A by $\tilde{\pi}(a) = \pi^{b}(Ta)$. Now for $x, y \in I$, $\xi \in H$. $\pi^{b}(Tx)\pi(y)\xi = \pi(Tx)y)\xi = \pi(x)\pi(y)\xi$, so $\pi(x) = \pi^{b}(Tx) = \tilde{\pi}(x)$. Then if $a \in A$ and $x \in I$, $\pi(ax) = \pi((Ta)x) = \pi^{b}(Ta)\pi(x) = \tilde{\pi}(a)\pi(x)$. Let σ be a $*$-representation of A on H extending π. Then if $a \in A$ and $x \in I$, $\sigma(ax) = \sigma(a)\sigma(x) = \sigma(a)\pi(x)$, so $(\sigma(a) - \tilde{\pi}(a))\pi(I)H = \{0\}$ so $\sigma(a) = \pi(a)$. □

Corollary 2.2.8

Let A and B be Banach $*$-algebras. Then $C^*(A \otimes B) = C^*(A \hat{\otimes} B)$.

Proof

$A \otimes B$ is dense in $A \hat{\otimes} B$. Every $*$-representation of $A \hat{\otimes} B$ restrictes to a $*$-representation of $A \otimes B$. Every $*$-representation π of $A \otimes B$ gives rise to $*$-representations π_A of A and π_B of B such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$, which extends to $A \hat{\otimes} B$ by the continuity of π_A and π_B and the nature of the norm of $A \hat{\otimes} B$. □

A norm $\|\ \|$ on the tensor product of normed spaces A and B is called a cross-norm if $\|a \otimes b\| = \|a\|\,\|b\|$ for all $a \in A$ and $b \in B$.

Corollary 2.2.9 (Guichardet [12])

The greatest B^*-seminorm on the tensor product of C^*-algebras is a cross-norm.

Definition 2.2.10

The completion of the tensor product of C^*-algebras A and B in the greatest B^*-seminorm (which is a norm) will be denoted by $A \otimes_{max} B$.

Corollary 2.2.11

$C^*(A \otimes B) = C^*(A) \otimes_{max} C^*(B)$ for uniform admissibility algebras.

Proof

$(A/_{*\text{-rad}(A)}) \otimes (B/_{*\text{-rad}(B)})$ is dense in $C^*(A) \otimes C^*(B)$. Every *-representation of $C^*(A) \otimes_{max} C^*(B)$ restricts and then lifts to a *-representation of $A \otimes B$. Every *-representation π of $A \otimes B$ gives rise to *-representations π_A of A and π_B of B such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$, which induces a *-representation of $C^*(A) \otimes C^*(B)$, and then extends to $C^*(A) \otimes_{max} C^*(B)$. □

Leptin et alii [2] established the next result for Banach *-algebras.

Let I be a *-ideal of A. Now let $| \;|_A$ be the maximal B^*-seminorm on A, and $| \;|_I$ be the maximal B^*-seminorm on I. Then for $x \in I$, $|x|_I = |x|_A$, so $C^*(I)$ naturally embeds as a *-ideal of $C^*(A)$. If $A \to A/I$ is the quotient homomorphism, then $A \to A/I \to C^*(A/I)$ is a *-homomorphism where the second map is the natural one to the enveloping C^*-algebra. Then this induces a natural map $C^*(A) \to C^*(A/I)$.

Corollary 2.2.12

Let I be a *-ideal of uniform admissibility algebra A. Then if all the maps are canonical,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^*(I) & \longrightarrow & C^*(A) & \longrightarrow & C^*(A/I) & \longrightarrow & 0 \end{array}$$

commutes and the horizontal sequences are exact.

Proof

Let us label some of the maps as follows:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "i_1"] \arrow[d, "\pi_I"] & A \arrow[r, "p_1"] \arrow[d, "\pi_A"] & A/I \arrow[r] \arrow[d, "\pi_Q"] & 0 \\
0 \arrow[r] & C^*(I) \arrow[r, "i_2"'] & C^*(A) \arrow[r, "p_2"'] & C^*(A/I) \arrow[r] & 0
\end{tikzcd}$$

All that remains to be established is that $i_2(C^*(I)) = \operatorname{Ker} p_2$. Now $\pi_Q p_1 i_1 = 0$, so $p_2 i_2 \pi_I = 0$, so $i_2\pi_I(I) \subset \operatorname{Ker} p_2$. But $\pi_I(I)$ is dense in $C^*(I)$, so $i_2(C^*(I)) \subset \operatorname{Ker} p_2$.

Let $x \in \operatorname{Ker}(p_2)$. Then there exists $(a_n) \subset A$ such that $\pi_A(a_n) \to x$. Then $\pi_Q p_1(a_n) \to 0$. Any $*$-representation π of A such that $\operatorname{Ker}_\pi \supset I$ gives a $*$-representation of $C^*(A/I)$. Define ψ on A by $\psi(z) = \pi_A(z) + \overline{\pi_A(I)}$, so $\psi : A \to \pi_A(A)/\pi_A(I)$. Then $\psi(a_n) \to 0$. Therefore there exists $(i_n) \subset I$ such that $\|\pi_A(a_n) - \pi_A(i_n)\| \to 0$. Therefore $x \in \pi_A(I) = i_2(C^*(I))$. □

§3 Inverse Semigroup Rings

We now apply some of this theory to inverse semigroup rings.

Theorem 2.3.1

Let S be an involutive semigroup in which for all $s \in S$, s^*s is an idempotent. Then

(i) if $k(S)$ has proper involution, i.e. $x^*x = 0$ only if $x = 0$, then S is an inverse semigroup, and

(ii) $k(S)$ is a uniform admissibility algebra.

Proof

(i) Let $s \in S$. Then

$(s - ss^*s)^*(s - ss^*s) = (s^* - s^*ss^*)(s - ss^*s) = s^*s - 2(s^*s)^2 + (s^*s)^3 = 0$.

Thus $s = ss^*s$. Let $e = e^2 \in S$. Then

$$(e - e^*)^3 = (e - e^*)(e - e^*e - ee^* + e^*) = e - ee^*e - ee^* + ee^*$$
$$- e^*e + e^*e - e^*ee^* + e^* = 0 .$$

But $h = i(e - e^*)$ is self-adjoint, and $h^4 = 0$. So $h^2 = 0$, so $h = 0$, so $e = e^*$. Thus the idempotents commute. Suppose $aba = a$, $bab = b$ and $aca = a$, $cac = c$. Then

$ab = (aca)b = (ac)(ab) = abac = ac$, and similarly $ba = ca$. Then $b = bab = bac = cac = c$, so S is an inverse semigroup.

(ii) Let $s \in S$ and $x \in k(S)$ and f be a positive functional on $k(S)$. Then $(1 - s^*s)^2 = (1 - s^*s)$, so

$f(x^*(1 - s^*s)x) = f(x^*x) - f(x^*s^*sx) \geq 0$ for all $x \in k(S)$. Then $k(S)$ is a uniform admissibility algebra by theorem 2.2.2. □

Some conditions must be imposed to force inverseness, since there exists semigroups such that s^*s is idempotent but $s = ss^*s$ may fail, and semigroups such that s^*s is idempotent and $s = ss^*s$ yet are not inverse semigroups.

Proposition 2.3.2

Let S be an ~~inverse~~ involutive semigroup with $S = S^2$. Then every positive functional on $k(S)$ is Hermitian.

Proof

Let $s = tu$. Then $f(s)^* = f(tu)^* = f(u^*t^*) = f(s^*)$, the central equality resulting from [4] lemma 37.6(ii).

Definition 2.3.3

A net (u_α) is an ultimate identity if $u_\alpha x = xu_\alpha = x$ eventually.

Lemma 2.3.4

Let S be an inverse semigroup. Then $k(S)$ has a self-adjoint idempotent ultimate identity.

Proof

Let ξ be the set of finite subsets of E_S ordered by inclusion. For $F \in \xi$, let $u_F = 1 - \prod_{e \in F} (1 - e)$. Then $(u_F)_{F \in \xi}$ is a self-adjoint idempotent ultimate identity. $\square$

This contrasts strongly with the fact that $\ell^1(S)$ may lack a bounded approximate identity, for Duncan and Namioka [9] proved that $\ell^1(S)$ has a bounded approximate identity if and only if there is a finite k such that every finite subset of E_S lies in the union of k principal ideals of E_S. When it exists, their bounded approximate identity is an ultimate identity for $k(S)$. We can use these self-adjoint idempotent ultimate identities to test for representability.

Lemma 2.3.5

Let S be an inverse semigroup and (u_α) be a self-adjoint idempotent ultimate identity for $k(S)$. Then for positive f and $\kappa \geq 0$, the following are equivalent:

(i) $|f(x)|^2 \leq \kappa f(x^*x)$ for all $x \in k(S)$;

(ii) $\lim_\alpha f(u_\alpha) \leq \kappa$;

(iii) $\sup_\alpha f(u_\alpha) \leq \kappa$.

Proof

(i) $\Longrightarrow$ (iii)

Assume (i) holds. Let $\tilde{f}$ be an extension of f with $\tilde{f}(1) = \kappa$. Then $\tilde{f}(1 - u_\alpha) \geq 0$, so $\sup_\alpha f(u_\alpha) \leq \tilde{f}(1) = \kappa$.

(iii) $\Longrightarrow$ (ii)

Assume (iii) holds. Given α there exists β such that for all $\gamma \geq \beta$, $u_\gamma u_\alpha = u_\alpha u_\gamma = u_\alpha$. Then $(u_\gamma - u_\alpha)^2 = u_\gamma - u_\alpha$, so $f(u_\gamma) \geq f(u_\alpha)$. Therefore $\lim_\alpha f(u_\alpha) = \sup_\alpha f(u_\alpha) \leq \kappa$.

(ii) $\Longrightarrow$ (i)

Assume (ii) holds. Given $x \in k(S)$, $u_\alpha x = xu_\alpha = x$ for α large enough. Then

$$0 \le f((u_\alpha - \lambda x)^*(u_\alpha - \lambda x)) = f(u_\alpha) - \lambda f(x) - \lambda^* f(x)^* + |\lambda|^2 f(x^*x)$$

so $|f(x)|^2 \le f(u_\alpha)f(x^*x)$. Thus

$$|f(x)|^2 \le \lim_\alpha f(u_\alpha)f(x^*x) \leqq \kappa f(x^*x) . \qquad \square$$

We now extend a result of Godement [13] from groups to involutive semigroups.

Lemma 2.3.6

Let (a_{ij}) and (b_{ij}) be positive $n \times n$ matrices, i.e. $\sum_{i,j} \xi_i^* b_{ij} \xi_j \ge 0$ for all $(\xi_i)_{i=1}^n$. Then $(a_{ij}b_{ij})$ is also positive.

Proof

If (a_{ij}) is positive, then $(a_{ij}) = (c_{ij})^*(c_{ij})$ for some matrix (c_{ij}), where $(c_{ij})^* = (c_{ji}^*)$, and if (b_{ij}) is positive then $(b_{ij}) = (d_{ij})^*(d_{ij})$, say. Then

$$\sum_{i,j} \xi_i^* a_{ij} b_{ij} \xi_j = \sum_{i,j,k,l} \xi_i^* c_{ki}^* c_{kj} d_{li}^* d_{lj} \xi_j$$

$$= \sum_{k,l} \sum_i (c_{ki} d_{li} \xi_i)^* \sum_j c_{kj} d_{ij} \xi_j$$

$$= \sum_{k,l} \Big|\sum_i c_{ki} d_{li} \xi_i\Big|^2 \ge 0 . \qquad \square$$

Corollary 2.3.7

For f and g positive functionals on $k(S)$, define fg by $fg(s) = f(s)g(s)$. Then fg is a positive functional.

Proof

Let $x \in k(S)$ and f and g be positive functionals on $k(S)$. Let $I = \mathrm{supp}(x)$. Then let $a_{st} = f(s^*t)$ and $b_{st} = g(s^*t)$. Then $fg(x^*x) \geq 0$ by lemma 2.3.6. □

§4 The Left Regular *-Representation of Inverse Semigroups

In this section S will be an inverse semigroup. We will study the analogue of the left regular representation of a group.

By the proof of theorem 2.3.1, every *-representation of $k_\theta(S)$ extends to $\ell^1_\theta(S)$, and vice versa by restriction. We will now show that $\ell^1_\theta(S)$ is *-semisimple. The proof is very similar to that for groups.

For a homomorphism T from one algebra to another, T^* will denote the corresponding algebraic adjoint, and will be used solely as a notational device. For Banach space homomorphisms it will denote the topological adjoint.

Definition 2.4.1 (Barnes [1])

We can define the left regular "*-representation" of $k_\theta(S)$ on $k_\theta(S)$ by

$$\lambda_S(a)b = \begin{cases} ab & \text{if } a^*ab = b \\ 0 & \text{otherwise} \end{cases}$$

for a and b in S, and extending by linearity. Then we extend it to the left regular *-representation of $\ell^1_\theta(S)$ on $\ell^2_\theta(S)$ by continuity. Note that if $ab = \theta$ and $a^*ab = b$ then $b = \theta$. Now for $a, b, c \in S$, $\lambda_S(a)b = c \Longleftrightarrow ab = c$ and $a^*ab = b \Longleftrightarrow b = a^*c$ and $aa^*c = c \Longleftrightarrow b = \lambda_S(a^*)c$, so it is *-representation. We shall sometimes write λ_x for $\lambda_S(x)$.

Theorem 2.4.2 (Wordingham [30])

The left regular *-representation of $\ell^1_\theta(S)$ on $\ell^2_\theta(S)$ is faithful.

Proof

We shall regard $\ell^1_\theta(S)$ as a subspace of $\ell^2_\theta(S)$.

Let $x \in \ell^1_\theta(S)$ and suppose $\lambda_x = 0$. Then for all $e \in E\backslash\{\theta\}$, $\lambda_x e = 0$. For $e \in E\backslash\{\theta\}$, define x_e by $x_e(s) = x(s)$ if $s^*s = e$ and 0 otherwise. Then $x_e \in \ell^1_\theta(S)$ and $x = \sum_{e \in E\backslash\{\theta\}} x_e$. For $f \in E$,

$$\lambda_{x_e} f = \begin{cases} x_e f & \text{if } e \geq f \\ 0 & \text{otherwise} \end{cases} .$$

Pick $u \in E\backslash\{\theta\}$ and let $F = \{e \in E : e \geq u\}$. Then F is a semilattice. Then for $f \in F$, $\lambda_x f = \sum_{e \geq f} x_e f = 0$, so $0 = (\lambda_x f)u = \sum_{e \geq f} x_e fu = \sum_{e \geq f} x_e u$. Now

$$\sum_{f \in F} ||x_f u|| \leq \sum_{f \in F} ||x_f|| \leq ||x|| ,$$

so let us define $w_s \in l^1(F)$ by $\omega_s(f) = (x_f u)(s)$. But for all $f \in F$, $\sum_{e \geq f} w_s(e) = \sum_{e \geq f} x_e u(s) = 0$, so by lemma 1.4 $w_s = 0$.

Now $x_u = x_u u$, so $x_u(s) = x_u u(s) = w_s(u) = 0$, so $x_u = 0$. But u was arbitrary, so $x = 0$. $\square$

Barnes [1] proved $\ell^1_\theta(S)$ had a faithful *-representation by imbedding it in an inverse semigroup algebra whose semilattice was a lattice, and proving that the latter's left regular *-representation was faithful. The fidelity of the corresponding representation of $k_\theta(S)$ is easier to show, (W.D. Munn, personal communication).

The following theorem makes the structure of the left regular *-representation easier to examine. Recall the Green's equivalences $\mathcal{L}$ and $\mathcal{D}$.

Theorem 2.4.3

For $e \in E$, let $L_e = \{s \in S : s^*s = e\}$. Each $\ell^2(L_e)$ with $e \neq \theta$ is a Hilbert $\ell^1(S)$-module under the left regular *-representation, and if $e \mathcal{D} f$ then $\ell^2(L_e)$ and $\ell^2(L_f)$ are isomorphic Hilbert modules.

Proof

Let $s \in S$ and $e \in E$. Then if $t \in L_e$, $\lambda_s t = 0$ or st . If $\lambda_s t = st$, then $s^*st = t$, so $t^*s^*st = t^*t$, so $st \in L_e$. Therefore $\ell^2(L_e)$ is an $\ell^1_\theta(S)$-module, and thus a Hilbert $\ell^1_\theta(S)$-module.

If $e \mathcal{D} f$ then by definition there exists $x \in S$ such that $e = xx^*$ and $f = x^*x$. Then if $s^*s = e$, $x^*s^*sx = x^*ex = f$. Let $\pi: L_e \to L_f$ by $\pi(s) = sx$. Now $\pi(s)x^* = sxx^* = se = s$ for $s \in L_e$, so π is an injection. If $t \in L_f$, $tx^* \in L_e$ and $\pi(tx^*) = tx^*x = tf = t$, so π is a bijection. Thus π lifts to a Hilbert space isomorphism. Let $s \in L_e$ and $t \in S$. Then

$$\lambda_t(\pi(s)) = \lambda_t(sx) = \begin{cases} tsx & \text{if } t^*t \geq sxx^*s^* = ss^* \\ 0 & \text{otherwise} \end{cases} = \pi(\lambda_t s) ,$$

so $\ell^2(L_e)$ and $\ell^2(L_f)$ are isomorphic Hilbert modules. $\square$

Note that if I is an ideal and $e \in E$, then $L_e \subset I$ or $L_e \cap I = \emptyset$. Then $\ell^2(S) = \ell^2(S\backslash I) \oplus \ell^2(I)$ is a Hilbert $\ell^1(S)$-module decomposition.

The claim of the left regular *-representation to be a generalisation of the left regular representation of a group is further strengthened by the following two results.

<u>Proposition 2.4.4</u>

Let $\lambda^{(p)}$ denote the left regular representation of S on $\ell^p(S)$ defined as in definition 2.4.1. Then for $1 < p < \infty$, $\lambda_s^{(p)*} = \lambda_{s^*}^{(q)}$, where

$$\frac{1}{p} + \frac{1}{q} = 1 \ , \quad \text{and} \quad \lambda_s^{(\infty)*}\Big|_{\ell^1(S)} = \lambda_{s^*}^{(1)} \ .$$

<u>Proof</u>

Let $f \in \ell^p(S)$ and $g \in \ell^q(S)$.

Then

$$\langle g, \lambda_s^{(p)} f\rangle = \sum\{g(t)f(s^*t) : ss^*t = t\}$$

$$= \sum\{g(su)f(u) : s^*su = u\} = \langle\lambda_{s^*}^{(q)} g, f\rangle \ . \quad \square$$

<u>Theorem 2.4.5</u>

Let ℓ_s be left translation by s . (Then for $f \in \ell^1(S)$, $\ell_s(f)(t) = \sum\{f(u) : t = su\}$.) Let μ be a mean on S . Then μ is ℓ-invariant if and only if it is λ-invariant.

<u>Proof</u>

Let $s, x \in S$. Then $\ell_{x^*x} s = x^*xs$. Now $(x^*x)x^*xs = x^*xs$ so $\ell_{xx^*x} s = xx^*xs = xs = \ell_x s$.

$$\lambda_{x^*x} s = \begin{cases} x^*xs & \text{if } x^*xs = s \\ 0 & \text{otherwise} \end{cases} \ .$$

$$\ell_x \lambda_{x^*x} s = \begin{cases} xs & \text{if } x^*xs = x \\ 0 & \text{otherwise} \end{cases} = \lambda_x s \ .$$

Therefore $\lambda_x \ell_{x^*x} = \ell_x$ and $\ell_x \lambda_{x^*x} = \lambda_x$. Therefore $\lambda^{**}_x \ell^{**}_{x^*x} = \ell^{**}_x$ and $\ell^{**}_x \lambda^{**}_{x^*x} = \lambda^{**}_x$. Let μ be an ℓ-invariant mean. Then

$$\begin{aligned} \lambda^{**}_x \mu &= \lambda^{**}_x \ell^{**}_{x^*x}\mu \quad \text{by } \ell\text{-invariance} \\ &= \ell^{**}_x \mu \\ &= \mu \quad \text{by } \ell\text{-invariance,} \end{aligned}$$

so μ is λ-invariant. Similarly, if μ is λ-invariant, μ is ℓ-invariant. □

Let $C^*(S)$ denote $C^*(k(S)) = C^*(\ell^1(S))$ and $C^*_\theta(S)$ denote $C^*(k_\theta(S)) = C^*(\ell^1_\theta(S))$. We may write $C^*(S\backslash I)$ for $C^*_\theta(S/I)$. Let $C^*_r(S)$ denote the completion of $\lambda_S(\ell^1(S))$, and $C^*_{r,\theta}(S)$ denote the completion of $\lambda_S(\ell^1_\theta(S))$, etc.. The question naturally arises of when $C^*(S) = C^*_r(S)$, or $C^*_\theta(S) = C^*_{r,\theta}(S)$.

A *-representation S of *-algebra is said to weakly contain another *-representation T if there is a *-homomorphism U such that

$$\begin{tikzcd} & \overline{S(A)} \arrow[dd, "U"] \\ A \arrow[ur, "S"] \arrow[dr, "T"'] & \\ & \overline{T(A)} \end{tikzcd}$$

commutes. Proposition 2.4.8 and theorem 2.4.9, from Fell [10], are the crucial lemmas in the discussion of this question. Recall my definition of essential norm, definition 2.1.7.

<u>Notation 2.4.6</u>

For A a *-algebra, let A^{*+} be the set of positive functionals on A and let $P(A)$ be the set of positive functionals on A of essential norm ≤ 1.

The importance of these results is that we need only consider positive functionals on the *-algebras. If T is a *-representation of A and f a positive functional on $\overline{T(A)}$, then f induces a positive functional on A, namely $x \to f(Tx)$, of the same essential norm, as we show below. The set of such functionals is denoted $T^*(\overline{T(A)}^{*+})$ rather than $(\overline{T(A)}^{*+})T$.

<u>Proposition 2.4.7</u>

Let π be a *-representation of *-algebra A and let f be a positive functional on $\overline{\pi(A)}$. Then $||\pi^*(f)|| = ||f||$.

<u>Proof</u>

Clearly $||\pi^*(f)|| \leq ||f||$. Now

$$|f(\pi(x))|^2 \leq ||\pi^*(f)|| f(\pi(x^*x)) .$$

Therefore $||f|| \leq ||\pi^*(f)||$ by continuity of f on $\overline{\pi(A)}$. □

<u>Proposition 2.4.8</u>

Let π be a *-representation of A on H. Then $\pi^*(P(\overline{\pi(A)}))$ is the weak*-closure of the convex hull of $\{x \to \langle\pi(x)\xi, \xi\rangle : \xi \in H \text{ and } ||\xi|| \leq 1\}$.

<u>Proof</u>

Fell [10] theorem 1.1. □

<u>Theorem 2.4.9</u>

Let S and T be *-representations of A. The following are equivalent:

(i) $S^*(\overline{S(A)}^{*+}) \supset T^*(\overline{T(A)}^{*+})$

(ii) $S^*(P(\overline{S(A)})) \supset T^*(P(\overline{T(A)}))$

(iii) S weakly contains T.

Proof

(i) $\Longrightarrow$ (ii)

By proposition 2.4.7.

(ii) $\Longrightarrow$ (iii)

Suppose (ii) holds. Then

$$\|Tx\| = \sup\{f(x^*x) \ : f \in T^*(P(\overline{TA}))\} \le \sup\{f(x^*x) \ : f \in S^*(P(\overline{S(A)}))\}$$
$$= \|Sx\| \ .$$

Define $U : S(A) \to T(A)$ by $U(S(x)) = Tx$. Then U is continuous, so it can be extended to $\overline{S(A)}$ by continuity.

(iii) $\Longrightarrow$ (i)

Suppose (iii) holds. Then

$$\begin{tikzcd} & A \arrow[dl, "S"'] \arrow[dr, "T"] & \\ \overline{S(A)} \arrow[rr, "U"'] & & \overline{T(A)} \end{tikzcd}$$

commutes, and U is a $*$-homomorphism. Let $\phi \in \overline{T(A)}^{*+}$. Then $u^*\phi \in \overline{S(A)}^{*+}$. Then for $x \in A$, $T^*\phi(x) = \phi(Tx) = \phi(USx) = S^*U^*\phi(x)$, so $T^*\phi = S^*(U^*\phi)$. □

One says an inverse semigroup S has the weak containment property (abbreviated w.c.p.) if λ_S weakly contains all other $*$-representations of S , i.e. if $C^*(S) = C_r^*(S)$, regarding the algebras as completions of $\ell^1(S)$. I will not equate them if they should be isomorphic but with no isomorphism of this form.

Lemma 2.4.10

Let A be a C*-algebra, I an ideal thereof, and H a faithful Hilbert A-module. Then if $x \in \mathrm{ann}_A(I)$, either $x = 0$ or there exists $\xi \in H$ such that $x\xi \notin \overline{IH}$.

Proof

$\mathrm{ann}_A(I)$ is a closed ideal, so it is a *-ideal. Let $x \in \mathrm{Ann}_A(I)$, $\xi \in H$. Suppose $x\xi \in \overline{IH}$ for all $\xi \in H$. Then

$$x\xi = \lim_{n\to\infty} \sum_{r=1}^{N_n} a_{n,r}\,\xi_{n,r}$$

where $(a_{n,r}) \subset I$ and $(\xi_{n,r}) \subset H$. Then

$$x^*x\xi = \lim_{n\to\infty} \sum_{r=1}^{N_n} x^*a_{n,r}\xi_{n,r} = 0 \quad .$$

Therefore $x^*x = 0$, so $x = 0$. □

Theorem 2.4.11

Let I be an ideal of inverse semigroup S . If $C^*(I) = C_r^*(I)$ and $C_\theta^*(S/I) = C_{r,\theta}^*(S/I)$ then $C^*(S) = C_r^*(S)$.

Proof

Suppose $C^*(I) = C_r^*(I)$ and $C_\theta^*(S/I) = C_{r,\theta}^*(S/I)$.

Let $x \in C^*(S)$. Now there exist $(y_n) \subset k_\theta(S/I)$ and $(z_n) \subset k(I)$ such that $x = \lim_n (y_n + z_n)$ in $C^*(S)$. Suppose $\lambda_S(x) = 0$. Then $\lambda_S(xa) = 0$ for all $a \in C^*(I)$. $xa \in C^*(I)$, so $\lambda_I(xa) = \lambda_S(xa)|_{\ell^2(I)} = 0$. But $C^*(I) = C_r^*(I)$, so $xa = 0$. Similarly $ax = 0$ for all $a \in C^*(I)$. Thus $x \in \mathrm{ann}_{C^*(S)} C^*(I)$.

By corollary ~~6.2~~ 2.2.12, $C^*(S)/C^*(I) = C_\theta^*(S/I)$. Let $\xi \in \ell^2(S/I)$. Then

$$\lambda_{S/I}(x + C^*(I))\xi = \lim_{n\to\infty} \lambda_{S/I}(y_n + z_n + C^*(I))\xi = \lim_{n\to\infty} \lambda_{S/I}(y_n)\xi$$

$$= \lim_{n\to\infty} \lambda_S(y_n + z_n)\xi = \lambda_S(x)\xi = 0 \ .$$

Therefore $x \in C^*(I)$. Therefore $x = 0$. □

Theorem 2.4.12

Let S be an inverse semigroup with zero. Then $C^*(S) = C^*_r(S)$ if and only if $C^*_\theta(S) = C^*_{r,\theta}(S)$.

Proof

$C^*_\theta(S) = C^*_{r,\theta}(S)$ implies $C^*(S) = C^*_r(S)$ by theorem 2.4.11.

Suppose $C^*(S) = C^*_r(S)$. Let $x \in C^*(S)$ and $\xi \in \ell^2_\theta(S)$. We shall use coset notation. $\lambda_{S\setminus\{\theta\}}(x + \mathbb{C}\,\theta)(\xi + \mathbb{C}\,\theta) = \lambda_S(x)\xi + \mathbb{C}\,\theta$. Suppose $\lambda_{S\setminus\{\theta\}}(x + \mathbb{C}\,\theta)\xi = 0$ for all $\xi \in \ell^2(S\setminus\{\theta\})$. Then $\lambda_S(x)\xi \in \mathbb{C}\,\theta$ for all $\xi \in \ell^2(S)$. Let $\mu\theta = \lambda_S(x)\theta$. Then $\lambda_S(x - \mu\theta) \in \mathrm{ann}_{C^*(S)}(\mathbb{C}\theta)$. But $\lambda_S(x - \mu\theta)\xi \in \mathbb{C}\theta$ for all $\xi \in \ell^2(S)$, so by lemma 2.4.10, $x = \mu\theta$. Therefore $\lambda_{S\setminus\{\theta\}}$ is faithful on $C^*_\theta(S)$. □

Theorem 2.4.13

Let I be an ideal of inverse semigroup S . If $C^*(S) = C^*_r(S)$, then $C^*(I) = C^*_r(I)$.

Proof

Suppose $C^*(S) = C^*_r(S)$.

Now $\ell^2(S) = \ell^2(S\setminus I) \oplus \ell^2(I)$, and this is a Hilbert module decomposition. For $x \in \ell^1(S)$,

$$||x||_{C^*(S)} = ||\lambda_S x|| = \sup\{||\lambda_x \xi||_2 : \xi \in \ell^2(S) \text{ and } ||\xi||_2 = 1\} .$$

So for $x \in \ell^1(I)$, $||x||_{C^*(I)} = ||x||_{C^*(S)}$ by corollary 2.2.7, $= \max\{\sup\{||\lambda_2\xi||_2 : \xi \in \ell^2(I)$ and $||\xi||_2 = 1\}$, $=$ $\sup\{||\lambda_x\xi||_2 : \xi \in \ell^2(S\setminus I)$ and $||\xi||_2 = 1\} = ||\lambda_I(x)||$. □

We have not in general been able to decide whether Rees quotients of semigroups with the w.c.p. have the w.c.p..

Theorem 2.4.14

Let S have w.c.p. and I be an ideal of S with $C^*(I)$ having an identity u, and $S\backslash I$ a subsemigroup. Then $S\backslash I$ has the w.c.p.

Proof

Let S, I and u be as above. Let $R = S\backslash I$. Then $\ell^2(S) = \ell^2(R) \oplus \ell^2(I)$, and this decomposes λ_S. Now for $x \in k(R)$, $\lambda_S(x(1-u))|_{\ell^2(I)} = 0$, and $\lambda_S(x(1-u))|_{\ell^2(R)} = \lambda_R(x) = \lambda_S(x)|_{\ell^2(R)}$. Let $P : k(S) \to k(R)$ be the canonical homomorphism. Then P is a *-homomorphism, so $\lambda_S P$ is a *-representation of $k(S)$, so for all $z \in k(I)$, $||\lambda_S P(x)|| \le ||\lambda_S(x+z)||$. Now $\lambda_R P$ is a *-representation of S, so for $x \in k(R) \subset k(S)$,

$$||\lambda_S(x)|| \le ||\lambda_S(x(1-u))|| = ||\lambda_R(x)|| = ||\lambda_R P(x)|| \le ||\lambda_S(x)|| .$$

Let T be a *-representation of R. Then TP is a *-representation of S, so for $x \in k(R)$, $||Tx|| = ||TPx|| \le ||\lambda_S(x)|| = ||\lambda_k(x)||$, so R has w.c.p. □

Let us recall the structure of the minimal idempotents of $\ell^1(E)$ (proposition 1.5).

Corollary 2.4.15

Let S be a Clifford semigroup with the weak containment property. Then every subgroup of S is amenable if

(i) S is E-unitary, or

(ii) for all $e \in E_S$, there exists $e_1, \ldots, e_n \in E_S$ such that for all $f < e$, there exists $i \in \{1, \ldots, n\}$ such that $f \le e_i < e$ and each $e_i < e$.

Proof

(i) is [25] proposition 3.7(i) .

(ii) Theorem 2.4.14. □

Notation 2.4.16

For C*-algebras A and B , let $A \otimes_{\min} B$ be the closure of $A \otimes B$ in the least C*-norm, which exists by [28] 4.9.

Lemma 2.4.17

$$C^*_r(S) \otimes_{\min} C^*_r(T) = C^*_r(S \times T) .$$

Proof

By [28] 4.9, $(\lambda_S, \ell^2(S)) \otimes (\lambda_T, \ell^2(T))$ is a faithful *-representation of $C^*_r(S) \otimes_{\min} C^*_r(T)$. But for $s, u \in S$ and $t, v \in T$,

$$(\lambda_S(s) \otimes \lambda_T(t))(u \otimes v) = \lambda_S(s)u \otimes \lambda_T(t)v = \begin{cases} su \otimes tv & \text{if } s^*su = u \text{ and } t^*tv = u \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\lambda_{S\times T}((s, t))((u, v)) = \begin{cases} (su, tv) & \text{if } s^*su = u \text{ and } t^*tv = v \\ 0 & \text{otherwise.} \end{cases}$$

But $\ell^2(S \times T)$ is the Hilbert space tensor product of $\ell^2(S)$ and $\ell^2(T)$. □

For I an index set, let M_I be the Brandt semigroup $\mathcal{M}_0(I, \{1\})$. Now $k_\theta(M_I) = \cup\{k\ (M_F) : F \subset I$ and F is finite$\}$.

Let A be a C*-algebra. For F finite, $k_\theta(M_F) \otimes A$ is complete under any C*-norm. But

$k_\theta(M_I) \otimes A = \cup\{(k_\theta(M_F) \otimes A) : F \subset I$ and F is finite$\}$ which is an upwards directed union of C*-algebras, so $k_\theta(M_I)$ has a unique C*-norm and so does $C^*_\theta(M_I) \otimes A$. Thus in particular, $C^*_\theta(M_I) \otimes_{max} A = C^*_{r,\theta}(M_I) \otimes_{min} A$.

Lemma 2.4.18

Let $\mathcal{M}_0(I, G)$ be the Brandt semigroup with index set I and group G. Then $\mathcal{M}_0(I, G)$ has the weak containment property iff G is amenable.

Proof

$k_\theta(\mathcal{M}_0(I, G)) = k_\theta(M_I) \otimes k(G)$, so

$C^*_\theta(\mathcal{M}_0(I,G)) = C^*(k_\theta(M_I) \otimes k(G)) = C^*_\theta(M_I) \otimes_{max} C^*(G)$.

As in lemma 2.4.17,

$C^*_{r,\theta}(\mathcal{M}_0(I,G)) = C^*_{r,\theta}(M_I) \otimes_{min} C^*_r(G) = C^*_\theta(M_I) \otimes_{max} C^*_r(G)$.

If these two algebras induce the same C*-norm on $k(\mathcal{M}_0(I, G))$ then $C^*(G) = C^*_r(G)$, so G is amenable. Similarly, they are equal if G is amenable. □

This result can also be proved by following the method for groups, using lemma 2.3.5 with the ultimate identity $\sum_{i \in F} e_{ii}$.

Recall corollary 2.2.7.

Lemma 2.4.19

Let I be a *-ideal of uniform admissibility algebra A. Let S be a *-representation of A, and let T be a non-degenerate *-representation of I. Then if $S|_I$ weakly contains T, S weakly contains the extension $\tilde{T}$ of T to a *-representation of A on the same Hilbert space.

Proof

Let T represent I non-degenerately on H. Let U be the $*$-homomorphism such that

$$\begin{tikzcd} & \overline{S(I)} \arrow[dd, "U"] \\ I \arrow[ur, "S|_I"] \arrow[dr, "T"'] & \\ & \overline{T(I)} \end{tikzcd}$$

commutes. Then U is a $*$-representation of $\overline{S(I)}$ on H. Let $\tilde{U}$ be its extension to $\overline{S(A)}$. Let $x \in A$. Then for all $y \in I$, $\tilde{U}(Sx)Ty = \tilde{U}(Sx)\tilde{U}(Sy) = \tilde{U}((Sx*Sy)) = U(Sxy) = Txy = \tilde{T}xTy$. Then $\tilde{U}(Sx) = \tilde{T}x$, so

$$\begin{tikzcd} & \overline{S(A)} \arrow[dd, "\tilde{U}"] \\ A \arrow[ur, "S"] \arrow[dr, "\tilde{T}"'] & \\ & \overline{T(A)} \end{tikzcd}$$

commutes. □

Corollary 2.4.20

Let J and K be ideals of an inverse semigroup S which have the w.c.p. Then $J \cup K$ has the weak containment property.

Proof

Let T be an irreducible $*$-representation of $J \cup K$ on H. But $\overline{T(J)H}$ and $\overline{T(K)H}$ are invariant subspaces of H, so $H = \overline{T(J)H}$ or $H = \overline{T(K)H}$. Suppose the former. By hypothesis, $\lambda_{J\cup K}|_{K(J)} = \lambda_J$ weakly contains $T|_J$. But T is the extension of $T|_J$ to $J \cup K$, so $\lambda_{J\cup K}$ contains T by lemma ~~69~~ 2.4.19. Similarly in the other cases. Therefore $\lambda_{J \cup K}$ weakly contains every irreducible *-representation of $J \cup K$, and thus every *-representation thereof. $\square$

Theorem 2.4.21

Let $\mathcal{S}$ be an inverse semigroup. Then either no ideal of S has the weak containment property, or S has a greatest ideal with the weak containment property.

Proof

Suppose S does have an ideal with the w.c.p. Let $\mathcal{I} = \{I \subset S :$ I is an ideal and has the w.c.p.$\}$. Let $M = \cup \mathcal{I}$. Then M is an ideal of S.

Suppose M lacks the weak containment property. Then there exists *-representation T of M and $x \in k(M)$ such that $\|\lambda_M x\| < \|Tx\|$. Let $x = \sum_{i=1}^{n} \xi_i s_i$, and say $s_i \in I_i \in \mathcal{I}$. Then $J = \bigcup_{i=1}^{n} I_i$ has the weak containment property by corollary 2.4.20 so $\|\lambda_M x\| \geq \|\lambda_J x\| \geq \|Tx\| > \|\lambda_M x\|$, which is absurd. $\square$

Definition 2.4.22

A semilattice is well-founded if every non-empty subset has a minimal element.

Corollary 2.4.23

Let S be an inverse semigroup with well-founded semilattice and all of its subgroups be amenable. Then S has the w.c.p.

Proof

Without loss of generality, S has a zero element.

Now $\{\theta\}$ has the weak containment property.

Let M be the greatest ideal of S having the weak containment property. Suppose $M \neq S$. Now S/M is an inverse semigroup with well-founded semilattice, and all of its subgroups are amenable. Let e be a primitive idempotent of S/M. Let $I = (S/M)e(S/M)$. Then I is a Brandt semigroup by theorem 0.2, so has the w.c.p.. Then $I\setminus\{M\} \cup M$ is an ideal of S, and has the w.c.p. by theorem 2.4.10, contradicting the maximality of M. $\square$

CHAPTER 3

SYMMETRIC SEMIGROUP ALGEBRAS

§1 Symmetry and its Analogues

A *-algebra A is said to be symmetric if $-x^*x$ is quasi-regular, or, equivalently, if $Sp(x^*x) \subset \mathbb{R}^+ \equiv [0, \infty)$ for all $x \in A$. A *-algebra is said to be Hermitian if $Sp(h) \subset \mathbb{R}$ whenever $h = h^*$. For a Banach *-algebra these properties are equivalent. In general they are not. In a symmetric Banach *-algebra the spectral radius of a self-adjoint element is its norm under the greatest B*-seminorm, and this inequality implies symmetry for Banach *-algebras.

First we shall examine equivalent conditions to the last mentioned inequality. I start by establishing some technical results.

Definition 3.1.1

A non-empty subset W of a real vector space is a wedge if for all $x, y \in W$ and $\alpha \in \mathbb{R}^+$, αx, $x + y \in W$. We will write $x \geq y$ if $x - y \in W$. We do not require that $W \cap (-W) = \{0\}$.

Theorem 3.1.2 - The Krein Extension Lemma

Let M be a subspace of real vector space X with wedge W with $e \in M \cap W$ such that for each $x \in X$, $e + \lambda x \in W$ for small enough λ. Then if f is a linear functional on M with $f(x) \geq 0$ for all $x \in M \cap W$, f extends to a linear functional g on X with $g(x) \geq 0$ for all $x \in W$.

Proof

Bourbaki demands that W be a cone.

Let M, X, W, e and f be as above. Let $S = \{g \subset X \times \mathbb{R} ;\ g$ is a function, $\mathrm{dom}(g)$ is a linear subspace of X containing M, g is linear, $g|_M = f$, and $g(x) \geq 0$ for all $x \in \mathrm{dom}(g) \cap W\}$. Now $f \in S$. Then ordering S by inclusion, by Zorn's lemma S has a maximal element g. Suppose $\mathrm{dom}(g) \neq X$.

Now let $h \in X \backslash \mathrm{dom}(g)$. Without loss of generality, $e + x$, $e - x \in W$. Then if $m, n \in M$ and $m - x, x - n \in W$, $g(m) \geq g(n)$. Thus $\inf\{g(m) : m \in M$ and $m - x \in W\} \geq \sup\{g(m) : x - m \in W$ and $m \in M\}$. Let K be any number between these values. Then define $\tilde{g}$ on $\mathrm{dom}(g) + \mathbb{R}x$ by $g(y + \alpha x) = g(y) + \alpha K$ for $y \in \mathrm{dom}(g)$, $\alpha \in \mathbb{R}$. Then $\tilde{g} \in S$, contradicting the maximality of g. Thus $X = \mathrm{dom}(g)$. $\square$

This result can be applied to *-algebras because functionals on the self-adjoint part extend to the whole algebra by linearity.

Now for A a uniform admissibility algebra, let $K(A)$ be $\{h \in \mathrm{sym}(A) : f(h) \geq 0$ whenever f is a representable positive functional$\}$. Let $|\ |$ be the greatest B*-seminorm on A. Then if A has no representable positive functions, $K(A) = \mathrm{sym}(A)$. Now suppose A has an identity. If A has representable positive functionals, $|1| = 1$. Then if $h \in \mathrm{sym}(A)$, $|h| - h$ and $|h| + h$ lie in $K(A)$.

<u>Lemma 3.1.3</u>

Let A be a uniform admissibility algebra. Then if $h \in K(A)$ and $x \in A$ then $x^*hx \in K(A)$.

Proof

Let $h \in K(A)$ and $x \in A$. Let $T : A \to C^*(A)$ be the natural map. Then for every positive functional f on $C^*(A)$, $f(Th) \geq 0$. Thus $Th \geq 0$, so there exists $y \in C^*(A)$ such that $Th = y^*y$. Now every representable positive function f on A extends to a positive function $\tilde{f}$ on $C^*(A)$. Then
$f(x^*hx) = \tilde{f}((Tx)^*(Th)(Tx)) = \tilde{f}(Tx)^*y^*y(Tx)) \geq 0$, so $x^*hx \in K(A)$. □

The characterisation of symmetry of Banach *-algebras by positive functionals as below is due to Leptin [19].

Theorem 3.14

Let A be a uniform admissibility algebra. The following are equivalent:

(i) $Sp(h) \cap \mathbb{R} \subset \mathbb{R}^+$ for all $h \in K(A)$.

(ii) $Sp(h) \cap \mathbb{R} \subset \mathbb{R}^+$ for all $h \in K(\tilde{A})$.

(iii) $Sp(h) \subset \mathbb{R}^+$ for all $h \in K(A)$.

(iv) $Sp(h) \subset \mathbb{R}^+$ for all $h \in K(\tilde{A})$.

(v) Every proper left ideal of $\tilde{A}$ is annihilated by a non-zero positive functional.

(vi) Every proper modular left ideal of A is annihilated by a non-zero representable positive functional.

(vii) Every proper modular left ideal of A is annihilated by a non-zero positive functional.

(viii) $\rho(h) = |h|$ for all $h \in sym(A)$, where $\rho(h) = 0$ if $Sp(h) = \emptyset$.

Proof

(i) $\Longrightarrow$ (ii)

As Doran [8], we remove characters from the argument of Civin and Yood [5]. Suppose (i) holds. Let $\alpha + h \in K(\tilde{A})$ with $\alpha \in \mathbb{R}$ and $h \in sym(A)$.

Since $\phi(x + \lambda) = \lambda$ for $x \in A$ and $\lambda \in \mathbb{C}$ is a positive function on $\tilde{A}$, $\alpha \geq 0$. (a)

Also $h(\alpha + h)h \in K(A)$. (b)

Suppose $\lambda \in Sp(h) \cap \mathbb{R}$. Then $\lambda(\alpha + \lambda)\lambda \geq 0$ by hypothesis, so $\lambda = 0$ or $\alpha + \lambda \geq 0$, so $Sp(\alpha + h) \cap \mathbb{R} \subset \mathbb{R}^+$.

<u>(ii) $\Longrightarrow$ (iii)</u>

Suppose (ii) holds and $\alpha + i\beta \in Sp(h)$, $\alpha, \beta \in \mathbb{R}$ and $h \in K(A)$. Then $i\beta \in Sp(h - \alpha)$, so $-\beta^2 \in Sp((h - \alpha)^2)$. But $(h - \alpha)^2 \in K(\tilde{A})$, so $-\beta^2 \geq 0$, so $\beta = 0$. Then $\alpha \geq 0$, so $Sp(h) \subset \mathbb{R}^+$.

<u>(iii) $\Longrightarrow$ (iv)</u>

As (i) $\Longrightarrow$ (iii).

<u>(iv) $\Longrightarrow$ (v)</u>

Assume (iv) and let L be a proper left ideal of $\tilde{A}$. Define f on $sym(L + \mathbb{R}1)$ by $f(x + \lambda 1) = \lambda$ for $x \in sym(L)$, $y \in \mathbb{R}$. Now either $f(z) \geq 0$ when $z \in sym(L + \mathbb{R}1) \cap K(\tilde{A})$ or there exists $x \in sym(L)$ and $\lambda > 0$ such that $x - \lambda = w \in K(\tilde{A})$. Then $x = \lambda + w$ which is invertible, so $x \notin L$. Then f extends to a positive function on $\tilde{A}$ by theorem 3.1.2. Then for $y \in L$, $|f(y)|^2 \leq f(1)f(y^*y) = 0$.

<u>(v) $\Longrightarrow$ (vi)</u>

Assume (v) and let L be a proper left ideal of A with right modular unit e. Then $L + \mathbb{C}(1 - e)$ is a proper left ideal of $\tilde{A}$. Let f be a non-zero positive functional on $\tilde{A}$ annihilating $L + \mathbb{C}(1 - e)$. Then $f(e) = f(1) \neq 0$, so $f|_A$ is a non-zero representable positive functional on A annihilating L.

(vi) $\Longrightarrow$ (vii)

A fortiori.

(vii) $\Longrightarrow$ (vi)

Assume (vii) and let L be a proper left ideal of A with right modular unit e. Let f be a non-zero positive functional on A annihilating L. Then for $x \in A$, $f(x) = f(xe)$. Then $f(x) = f(xe) = (f(e^*x^*))^* = (f(e^*x^*e))^* = f(e^*xe)$, so f is a representable positive functional.

(vi) $\Longrightarrow$ (i)

Let $h \in K(A)$ and suppose $-h$ is left quasisingular. Then $L = A(1 + h)$ is a proper modular ideal. Suppose f is a representable positive functional annihilating L. $f(h + h^2) = 0$. But $f(h)$, $f(h^2) \geq 0$, so $f(h) = f(h^2) = 0$. Then for $x \in A$, $|f(xh)|^2 \leq f(xx^*)f(h^2) = 0$, so $f(x) = f(x + xh) = 0$, so $f = 0$. Thus (vi) fails.

(iv) $\Longrightarrow$ (viii)

Assume (iv). Let $h \in \operatorname{sym}(A)$. If $\mu > |h|^2$, then there exists $\nu \in \mathbb{R}$ such that $\mu > \nu > |h|^2$,

$$\text{so } \nu - |h|^2 \in K(\tilde{A}) . \qquad \text{(c)}$$

Then $\mu - |h|^2 = (\mu - \nu) + (\nu - |h|^2)$ which is invertible, so $\mu \notin Sp(h^2)$. Now if $|h| \neq 0$, $|h|^2 \in Sp(h^2)$. Then $|h|^2 = \rho(h^2) = \rho(h)^2$.

(viii) $\Longrightarrow$ (i)

Let $h \in K(A)$. Suppose $\alpha \in Sp(h)$ and $\alpha < 0$. Then there exists real polynomial f such that $f(0) = 0$ and $|f(\alpha)| > |f(x)|$ for $x \in [0, \rho(h)]$.

Let $T : A \to C^*(A)$ be the natural map. Now $|f(h)^2| = \rho(f(h)^2) \geq f(\alpha)^2 > \rho(f(Th)^2) = |f(h)^2|$, which is impossible, the strict inequality following from $Sp(Th) \subset \mathbb{R}^+$. $\square$

$K(A)$ lacks a pleasant algebraic description. The convex hull $K_0(A)$ of $\{x^*x : x \in A\}$ is more natural. To replace $K(A)$ by $K_0(A)$ in theorem 3.1.4, we must be able to use theorem 3.1.2 and justify assertions (a), (b) and (c) of the proof of theorem 3.1.4. Then the proof holds with $K_0(A)$ in place of $K(A)$.

Definition 3.1.5

A *-algebra A is a positive neighbourhood algebra if for all $h \in \mathrm{sym}(A)$ then $1 - \lambda h \in K_0(\tilde{A})$ for small enough real λ, or equivalently since $1 - \lambda h = \frac{1}{2}(1 - \lambda) + \frac{1}{2}(1 - \lambda h^2) + \frac{1}{2}\lambda(1 - h)^2$ for $h \in \mathrm{sym}(A)$, $1 - \lambda x^*x \in K_0(\tilde{A})$ for small enough real λ.

Theorem 3.1.6

Let G generate *-algebra A. Then if for all $g \in G$ there exists $K_g > 0$ such that $K_g^2 - g^*g \in K_0(\tilde{A})$, A is a positive neighbourhood algebra.

Proof

As for theorem 2.2.2 with $f(x^* \ldots x)$ stripped from the expressions.

As with uniform admissibility algebras, unitisations, direct sums and directed unions of positive neighbourhood algebras are positive neighbourhood algebras. The examples I gave of uniform admissibility algebras are all positive neighbourhood algebras, and I do not know whether the classes are distinct.

Theorem 3.1.7

Let A be a $*$-algebra and I a $*$-ideal thereof. Then A is a positive neighbourhood algebra if and only if A/I and I are.

Proof

$\Longrightarrow$

Suppose A is a positive neighbourhood algebra. It is immediate that A/I is a positive neighbourhood algebra. Let $y \in I$. Then there exists $\mu > 0$ such that $1 - \mu yy^* = \sum_{i=1}^{n} x_i^* x_i$ where $x_i \in \tilde{A}$. Then $y^*y - \mu(y^*y)^2 = \sum_{i=1}^{n} y^* x_i^* x_i y$. But $x_i y \in I$.

Now $(1 - \mu y^*y)^2 = 1 - 2\mu y^*y + \mu^2 y^*yy^*y$, so $1 - \mu y^*y = \mu(y^*y - \mu(y^*y)^2) + (1 - \mu y^*y)^2 \in K_0(\tilde{I})$.

$\Longleftarrow$

Suppose A/I and I are positive neighbourhood algebras. Let $y \in A$. Then there exists $M \geq 0$ such that $M - y^*y + I = \sum_{i=1}^{n} x_i^* x_i + I$. Let $h = y^*y + \sum_{i=1}^{n} x_i^* x_i - M$. Then h is a self-adjoint element of I. Then $N - h \in K_0(\tilde{I})$ for some $N \geq 0$. Then $M + N - y^*y = N - h + \sum_{i=1}^{n} x_i^* x_i \in K_0(\tilde{A})$. □

Corollary 3.1.8

A tensor product of positive neighbourhood algebras is a positive neighbourhood algebra.

Proof

Let A and B be positive neighbourhood algebras. $A \otimes B$ is an ideal of $\tilde{A} \otimes \tilde{B}$. Let $u \in \tilde{A}$ and $v \in \tilde{B}$. Then there exists $M, N \in \mathbb{R}^+$ such that $M - u^*u \in K_0(\tilde{A})$ and $N - v^*v \in K_0(\tilde{B})$.

Then $MN(1 \otimes 1) - u^*u \otimes v^*v = M1 \otimes (N - v^*v) + (M - u^*u) \otimes v^*v \in K(\tilde{A} \otimes \tilde{B})$,

so $\tilde{A} \otimes \tilde{B}$ is a positive neighbourhood algebra. □

Theorem 3.1.9

Let A be a positive neighbourhood algebra. Then theorem 3.1.4 holds with $K_0(\ )$ in place of $K(\ )$.

Proof

We now justify the assertions (a), (b) and (c) of the proof of theorem 3.1.4.

(a) Suppose $\alpha + h \in K_0(\tilde{A})$ with $\alpha \in \mathbb{C}$ and $h \in A$. Then there exists $(\xi_i) \subset \mathbb{C}$ and $(x_i) \subset A$ and $n \in \mathbb{N}$ such that $\alpha + h = \sum_{i=1}^{n} (\xi_i + x_i)^*(\xi_i + x_i)$, so $\alpha = \sum_{i=1}^{n} \xi_i^*\xi_i \geq 0$.

(b) Then $h(\alpha + h)h = \sum_{i=1}^{n} (\xi_i h + x_i h)^*(\xi_i h + x_i h) \in K_0(A)$.

More complicated is (c). We evaluate the maximal B*-seminorm in terms of $K_0(\tilde{A})$.

Let $x \in A$ and let $\mu = \inf\{M > 0 : M - x^*x \in K_0(\tilde{A})\}$.

Now if f is a positive functional on $\tilde{A}$ with $f(1) = 1$ and $M - x^*x \in K_0(\tilde{A})$, $f(x^*x) \leq M$. Thus $|x|^2 \leq M$, so $|x|^2 \leq \mu$.

Conversely we will produce a positive functional on $\tilde{A}$ with $f(1) = 1$ and $f(x^*x) = \mu$. Then $|x|^2 \geq \mu$. Define f on the real span of $\{1, x^*x\}$ by $f(\alpha 1 + \beta x^*x) = \alpha + \beta\mu$. If $\alpha < 0$ then $\alpha 1 + \beta x^*x \notin K_0(\tilde{A})$. If $\alpha \geq 0$, $\beta \geq 0$ then $f(\alpha 1 + \beta x^*x) \geq 0$. This only leaves the case $\alpha \geq 0$, $\beta < 0$, so we need only consider the case of $\alpha - x^*x$ with $\alpha \geq 0$. If $\alpha 1 - x^*x \in K_0(\tilde{A})$, $\alpha \geq \mu$, so $f(\alpha 1 - x^*x) \geq 0$. Then by theorem 3.1.2, f extends to a positive functional on $\tilde{A}$. □

A *-algebra is called completely symmetric if every element of $-K(A)$ is quasiregular. It is called k-symmetric if $-(x^*_1x_1 + \dots + x_k{}^*x_k)$ is quasiregular for all such expressions. Recall that the quasi-inverse of an element of an ideal lies in that ideal. Thus *-ideals of completely (respectively k-) symmetric *-algebras are themselves completely (respectively k-) symmetric *-algebras. Now if I is a *-ideal of uniform admissibility algebra A, $K_0(I) \subset K_0(A)$ so if A satisfies the conditions of theorem 3.1.4, so does I. For Banach *-algebras a key point is that symmetry implies complete symmetry ([4] lemma 41.4). The next result was proved by Leptin for Banach *-algebras [20].

<u>Theorem 3.1.10</u>

Let I be a *-ideal of uniform admissibility algebra A. Then A satisfies the conditions of theorem 3.1.4 if and only if A/I and I do.

<u>Proof</u>

Let A and I be as above.

Suppose A satisfies the conditions. Let L be a modular left ideal of I with right modular unit e. Then $L + A(1 - e)$ is a proper modular left ideal of A, so there exists non-zero positive functional f annihilating $L + A(1 - e)$. Now if $f(e^*e) = 0$, $f(xe) = 0$ for all $x \in A$, so $f(x) = f(x - xe) = 0$, which is not so. Thus $f(e^*e) \neq 0$, so $f|_I$ is a non-zero positive functional on I annihilating L.

Let L be a proper modular left ideal of A/I (regarded as cosets of I) with right modular unit $e + I$. Then $M = \cup L$ is a proper modular left ideal of A. Let f be a non-zero positive functional on A annihilating M. But $M \supset I$, so f induces a non-zero positive functional on A annihilating L.

Suppose I and A/I satisfy the conditions and let L be a maximal modular left ideal of A with right modular unit e. Suppose $I \subset L$. Then L/I is a proper modular left ideal of A/I with modular unit $e + I$, so there exists non-zero positive functional f on A/I annihilating L/I, which induces a non-zero positive functional on A annihilating L. Suppose $I \not\subset L$. Then $A = L + I$, so if $e = x + j$ with $x \in L, j \in I$, j is a left modular unit for L and indeed for $L_0 = L \cap I$. Then L_0 is a proper modular left ideal of I, so there exists non-zero positive function f_0 on I annihilating L_0. Then define positive function f on A by $f(z) = f_0(j^*zj)$. Then f extends f_0. If $x \in L$, $f(x) = f_0(j^*xj) = f_0(j^*x)$, because $j^*x \in I$, $= 0$ because $j^*x \in I \cap L = L_0$.

There is also an algebraic version.

<u>Theorem 3.1.11</u> (Wichmann [29])

Let I be a $*$-ideal of $*$-algebra A. Then A is k-symmetric if and only if A/I and I are, and hence the same holds for complete symmetry.

From this and the k-symmetry of radical *-algebras it follows that completely symmetric *-algebras and k-symmetric *-algebras form hereditary radical classes of *-algebras. He then proves that the symmetric radical of a Banach *-algebra is closed, and thus that symmetric Banach *-algebras form a hereditary radical class of Banach *-algebras.

§2 Hermitian Inverse Semigroups

We shall now investigate the symmetry of inverse semigroup algebras. We shall call an inverse semigroup S Hermitian if $\ell^1(S)$ is symmetric. Then as images of symmetric *-algebras are symmetric and closed subalgebras of symmetric Banach *-algebras are symmetric, S is Hermitian only if all its subgroups and G_S are Hermitian. If ϕ is a character on S, then for $s \in S$ $\phi(s) = \phi(s)\phi(s^*)\phi(s)$, so $\phi(s) = \phi(s^*) = 0$ or $\phi(s)^* = \phi(s^*)$, so commutative inverse semigroups are Hermitian. Finite inverse semigroups are Hermitian because they have an equivalent C*-norm.

Theorem 3.2.1

If A is a commutative symmetric Banach *-algebra, and B is a symmetric Banach *-algebra, then $A \hat{\otimes} B$ is symmetric.

Proof

Bonic [3] Corollary 3.3. □

Corollary 3.2.2

Let S be an E-unitary semilattice of groups. Then if $\ell^1(G_S)$ is symmetric, so is $\ell^1(S)$.

Proof

S is a subsemigroup of $E_S \times G_S$. Therefore $\ell^1(S)$ is a closed algebra of $\ell^1(E_S \times G_S) \cong \ell^1(E_S) \hat{\otimes} \ell^1(G_S)$, which is symmetric. Therefore $\ell^1(S)$ is symmetric. □

The key lemma in this section, used by Leptin in [21], is:

Lemma 3.2.3

Let A be a Banach *-algebra with a family of closed * sub-algebras $\{A_\alpha : \alpha \in \mathcal{A}\}$ such that:

(i) $\sum_{\alpha\in\mathcal{A}} AA_\alpha$ is dense in A ;

(ii) $A_\alpha AA_\alpha \subset A_\alpha$.

(iii) A_α is symmetric;

(iv) each A_α has an approximate identity $(e_{\alpha\lambda})$ such that $e_{\alpha\lambda}x$ converges for all $x \in AA_\alpha$;

(v) $f(x^*x) \geq 0$ for all $x \in AA_\alpha$ whenever f is a continuous positive functional on A_α .

Then A is symmetric.

Proof

Assume conditions (i) to (v) hold for A . As they hold for $A/\mathrm{rad}(A)$, we may assume A is semisimple and has isometric involution [4] theorem 25.9.

Let L be a maximal modular left ideal of A . Then there exists α such that $L \not\supset AA_\alpha$. Therefore $A = L + AA_\alpha$, so L has a modular right unit $e \in AA_\alpha$. $L \cap A_\alpha$ is a proper left ideal of A_α . For $x \in A_\alpha$,

$$x - xe = x - (\lim_\lambda xe_{\alpha\lambda})e = x - x \lim_\lambda(e_{\alpha\lambda}e) = x - x\eta$$

where $\eta = \lim_\lambda e_{\alpha\lambda}e \in A_\alpha$. Thus $L \cap A_\alpha$ is a modular left ideal of A_α .

Let f be a non-zero continuous positive functional on A_α annihilating $L \cap A_\alpha$. (Such a function exists by theorem 3.1.4.) Define F on A by $F(x) = f(e^*xe)$. If $x \in L$, then $xe \in L$, $e^*xe \in L \cap A_\alpha$, so $F(x) = f(e^*xe) = 0$. By (v), F is positive on A . Suppose $F = 0$ and let $x \in A_\alpha$.

$$f(\eta^*\eta) = \lim_\lambda f(\eta^*e_{\alpha\lambda}e) = \lim_\lambda \lim_\mu f(e^*e^*_{\alpha\mu}e_{\alpha\lambda}e) = \lim_\lambda \lim_\mu F(e^*_{\alpha\mu}e_{\alpha\lambda}) = 0 .$$

Then $|f(x\eta)|^2 \leq f(xx^*)f(\eta^*\eta) = 0$. But $f(x - x\eta) = 0$, so $f(x) = 0$. Therefore $f = 0$, which is a contradiction.

Thus F is a positive functional on A annihilating L . Then A is symmetric by theorem 3.1.4.

Corollary 3.2.4

A Brandt semigroup has symmetric ℓ^1-algebra if and only if its subgroups are Hermitian.

Proof

Let M be a Brandt semigroup with associated group G. We prove $\ell^1_\theta(M)$ is symmetric if $\ell^1(G)$ is. We take as subalgebras the $(1)_{ii}\ell^1_\theta(M)(1)_{ii}$ where $(1)_{ii} \in M$. They are isomorphic to $\ell^1(G)$, and as in Leptin [21] satz 2 we verify that they satisfy conditions (i), (ii), (iv) & (v) above. □

Recall that a semilattice is well-founded if every non-empty subset has a minimal element.

Theorem 3.2.5

Let S be an inverse semigroup with well-founded semilattice and all of whose subgroups are Hermitian. Then $\ell^1(S)$ is symmetric.

Proof

Suppose $\ell^1(S)$ were not symmetric.

Let Σ be its greatest symmetric ideal, which is closed. (It exists by Wichmann [29].) Let $M = \{s \in S : s \in \Sigma\}$. M is non-empty as it contains the minimal element of E_S . Then M is an ideal of S . Suppose $M \neq S$.

Now S/M is an inverse semigroup with a well-founded semilattice, and all of its subgroups are Hermitian. Let e be a primitive idempotent of S/M . Let $I = (S/M)e(S/M)$. Then $\ell^1_\theta(I)$ is symmetric by theorem 0.1 and corollary 3.2.4.

But $T = (I\backslash\{M\}) \cup M$ is an ideal of S , and $\ell^1(T)/\ell^1(M) \cong \ell^1_\theta(I)$, so $\ell^1(T)$ is symmetric. But $T \subset M$, which contradicts the existence of e . □

Comment 3.2.6

A semilattice of groups with Hermitian group algebras need not be symmetric. Using the notation of generators and relations, let $G = \langle x_i : i \in \mathbb{Z}^+,\ x_i^2 = 1,\ x_i x_k x_j x_k = x_k x_j x_k x_i \text{ for } i, j < k \rangle$. Fountain, Ramsay and Williamson [11] have proved that $\ell^1(G)$ is not symmetric, although $\ell^1(G_n)$ is, where $G_n = \langle x_i \in G : i \le n \rangle$.

Definition 3.2.7

Let A be a Banach $*$-algebra with isometric involution and G be a discrete group acting on A by $*$-isometries. Then the Leptin algebra $\ell^1(G, A)$ is $\{f : G \to A : \sum_{g \in G} \| f(g) \| < \infty\}$ with norm $\| f \| = \sum_{g \in G} \| f(g) \|$, multiplication induced by $(x, g)(y, h) = (xg(y), gh)$ where $x, y \in A$, $g, h \in G$, and involution given by $f^*(g) = g(f(g^{-1}))^*$.

Now Civin and Yood [5] have proved that any Banach $*$-algebra with dense socle and proper involution (i.e. $x^*x = 0$ only if $x = 0$) is symmetric. Merely as a foretaste of the main theorem to come, we establish the next theorem. Recall that if H is a subgroup of G, then $T \subset G$ is a right transversal for H in G if $G = \cup\{Ht : t \in T\}$ and for $s, t \in T$, $Hs = Ht$ only if $s = t$.

Theorem 3.2.8

If A is a commutative Banach $*$-algebra with dense socle and isometric proper involution (i.e. $x^*x = 0$ implies $x = 0$), $\ell^1(G, A)$ is symmetric if $\ell^1(G)$ is.

Proof

Let A be as above and let $\ell^1(G)$ be symmetric. Let $E = \{e \in A : e = e^* \text{ is a minimal idempotent}\}$ and let

$H_e = \{g \in G : g(e) = e\}$. Then for $e \in E$,
$\ell^1(H_e, \mathbb{C}e) \cong \ell^1(H_e)$, which is symmetric. We shall apply lemma 3.2.3 using these subalgebras.

$$(e \otimes 1)(a \otimes g)(e \otimes 1) = aeg(e) \otimes g = \begin{cases} 0 & \text{if } g \notin H_e \\ eae \otimes g & \text{if } g \in H_e \end{cases}$$

so $(e \otimes 1)\ell^1(G, A)(e \otimes 1) \subset \ell^1(H_e, \mathbb{C}e)$.

$(e \otimes g)(g^{-1}(e) \otimes 1) = e \otimes g$, so $\sum_{e \in E} \ell^1(G, A)\ell^1(H_e, \mathbb{C}e)$ is dense in $\ell^1(G, A)$ by [26] 4.10.1.

$$(e \otimes 1)(\sum_{g \in G} a_g \otimes g) = \sum_{g \in G} \lambda_g e \otimes g = \sum\{\lambda_{ht} e \otimes ht : h \in H_e, t \in T\}$$

where T is a right transversal for H_e in G . Then

$$(\sum_{g \in G} \lambda_g e \otimes g)(\sum_{g \in G} \lambda_g e \otimes g)^* = \sum_{t \in T} (\sum_{h \in H_e} \lambda_{ht} e \otimes h)(\sum_{h \in H_e} \lambda_{ht} e \otimes h)^* .$$

Therefore $\ell^1(G, A)$ is symmetric. □

Theorem 3.2.9 (McAlister [22])

Let S be an E-unitary inverse semigroup. Then E_S can be imbedded as an ideal of some poset X and G_S made to act on X by order automorphisms in such a way that
$S \cong \{(e, g) \in E_S \times G_S : g^{-1}(e) \in E_S\}$, and
$E_S \cong \{(e, 1) : e \in E_S\}$ under the same automorphism, where multiplication is defined by $(e, g)(f, h) = (\text{g.l.b.}\{e, g(f)\}, gh)$, this being well defined because $g^{-1}(e)$ and $f = g^{-1}(g(f))$ are elements of E_S .

Definition 3.2.10

Let $(X, \leq)$ be a poset.

If $x, y \in X$, x is said to cover y if

(i) $x > y$

and (ii) for all $z > y$, $z \geq x$,

and x is said to support y if it covers y in $(X, \geq)$.

Poset $(X, \leq)$ is tree-like if for all $x, y \in X$ there exist z and finite sequences (u_i) and (v_i) starting in x and y respectively and ending in z such that u_{i+1} covers u_i and v_{i+1} covers v_i . Poset $(X, \leq)$ is dually tree-like if $(X, \geq)$ is tree-like.

<u>Notation 3.2.11</u>

For vector spaces A and B with subsets C and D we shall identify $C \times D$ and $\{c \otimes d \in A \otimes B : c \in C$ and $d \in D\}$ and vector space $\ell^1(U) \hat{\otimes} \ell^1(V)$ with $\ell^1(U \times V)$.

Let X be a partially ordered set, with partial multiplication defined by $x \circ y$ being the greatest lower bound of x and y if it exists. Let $E = \bigtimes_{i=1}^{n} E_i$ be a subsemilattice and ideal of X with the E_i dually tree-like semilattices and let G be a group acting on X by order automorphisms.

$\ell^1(X \times G)$ will be endowed with the partial multiplication induced by $(x, g)(y, h) = (xg(y), gh)$ whenever $xg(y)$ is defined. Let $S = P(G, X, E)$ as defined by McAlister [22], i.e. $\{(x, g) \in E \times G : g^{-1}(x) \in E\}$, with the multiplication of $\ell^1(X \times G)$.

To motivate the general case we do the case of E_S a dually tree-like semilattice.

Theorem 3.2.12

Let S be an E-unitary inverse semigroup with dually tree-like semilattice and $\ell^1(G_S)$ symmetric. Then $\ell^1(S)$ is symmetric.

Proof

Let E_S be dually tree-like and $\ell^1(G_S)$ be symmetric.

Let $\tilde{\chi}$ extend $\chi : S \to G_S$ to $\ell^1(S) \to \ell^1(G_S)$. Then by theorem 3.1.10 it will suffice to prove $\ker(\tilde{\chi})$ symmetric. For $u \in E_S$ let $H_u = \{g \in G : g(u) = u\}$. Then H_u is a group. Let A_u be the linear span of $\{(u - v) \otimes g : g \in H_u\}$ where v supports u . Now $((u - v) \otimes g)((u - v) \otimes h) = (u - v) \otimes gh$ if $g, h \in H_u$, so $A_u \cong l^1(H_u)$, so A_u is symmetric. We shall apply lemma 3.2.3 to the set of subalgebras A_u .

Now for $e \otimes g \in S$,

$$((u - v) \otimes 1)(e \otimes g)((u - v) \otimes 1) = ((u - v) \otimes 1)(e \otimes g)(g^{-1}(e) \otimes 1)((u - v) \otimes 1)$$

$$= \begin{cases} ((u - v) \otimes 1)(e \otimes g)((u - v) \otimes 1) & \text{if } e \geq g(u)) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} ((u - v) \otimes 1)((g(u) - g(v)) \otimes g) & \text{if } e \geq g(u)) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (u - v) \otimes g & \text{if } u = g(u) \\ 0 & \text{otherwise,} \end{cases}$$

so $A_u A A_u = A_u$.

Now $\ker\tilde{\chi}$ is the closed linear span of $\{(u - v) \otimes g : u, g^{-1}(u) \in E_S$ and v supports $u\}$. But $((u - v) \otimes g)((g^{-1}(u) - g^{-1}(v)) \otimes 1) = (u - v) \otimes g$, so $\sum_{u \in E} AA_u$ is dense in A .

Now

$$((u - v) \otimes 1) \sum_{\substack{e \in E_S \\ g \in G_S \\ \text{and } g^{-1}(e) \in E_S}} \alpha_{e,g}(e \otimes g) = ((u - v) \otimes 1) \sum_{\substack{e \in E_S u \\ g \in G_S \\ \text{and } g^{-1}(e) \in E_S}} \beta_{e,g}(e \otimes g)$$

$$= ((u - v) \otimes 1) \sum_{\substack{e \in E_S \\ g \in H_u \\ t \in T}} \beta_{e,gt}\, e \otimes gt$$

where $\beta_{e,gt} = 0$ unless $e = u$ and $t^{-1}g^{-1}(e) \in E_S$, and T is a right transversal of H_u in G. Then

$$((u - v) \otimes 1)\left(\sum_{g \in H_u} \beta_{e,gt}\, u \otimes gt\right)\left(\sum_{g \in H_u} \beta_{e,gt}\, u \otimes gt\right)^*(u - v) \otimes 1$$

$$= \sum_{g,h \in H_u} \beta_{e,gt_1} \beta^*_{e,gt_2}((u - v) \otimes 1)(ugt_1t_2^{-1}h^{-1}(u) \otimes gt_1t_2^{-1}h^{-1})$$
$$((u - v) \otimes 1)$$

$= 0$ if $t_1 \neq t_2$, and if $t_1 = t_2$ it is

$$\sum_{g,h \in H_u} \beta_{e,gt_1} \beta^*_{e,gt_1}((u - v) \otimes 1)(u \otimes gh^{-1})((u - v) \otimes 1)$$

$$= \left(\sum_{g \in H_u} \beta_{e,gt_1}((u - v) \otimes g)\right)\left(\sum_{g \in H_u} \beta_{e,gt_1}((u - v \otimes g)\right)^* . \qquad \square$$

Let $\min(E_i)$ be the set of minimal idempotents of $\ell^1(E_i)$ other than the zero of E_i (if it exists), which have been determined in proposition 1.5.

Let $M = \bigtimes_{i=1}^{n} (E_i \cup \min(E_i)) \cup \{0\} \quad \ell^1(E)$. Extend order $\leq$ on E to $\leq$ on M by $e \leq f \iff e = ef$. For $x_i \in \min(E_i)$ or $x_i = E_i$, let $F_{x_1 \ldots x_n} = \bigtimes_{i=1}^{n} Z_i$ where $Z_i = E_i$ if $x_i = E_i$ and $Z_i = \{x_i\}$ if $x_i \in \min(E_i)$.

Corollary 3.2.13

$M \cup \{0\}$ and $F_{x_1 \dots x_r}$ are semilattices under $\le$.

Let $M_m = \{\prod_{r=1}^{n-m} (u - v_r) : u, v_r \in E$, the v_r are distinct, $u > v_r$ and there is no $w \in E$ such that $u > w > v_r\}$.

Then $M_m = \cup\{F_{x_1 \dots x_n} : F_{x_1 \dots x_n} \subset M_m\}$, $M_m M_p \subset \bigcup_{r \wedge m \le p} M_r \cup \{0\}$,

and if $F_{x_1 \dots x_n}, F_{y_1 \dots y_n} \subset M_m$, then

$$F_{x_1 \dots x_n} F_{y_1 \dots y_n} \subset \begin{cases} F_{x_1 \dots x_n} & \text{if } F_{x_1 \dots x_n} = F_{y_1 \dots y_n} \\ \bigcup_{r<m} M_r \cup \{0\} & \text{otherwise.} \end{cases}$$

Proof

Immediate.

Lemma 3.2.14

For all $g \in G$, $g(M_m) \cap \ell^1(E) \subset M_m$.

Proof

$$M_m = \{\prod_{r=1}^{n-m} (u - v_r) : u, v_r \in E \text{ , the } v_r \text{ are distinct, } u > v_r \text{ and there is no } w \in E \text{ such that } v_r < w < u\} .$$

If $g(\prod_{r=1}^{n-m} (u - v_r)) \in \ell^1(E)$, then $g(u) \in E$, so

$$g(\prod_{r=1}^{n-m} (u - v_r)) = \prod_{r=1}^{n-m} (g(u) - g(v_r)) \in M_m . \qquad \square$$

Lemma 3.2.15

$\{g \in G :$ There exists u such that

$u, g(u) \in F_{x_1 \dots x_n}\} = \{g \in G : \emptyset \ne g(F_{x_1 \dots x_n}) \cap \ell^1(E) \subset F_{x_1 \dots x_n}\}$,

and this set, $H_{x_1 \dots x_n}$, is a subgroup of G .

Proof

$\{g \in G :$ There exists u such that $u, g(u) \in F_{x_1 \ldots x_n}\}$

$$\supset \{g \in G : \emptyset \neq g(F_{x_1 \ldots x_n}) \cap l^1(E) \subset F_{x_1 \ldots x_n}\} .$$

Suppose $u, g(u) \in F_{x_1 \ldots x_n} \subset M_m$

(i) If $u \le v \in F_{x_1 \ldots x_n}$, $g(u) = g(uv) = g(v)g(u)$. Then if $g(v) \in \ell^1(E)$, $g(v) \in M_m$ by lemma 3.2.14, so $g(v) \in F_{x_1 \ldots x_n}$ by corollary 3.2.13.

(ii) If $u \ge v \in F_{x_1 \ldots x_n}$, $\mathrm{supp}(g(v))$ has a greatest element, which is bounded above by an element of $\mathrm{supp}(g(u)) \subset E$, so $g(v) \in l^1(E)$. $v \in F_{x_1 \ldots x_n} \subset M_m$, so $g(v) \in M_m$. Therefore $g(uv) = g(u)g(v) = g(v) \in F_{x_1 \ldots x_n}$.

Now, if $v \in F_{x_1 \ldots x_n}$, then $uv \in F_{x_1 \ldots x_n}$ and $uv \le u$, $uv \le v$. By (ii), $g(uv) \in F_{x_1 \ldots x_n}$. Then by (i), if $g(v) \in l^1(E)$, $g(v) \in F_{x_1 \ldots x_n}$. Thus the two sets are equal.

Let $g, h \in H_{x_1 \ldots x_n}$. Say $u, g(u), v, h(v) \in F_{x_1 \ldots x_n}$. $g(u), g^{-1}(g(u)) \in F_{x_1 \ldots x_n}$, so $g^{-1} \in H_{x_1 \ldots x_n}$. $uh(v) \in F_{x_1 \ldots x_n}$. Now $h^{-1}(u)v \le v \in l^1(E)$, so $h^{-1}(u)v \in l^1(E)$, so $h^{-1}(u)v \in F_{x_1 \ldots x_n}$. $gh(h^{-1}(u)v) = g(u)gh(v) \in l^1(E)$, so $g(u)gh(v) \in M_m$, so $g(u)gh(v) \in F_{x_1 \ldots x_n}$. Therefore $gh \in H_{x_1 \ldots x_n}$. $\square$

Lemma 3.2.16

Let $F_{x_1 \ldots x_n} \subset M_m$.

Let $B_{x_1 \ldots x_n} = \{u \otimes g \in F_{x_1 \ldots x_n} \times H_{x_1 \ldots x_n} :$

$g^{-1}(u) \in F_{x_1 \ldots x_n}\}$.

Then (i) $B_{x_1 \ldots x_n} \subset \ell^1(S)$;

(ii) $B_{x_1 \ldots x_n}$ is an E-unitary inverse semigroup whose semilattice of idempotents is the product of m dually tree-like semilattices.

(iii) If $u \otimes g \in (M_m \times G) \cap \ell^1(S)$, then $B_{x_1 \ldots x_n} (u \otimes g) B_{x_1 \ldots x_n} \subset \bigcup_{r<m} (M_r \times G) \cup \{0\}$ unless $u \otimes g \in B_{x_1 \ldots x_n}$.

Proof

Let $F = F_{x_1 \ldots x_n}$, $H = H_{x_1 \ldots x_n}$, and $B = B_{x_1 \ldots x_n}$.

(i) If $u \otimes g \in B$, $g^{-1}(u) \in F$, so $u \otimes g \in \ell^1(S)$.

(ii) Let $u \otimes g$, $v \otimes h \in B$.

Then $g^{-1}(u)v \in F$. Now $ug(v) \in \ell^1(E)$, so $ug(v) \in M_m$, so $ug(v) = u^2g(v) \in F$. $h^{-1}(v) \in F$, so $h^{-1}g^{-1}(u)h^{-1}(v) \in \ell^1(E)$, so $h^{-1}g^{-1}(u)h^{-1}(v) \in M_m$, so $h^{-1}g^{-1}(u)h^{-1}(v) \in F$. Therefore $ug(v) \otimes gh \in B$, so B is a semigroup.

Suppose $u \otimes g$, $v \otimes h \in B$ and $(u \otimes g)(v \otimes h)(u \otimes g) = (u \otimes g)$, and $(v \otimes h)(u \otimes g)(v \otimes h) = (v \otimes h)$. Then $ghg = g$, so $h = g^{-1}$. Then $ug(v) = u$ and $vg^{-1}(u) = v$, so $u < g(v)$ and $v \leq g^{-1}(u)$, so $v \otimes h = g^{-1}(u) \otimes g^{-1}$. But this is an element of B , so B is an inverse semigroup. The rest is clear.

(iii) Let $v \otimes h,\ w \otimes k \in B$ and $u \otimes g \in (M_m \times G) \cap \ell^1(S)$. $g^{-1}(u) \in M_m$, so $g^{-1}(u)w \in \bigcup_{r\le m} M_r \cup \{0\}$. But $ug(w) \in \ell^1(E)$, so $ug(w)e \in \bigcup_{r\le m} M_r \cup \{0\}$. $h^{-1}(v) \in M_m$, so $h^{-1}(v)ug(w) \in \bigcup_{r\le m} M_r \cup \{0\}$. Then $vh(u)hg(w) \in \bigcup_{r\le m} M_r \cup \{0\}$.

Suppose $vh(u)hg(w) \in M_m$. Then $h^{-1}(v)ug(w) \in M_m$, so $h^{-1}(v)ug(w) \in F$, so $ug(w) \in F$. Then $g^{-1}(u)w \in M_m$, so $g^{-1}(u)w \in F$, so $g^{-1}(u) \in F$.

Now $k^{-1}g^{-1}h^{-1}(v)k^{-1}g^{-1}(u)k^{-1}(w) \in l^1(E)$ and so is an element of F , so $hgk \in H$, so $g \in H$. But $u \in l^1(E)$, so $u = g(g^{-1}(u)) \in F$. Therefore $u \otimes g \in B$. □

Theorem 3.2.17

If S is E-unitary, $\ell^1(G_S)$ is symmetric and E_S is the product of finitely many dually tree-like semilattices, then $\ell^1(S)$ is symmetric.

Proof

Let E_S be the product of n dually tree-like semilattices and $\ell^1(G_S)$ be symmetric. Let A_m be the closed subspace of $\ell^1(S)$ generated by $\bigcup_{r\le m} (M_r \times G) \cap \ell^1(S)$ for $m \ge 0$, and let $A_{x_1 \ldots x_n}$ be the closed subspace of $\ell^1(S)$ generated by $B_{x_1 \ldots x_n}$.

Let $u \otimes g \in (M_s \times G) \subset \ell^1(S)$ and $v \otimes h \in (M_t \times G) \subset \ell^1(S)$. Then $ug(v) \otimes gh \in \ell^1(S)$. $g^{-1}(e) \in M_s$, so $g^{-1}(ug(v)) = g^{-1}(u)v \in M_sM_t \subset \bigcup_{r\le s\cap t} M_r \cup \{0\}$. $ug(v) \in \ell^1(E)$ so

$ug(v) \in \bigcup_{r \leq s \wedge t} M_r \cup \{0\}$, so A_m is a closed *-subalgebra of $\ell^1(S)$ and A_r is an ideal of A_s if $r < s$.

$A_{x_1 \ldots x_n}$ is a closed *-subalgebra by lemma 3.2.16

Let $R_m = A_m/A_{m-1}$ and let π_m be the canonical homomorphism. Let $R_{x_1 \ldots x_n} = \pi_m(A_{x_1 \ldots x_n})$.

Let us define the product of 0 dually tree-like semilattices to be a singleton. Let P_r be "If S is E-unitary, $\ell^1(G_S)$ is symmetric and E_S is the product of r dually tree-like semilattices, then $\ell^1(S)$ is symmetric." Now $A_n/A_{n-1} \cong \ell^1(G_S)$, which is symmetric so it will suffice to prove A_{n-1} is symmetric.

I shall now inductively prove A_m is symmetric and P_{m+1} is true for $0 \leq m < n$. Now A_{-1} is trivially symmetric and P_0 is true a priori.

Suppose $0 \leq m < n$, A_{m-1} is symmetric and P_m . First I prove $R_m = \pi_m(A_m)$ is symmetric. I shall show that $\{R_{x_1 \ldots x_n} : F_{x_1 \ldots x_n} \subset M_m$ satisfies conditions (i) to (v) of lemma 3.2.3 with respect to R_m . By lemma 3.2.16 condition (ii) is satisfied, and as $A_{x_1 \ldots x_n} \cong \ell^1(B_{x_1 \ldots x_n})$, $R_{x_1 \ldots x_n}$ is symmetric by P_m , so condition (iii) is satisfied. Now $\pi_m(F_{x_1 \ldots x_n} \times \{1\})$ is a singleton, so condition (iv) is trivially satisfied. Now if $u \otimes g \in (M_m \times G) \cap \ell^1(S)$, then $g^{-1}(u) \in M_m$, so $g^{-1}(u) \otimes 1 \in B_{x_1 \ldots x_n}$ for some $F_{x_1 \ldots x_n} \subset M_m$. Then $(u \otimes g)(g^{-1}(u) \otimes 1) = u \otimes g$, so condition (i) is satisfied.

Now we come to the most difficult condition, condition (v). Let $F = F_{x_1 \ldots x_n}$ and $H = H_{x_1 \ldots x_n}$. Let $v \in F$. Then

$v \otimes 1 + A_{m-1}$ is an identity for $R_{x_1 \ldots x_n}$. Thus $R_m R_{x_1 \ldots x_n} = R_m \pi_m (v \otimes 1)$. Let T be a right transversal for H in G_S. Then elements of $R_m R_{x_1 \ldots x_n}$ are of the form $\sum \alpha_{u,h,t} (u \otimes ht)^*(v \otimes 1) + A_{m-1}$ where $u \in M_m$, $h \in H$ and $t \in T$. Now $((u \otimes ht)^*(v \otimes 1))^* = (v \otimes 1)(u \otimes ht) = vu \otimes ht$. Thus $u \in F$ or $(u \otimes ht)^*(v \otimes 1) \in A_{m-1}$. Suppose the former. Now $h \in H$, so there exists $w \in F$ such that $h^{-1}(w) \in F$. Then $uw \in F$ and $h^{-1}(uw) \in F$. Now $(u - uw) \otimes 1 \in A_{m-1}$, so $((u - uw) \otimes 1)(u \otimes ht) = (u \otimes ht) - (uw \otimes ht) \in A_{m-1}$, so $u \otimes ht + A_{m-1} = uw \otimes ht + A_{m-1}$. Let $x \in R_m R_{x_1 \ldots x_n}$. Then

$$x = \sum_{\substack{u \in F \\ h \in H \\ t \in T}} \alpha_{u,h,t} (u \otimes ht)^*(v \otimes 1) + A_{m-1} \quad \text{with } h^{-1}(u) \in F \text{ whenever } \alpha_{u,h,t} \neq 0 .$$

Then

$$\left(\sum_{\substack{u \in F \\ h \in H}} \alpha_{u,h,s} (u \otimes hs)^*(v \otimes 1) + A_{m-1} \right)^* \left(\sum_{\substack{u \in F \\ h \in H}} \alpha_{u,h,t} (u \otimes ht)^*(v \otimes 1) + A_{m-1} \right)$$

$$= \sum_{\substack{u,w \in F \\ g,h \in H}} \alpha_{u,g,s} \alpha_{w,h,t} (v \otimes 1)(u(gst^{-1}h^{-1}(w)) \otimes gst^{-1}h^{-1})(v \otimes 1) + A_{m-1}$$

$$= \delta_{st} \left(\sum_{\substack{u \in F \\ h \in H}} \alpha_{u,h,s} (u \otimes h)^*(v \otimes 1) + A_{m-1} \right)^* \left(\sum_{\substack{u \in F \\ h \in H}} \alpha_{u,h,s} (u \otimes h)^*(v \otimes 1) + A_{m-1} \right)$$

and the latter factor lies in $R_{x_1 \ldots x_n}$. Thus condition (v) is satisfied, and thus R is symmetric.

But A_{m-1} is symmetric, so A_m is symmetric. Then P_{m+1} is true. □

§3 Completely Symmetric Semigroup Rings

All the results of the preceding section can be pushed through for the complete symmetry of inverse semigroup rings. Now if S is a finite inverse semigroup, $k(S)$ is completely symmetric. A semigroup is called locally finite if every finitely generated subsemigroup is finite.

Theorem 3.3.1

Suppose S is a locally finite involutive semigroup and $\ell^1(S)$ is symmetric. Then $k(S)$ is completely symmetric.

Proof

Let S be as stated. Let us adjoin an identity to it. Let $x_1, \ldots x_n \in k(S)$ and let $h = \sum_{i=1}^{n} x_i^* x_i$. Then $\lambda - h$ is invertible in $\ell^1(S)$ for $\lambda \in \mathbb{C}\setminus[0, ||h||]$. Let $B = \text{supp}(h) \cup \{1\}$. Now for $s \in S$ let π_s be the coordinate projection. Let $R(\lambda) = (\lambda - h)^{-1}$ for $\lambda \in \mathbb{C}\setminus[0, ||h||]$. $R(\lambda)$ and hence $\pi_s R$ are analytic in $\mathbb{C}\setminus[0, ||h||]$. Let T be the subsemigroup of S generated by B . Then if $s \in S\setminus T$ and $|\lambda| > ||h||$, $\pi_s R(\lambda) = 0$, so $\pi_s R = 0$. Thus $\text{supp}(R(\lambda)) \subset T$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}^+$ so $R(\lambda) \in k(S)$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}^+$. □

Theorem 3.3.2

Let S be an involutive semigroup such that $k(S)$ is completely symmetric and T be an involutive subsemigroup. Then $k(T)$ is completely symmetric.

Proof

Let S and T be as stated. Adjoin a common identity element to S and T. Let $x_1, \ldots, x_n \in k(T)$ and let $h = \sum_{i=1}^{n} x_i^* x_i$. For $s \in S$ let π_s be the coordinate projection. Let $R(\lambda) = (\lambda - h)^{-1}$ for $\lambda \in \mathbb{C}\setminus[0, ||h||]$ in $\ell^1(S)$. R and hence $\pi_s R$ are analytic in $\mathbb{C}\setminus[0, ||h||]$. Then if $s \in S\setminus T$ and $|\lambda| > ||h||$, $\pi_s R(\lambda) = 0$, so $\pi_s R = 0$. Then $\text{supp}(R(\lambda)) \subset T \cup \{1\}$ for $\lambda \in \mathbb{C}\setminus[0, ||h||$. Then if $\lambda \in \mathbb{C}\setminus\mathbb{R}^+$, $(\lambda - h)^{-1} \in l^1(T) \cap k(S) = k(T)$. □

Now $k(\mathbb{Z})$ is not even symmetric, for consider its characters. From this and theorems 3.3.1 and 3.3.2 it follows that if G is an Abelian group, $k(G)$ is completely symmetric if and only if G is locally finite, or equivalently, if and only if G contains no copy of $\mathbb{Z}$.

Theorem 3.3.3

Let A be a positive neighbourhood algebra with a family of positive neighbourhood subalgebras $\{A_\alpha : \alpha \in \mathcal{A}\}$ such that:

(i) $\sum_{\alpha\in\mathcal{A}} AA_\alpha = A$

(ii) $A_\alpha AA_\alpha \subset A_\alpha$

(iii) A_α is completely symmetric

(iv) A_α has an identity

(v) $f(x^*x) \geq 0$ for all $x \in AA_\alpha$ whenever f is a positive functional on A_α. Then A is completely symmetric.

Proof

As theorem 3.2.3, save that it is slightly simpler. □

Corollary 3.3.4

The semigroup ring of a Brandt semigroup is completely symmetric if and only if the semigroup rings of its subgroups are.

Proof

Apply theorem 3.3.3 with subalgebras the semigroup rings. □

Theorem 3.3.5

Let S be an inverse semigroup with well-founded semilattice and all of whose subgroups have completely symmetric group rings. Then $k(S)$ is completely symmetric.

Proof

As theorem 3.2.5. □

Corollary 3.3.6

Let S be a semilattice of groups with completely symmetric group rings. Then $k(S)$ is completely symmetric.

Proof .

Let $x_1, \ldots, x_n \in k(S)$. Let T be the inverse-subsemigroup of S generated by $\bigcup_{i=1}^{n} \operatorname{supp}(x_i)$. Then T has finitely many idempotents and all its subgroups have completely symmetric group rings. □

Theorem 3.3.7

If S is E-unitary, $k(G_S)$ is completely symmetric and E_S is the product of finitely many dually tree-like semilattices, then $k(S)$ is completely symmetric.

Proof

As theorem 3.2.17, but slightly simpler. □

CHAPTER 4

SIMPLE SEMIGROUP ALGEBRAS

Certainly $\ell^1(S)$ is not a simple algebra, for it has the character $\Sigma\alpha_s s \to \Sigma\alpha_s$. But $\ell^1_\theta(S)$ may be simple. We shall give sufficient conditions on the semigroup for the algebra to be topologically simple, and give some recipes for creating such semigroups.

A semigroup possessing an identity is known as a monoid.

Definition 4.1

$u \in S$ is a relative left identity for $t \in S$ if $ut = t$; it is non-trivial if $u \neq 1$.

Definition 4.2

A semigroup is 0-simple if its only proper ideal consists of the zero element. A semigroup is 0-bisimple if $a \mathcal{D} b$ whenever neither a nor b is the zero element.

Definition 4.3

A semigroup with zero is strongly disjunctive if for every finite set $A = \{a_1, \ldots, a_n\}$ disjoint from $\{\theta\}$ there exist $u, v \in S$ such that $|uAv\backslash\{\theta\}| = 1$ and $ua_i v = ua_j v \neq \theta$ implies $i = j$.

For a semilattice E this may be formulated as: if $e_i < e$ for $1 \leq i \leq n$, then there exists $u \in E$ such that $ue \neq \theta$ and $ue_i = \theta$ for $1 \leq i \leq n$.

Definition 4.4

An inverse semigroup is fundamental if it has no idempotent separating homomorphisms but isomorphisms. The greatest idempotent

separating homomorphism is $\mu : S \to \mathrm{End}(E)$ by $\mu(s)(e) = ses^*$.
See [15] for further details.

Definition 4.5

A semigroup S is left cancellative if $ab = ac$ implies $b = c$. An inverse semigroup is quasicancellative if and only if for all $a, b, c \in S$ $ab = ac \neq \theta$ and $bb^* = cc^*$ imply $b = c$.

Perhaps a more revealing formulation is given by the next proposition.

Proposition 4.6

Inverse semigroup S with zero is quasicancellative if and only if $e^2 = e = es \neq \theta$ implies $s \in E_S$.

Proof

Suppose S is quasicancellative and $e^2 = e = es \neq \theta$. Then $es^* = ss^*e = ss^*es = ess^*s = es$, so $s = ss^* \in E_S$.

Suppose $e^2 = e = es \neq \theta$ implies $s \in E_S$. Suppose $at = au \neq \theta$ and $tt^* = uu^*$. Then $\theta \neq a^*(at)t^* = a^*aut^* = a^*aut^*(tt^*) = (a^*att^*)(ut^*)$ so $ut^* \in E_S$. Then $(t^*u)^3 = (t^*u)^2 \neq \theta$. If $x^3 = x^2 \neq \theta$, $(x^{*2}x^2)x = (x^{*2}x^2) \neq \theta$, so $x \in E_S$, so $t^*u \in E_S$. Then $u^*t = t^*u$. Then
$t = tt^*tt^*t = t(t^*u)(u^*t) = tt^*u = uu^*u = u$. □

Corollary 4.7

Suppose $\chi : S \to G$, a group, and $\chi(st) = \chi(s)\chi(t)$ whenever $st \neq \theta$, and $\chi(s) = 1$ implies $s \in E_S$. Then S is quasicancellative.

<u>Proof</u>

Immediate from the above. $\square$

It follows that the quotient of any E-unitary inverse semigroup by an ideal is quasicancellative; it has been conjectured that the converse is true. It would suffice to prove the existence of a partial homomorphism to a group as above. We now consider a weaker cancellation property.

<u>Theorem 4.8</u>

Let S be an inverse semigroup. Then the following are equivalent:

(i) if $ag = ah \neq \theta$ and g and h lie in the same subgroup then $g = h$;

(ii) if $egf = ehf \neq \theta$ and $g \mathcal{H} h$ then $g = h$.

<u>Proof</u>

<u>(ii) $\Longrightarrow$ (i)</u>

Suppose $ag = ah \neq \theta$ and g and h lie in the same subgroup, and (ii) holds. Then $\theta \neq ag = agg^*g = ahg^*g$, so $g = h$.

<u>(i) $\longrightarrow$ (ii)</u>

Suppose (i) holds, $g \mathcal{H} h$ and $egf = ehf \neq \theta$. Then $\theta \neq egf = egg^*gf = e(gh^*)hf = e(hh^*)hf$. Then $\theta \neq e^*e(gh^*)hf(hf)^* = e^*e(hh^*)hf(hf)^*$, so $vkv = v \neq \theta$ where $u = hh^*$, $k = gh^*$ and $v = e^*ehf(hf)^*(hh^*)$.

Now $k^*k = hg^*gh^* = hh^* = u$, and $kk^* = gh^*hg^* = gg^* = u$. Then $v(k^*vk)v = (vk^*v)(vkv) = v$, and $(k^*vk)v(k^*vk) = k^*(vkv)(vk^*v)k = k^*vk$, so $k^*vk = v^* = v$. Now $vk(vk)^* = vkk^*v = vuv = v$, and $(vk)^*vk = k^*vk = v$. But $\theta \neq v^2 = vkv$, so $v = vk$. But $\theta \neq v = vu = vk$, so $k = u$. Then $g = gg^*g = gh^*h = kh = uh = hh^*h = h$. $\square$

Definition 4.9

An inverse semigroup is said to have property (A) if it has the above properties.

Definition 4.10

The inverse hull of a left cancellative semigroup is the inverse subsemigroup of the symmetric inverse semigroup of partial one-one mappings of the semigroup generated by maps of the form $\lambda_s = \{(x, sx) : x \in S\}$ where $s \in S$.

We shall now examine some of the properties of inverse hulls in terms of the original semigroups. The relationship is most straightforward for 0-bisimple inverse monoids.

Theorem 4.11

There exists a one-one correspondence between 0-bisimple inverse monoids with a zero and left cancellative monoids U whose principal right ideals and $\emptyset$ form a semilattice under intersection. This semilattice is isomorphic to the semilattice of idempotents of the inverse semigroup. The inverse semigroup is the inverse hull of the left cancellative semigroup with a zero adjoined if need be, and the left cancellative semigroup is the left unit semigroup of the inverse semigroup, i.e. $\{x \in S : x^*x = 1\}$.

Proof

[6] lemma 8.41, corollary 8.43 and theorem 8.4.4. The omission of a zero from their proofs is readily rectified.

Proposition 4.12

Let Σ be the inverse hull of left cancellative monoid S . Then Σ is fundamental if 1 is the only invertible element u of S such that $uR = R$ for every (principal) right ideal R of S .

Proof

Let μ be the greatest idempotent separating homomorphism on Σ .

Suppose Σ is fundamental and $uR = R$ for every right ideal in S . Then for any idempotent P of Σ , $\lambda_u^* P \lambda_u = P$, so $\lambda_u \mu \lambda_1$, so $u = 1$.

Conversely, suppose 1 is the only such element of S . Let P and Q be non-zero elements of Σ such that $P\mu Q$. Then $P^*P = Q^*Q$, so $\mathrm{dom}(P) = \mathrm{dom}(Q) \neq \emptyset$. Let $x \in \mathrm{dom}(P)$. Let $y = Px$ and $z = Qx$. Then $\lambda_y = P\lambda_x$ and $\lambda_z = Q\lambda_x$, so $\lambda_y \mu \lambda_z$, so $\mathrm{rge}(\lambda_y) = \mathrm{rge}(\lambda_z)$, so there exists $u \in S$ such that $z = yu$, and similarly there exists $v \in S$ such that $y = zv$. Then $y1 = zv = yuv$, so $uv = 1$ and similarly $vu = 1$.

Then for $s \in S$, $\lambda_u \lambda_s \lambda_s^* \lambda_u^* = (\lambda_y^* \lambda_z) \lambda_s \lambda_s^* (\lambda_z^* \lambda_y) = \lambda_s \lambda_s^*$ as $\lambda_1 = \lambda_y^* \lambda_y \ \mu \ \lambda_y^* \lambda_z$. But $\lambda_{us} \lambda_{us}^* = \lambda_s \lambda_s^*$ implies $usS = sS$, so $uR = R$ for every principal right ideal of S , so $u = 1$, so $y = z$, so $P = Q$. Therefore Σ is fundamental. □

Proposition 4.13

Let Σ be the inverse hull of a left cancellative monoid S . Then Σ is quasicancellative if S can be imbedded in a group and only if S is cancellative.

Proof

Suppose Σ is quasicancellative and $ac = bc$. Then $\lambda_a \lambda_c = \lambda_b \lambda_c \neq \emptyset$ and $\lambda_a^* \lambda_a = \lambda_b^* \lambda_b$, so $\lambda_a = \lambda_b$, so $a = b$.

Suppose $\chi : S \to G$ is an imbedding. Extending χ to Σ by $\tilde{\chi}(\lambda_{s_1}^* \lambda_{t_1} \ \ldots \ \lambda_{s_n}^* \lambda_{t_n}) = \chi(s_1)^{-1} \chi(t_1) \ \ldots \ \chi(s_n)^{-1} \chi(s_n)$ if

$\lambda^*_{s_1}\lambda_{t_1} \dots \lambda^*_{s_n}\lambda_{t_n} \neq \emptyset$, and 1 if it is $\emptyset$. Suppose

$\lambda^*_{s_1}\lambda_{t_1} \dots \lambda^*_{s_n}\lambda_{t_n} = \lambda^*_{u_1}\lambda_{v_1} \dots \lambda^*_{u_m}\lambda_{v_m} \neq \emptyset$. Then there exists

$(a_i)_0^n$ and $(b_j)_0^n$ such that $s_i a_{i-1} = t_i a_i$ and $u_i b_{i-1} = v_i b_i$ for $i > 0$, $a_0 = b_0$ and $a_n = b_m$. Then
$\chi(a_0) = \chi(s_1)^{-1}\chi(t_1) \dots \chi(s_n)^{-1}\chi(t_n)\chi(a_n)$ and
$\chi(b_0) = \chi(u_1)^{-1}\chi(v_1) \dots \chi(u_m)^{-1}\chi(v_m)\chi(b_m)$, so $\tilde{\chi}$ is well defined.

Now suppose $P \in \Sigma\backslash\{\emptyset\}$ and $\tilde{\chi}(P) = 1$. Then for $s \in \mathrm{dom}(P)$, $\chi(Ps) = \tilde{\chi}(P)\chi(s)$, so $\chi(Ps) = \chi(s)$, so $Ps = s$. Thus P is an idempotent. Then by corollary 4.7, Σ is quasicancellative. □

A 0-bisimple inverse monoid is quasicancellative if and only if its left unit semigroup is cancellative, but for our purposes property (A) is more interesting.

Proposition 4.14

The left unit semigroup of a 0-bisimple inverse monoid has no non-trivial invertible relative left identities for any of its elements if and only if the inverse semigroup has property (A) .

Proof

Let U be the left unit semigroup of inverse semigroup S .

Suppose S has property (A) and $s \in U$ is an invertible relative left identity for $t \in U$. Then $st = 1t$, and $ss^* = s^*s = 1$, so $s = 1$ by property (A) .

Conversely, suppose U has no non-trivial invertible relative left identities, and suppose $sa = ta \neq \theta$ with s and t lying in the subgroup with unit e . Then $\theta \neq s^*sa = s^*ta$, and $(s^*t)^*(s^*t) = e \neq \theta$

There exists $g \in U$ such that $gg^* = e$. Then $(g^*s^*tg)(g^*t^*sg) = g^*s^*tet^*sg = 1$ and $(g^*t^*sg)(g^*s^*tg) = 1$, so g^*s^*tg lies in the subgroup of units. Now $\theta \neq s^*ta = s^*sa = gg^*a$, so $\theta \neq g^*a = g^*s^*ta = (g^*s^*tg)g^*a$, so $(g^*s^*tg)(g^*aa^*g) = g^*aa^*g \neq \theta$. But then there exists $u \in U$ such that $uu^* = g^*aa^*g$, so $(g^*s^*tg)u = u$. Then $g^*s^*tg = 1$. Then $s = se = sgg^* = sg(g^*s^*tg)g^* = ses^*te = ete = t$. □

We now come to a very useful lemma in the study of inverse semigroup rings.

<u>Lemma 4.15</u>

Let S be an inverse semigroup, F be a field, and let I be a non-zero ideal in $F_\theta S$. Then there exist $x \in I$ and $e \in E_S \backslash \{\theta\}$ such that $e \in \mathrm{supp}(x) \subset eSe$.

<u>Proof</u>

S has a partial order defined by $s \leq t$ if there exists $e \in E_S$ such that $s = et$. (This is known as the natural order. See [15] for details.)

Let I be a non-zero ideal and pick $x \in I \backslash \{\theta\}$. Then $\mathrm{supp}(x)$ has an element s which is maximal in $\mathrm{supp}(x)$ under this order. Then I claim $s^*s \in \mathrm{supp}(s^*xs^*s)$. Let $x = \alpha s + \sum \beta_i t_i$, $\alpha \neq 0$, $\beta_i \neq 0$. For suppose $t \in \mathrm{supp}(x)$ and $s^*ss^*s = s^*ts^*s$. Then $s^* = s^*ts^*$. But $s = ss^*s = s(s^*ts^*)s = ss^*t(t^*t)(s^*s) = ss^*(ts^*st^*)t$, so $s \leq t$. Therefore $s = t$. Then $s^*s \in \mathrm{supp}\ s^*xs^*s \subset s^*sSs^*s$. But $s^*xs^*s \in I$. □

Lemma 4.16

Let S be an inverse semigroup, and let I be a non-zero ideal of a *-algebra A contained in $C^*_{r,\theta}(S)$ and containing $\lambda_S(S)$. Then there exist $x \in I$ and $e \in E_S \backslash \{\theta\}$ such that $\langle xe, e\rangle \neq 0$ and $x \in \lambda_e A \lambda_e$ where λ_S is the regular left *-representation of $l^1_\theta(S)$ on $l^2_\theta(S)$.

Proof

Let I be as above, and let $x \in I\backslash\{0\}$. Then $x^*x \neq 0$. Suppose $\langle x^*xs, s\rangle = 0$ for all $s \in S$. Then $xs = 0$ for all $s \in S$, so $x = 0$. Therefore there exists $t \in S$ such that $\langle x^*xt, t\rangle \neq 0$. But $t = \lambda_t t^*t$, so $\langle(\lambda_t^* x^*x\lambda_t)t^*t, t^*t\rangle \neq 0$. But $\lambda_{t^*t}(\lambda_t^*x^*x\lambda_t)\lambda_{t^*t} = \lambda_t^*x^*x\lambda_t \in I$. $\square$

The next theorem was proved by Munn [23] with quasicancellativity in place of property (A) .

Theorem 4.17

Let S be a fundamental 0-simple inverse semigroup with property (A) and strongly disjunctive semilattice of idempotents, and let F be a field. Then $F_\theta S$ is a simple algebra.

Proof

Let F and S be as above, and let I be a non-zero ideal of $F_\theta S$. For $e \in E\backslash\{\theta\}$, let $M_e = \{x \in I : exe = x \text{ and } e \in \mathrm{supp}(x)\}$. By lemma ~~8~~ 4.15, there exists $e \in E\backslash\{\theta\}$ such that $M_e \neq \emptyset$.

Let $\alpha = \min \bigcup_{e \in E\backslash\{\theta\}} \{|\mathrm{supp}(x)| : x \in M_e\}$. Then $\alpha > 0$. Let $M = \bigcup_{e \in E\backslash\{\theta\}} \{x \in M_e : x(e) = 1 \text{ and } |\mathrm{supp}(x)| = \alpha\}$, and pick $x \in M$, say with $x \in M_e$. Let $V = \{ss^*, s^*s : s \in \mathrm{supp}(x)\}\backslash\{e\}$.

Suppose $V \neq \emptyset$. Then $v < e$ for all $v \in V$, so there exists $u \in E$ such that $uv = \theta$ for all $v \in V$ and $ue \neq \theta$. Then $|\mathrm{supp}(uxu)| < |\mathrm{supp}(x)|$. But if $s \in \mathrm{supp}(x)$ and $usu = ueu$, then $s^*s = ss^* = e$, so $s = e$. Then $eu \in \mathrm{supp}(uxu)$ and so $uxu \in M_{eu} \cap M$, contradicting the minimality of $|\mathrm{supp}(x)|$, so $V = \emptyset$. Thus $x \in M \cap M_e$ implies that $ss^* = s^*s = e$ for all $s \in \mathrm{supp}(x)$.

Suppose $\mathrm{supp}(x) \neq e$. Let $s \in \mathrm{supp}(x)\backslash\{e\}$. Then as S is fundamental there exists $f \in E$ such that $sfs^* \neq efe$. As they are unequal, neither is θ. If $sfs^* > ef$, $s^*(sfs^*)s = efe < sfs^*$. Let

$$u = \begin{cases} f & \text{if } sfs^* \not> ef\text{. Then } sus^* \not\geq eu \neq \theta\text{.} \\ sfs^* & \text{if } sfs^* > ef\text{. Then } s^*us \not\geq eu = \theta\text{.} \end{cases}$$

Let $z = uxu$. If $tt^* = t^*t$ and $utu = ueu \neq \theta$, then $t = e$. Therefore $z \in M \cap M_{ue}$. If $utu = uvu$ and $t^*t = tt^* = v^*v = vv^* = e$, then $t = v$. Thus $usu \in \mathrm{supp}(z)$, so $(usu)(usu)^* = eu$, so $sus^* \geq eu$, and $(usu)^*(usu) = eu$ so $s^*us \geq eu$. This is impossible by the choice of u. Therefore $\mathrm{supp}(x) = \{e\}$. But S is 0-simple, so $I \supset S$. □

<u>Theorem 4.18</u>

Let S be a fundamental 0-simple inverse semigroup with property (A) and strongly disjunctive semilattice of idempotents, and let A be a *-subalgebra of $C^*_{r,\theta}(S)$ containing $\lambda_S(S)$ with an algebra norm bounded by the ℓ^1-norm. Then A is topologically simple. In particular, $\ell^1_\theta(S)$ is topologically simple.

Let I be a non-zero ideal of A. Pick $\varepsilon > 0$. For $e \in E\backslash\{\theta\}$ let $M_e = \{x \in k(S) : exe = x, x(e) = 1$ and $d(\lambda_x, I) < \varepsilon\}$ where d is the distance in the norm of A. By lemma 4.16, there is $z \in I$ and $e \in E\backslash\{\theta\}$ such that $\lambda_e z \lambda_e = z$ and $\langle ze, e\rangle = 1$. Then there is $w \in k(S)$ such that

$$||\lambda_w - z||_A < \frac{\varepsilon}{1 + \varepsilon + ||z||}$$

and $ewe = w$. Now if $ese = s$ and $\langle\lambda_s e, e\rangle = 1$ then $se = e$ and $s^*se = e$. Then $s = ese = e$, so

$$|w(e) - 1| = \langle(\lambda_w - z)e, e\rangle < \frac{\varepsilon}{1 + \varepsilon + ||z||}$$

so $d(\frac{1}{w(e)} \lambda_w, I) < \varepsilon$. Then $M_e \neq \emptyset$.

Let

$$\alpha = \min \bigcup_{e \in E} \{|\mathrm{supp}(x)| : x \in M_e \text{ and } e \in \mathrm{supp}(x)\} .$$

This is well defined. Let

$$M = \bigcup_{e \in E} \{x \in M_e : x(e) = 1 \text{ and } |\mathrm{supp}(x)| = \alpha\} .$$

Let $x \in M$, say with $x \in M_e$. Then as in theorem 4.17, $x = e$, so $d(e, I) < \varepsilon$. Therefore $d(s, I) < \varepsilon$ for all $s \in S\backslash\{\theta\}$ as S is 0-simple. But ε is arbitrary, so $\bar{I} \supset S$. □

Now we produce some examples of fundamental 0-simple inverse semigroups with property (A) and strongly disjunctive semilattice of idempotents.

<u>Example 4.19</u>

$S = \mathcal{M}_0(I,\{1\})$ for I a non-empty index set. Its semi-lattice of idempotents is strongly disjunctive, and 0-bisimple. For e a non-zero idempotent, the ideal of $\ell^1_\theta(S)$ generated by e can be faithfully represented (theorems 2.4.2 and 2.4.3) as the subalgebra of $BL(\ell^2(I))$ whose kernels are orthogonal to finite / dimensional subspaces of $\ell^1(I)$ and whose range is in $\ell^1(I)$, and the ideal of $C^*_{r,\theta}(S)$ can be faithfully represented as the subalgebra of $BL(\ell^2(I))$ consisting of finite rank operators. Suppose I is infinite. Then by Baire's category theory applied to their ranks, neither algebra is a Banach algebra, so both are dense ideals.

We shall now produce some left cancellative monoids whose inverse hulls are fundamental, 0-bisimple, satisfy property (A) and have strongly disjunctive semilattices.

Example 4.20

Let $FS(I)$ be the free semigroup on an infinite set I. Then $FS(I)^1$, its unitisation, is a cancellative monoid whose principal right ideals and $\emptyset$ are closed under intersection. The corresponding semilattice is strongly disjunctive. Its subgroup is trivial, so its inverse hull is fundamental and satisfies property (A).

There are many images of $FS(I)$ with the same properties as listed above, save that the image is left cancellative and not right cancellative, e.g. the quotient under the congruence generated by $ab = b^2$ where $a, b \in I$.

We will now produce examples with non-trivial subgroups. Let G be a group acting on semigroup S by automorphisms. Then $S \rtimes G = S \times G$ with multiplication given by
$(s, g)(t, h) = (sg(t), gh)$.

Theorem 4.21

Let S be a monoid with group of invertibles U and group G acting upon it by automorphisms. Then

(i) S is left cancellative if and only if $S \rtimes G$ is left cancellative.

(ii) $R \to (R\backslash\{\theta\} \times G) \cup \{\theta\}$ is an isomorphism from the semilattice of right ideals of S^0 to the semilattice of right ideals of $(S \rtimes G)^0$ carrying principal right ideals to principal right ideals.

(iii) $U \rtimes G$ is the group of invertibles of $S \rtimes G$.

(iv) If J is the set of relative left identities of $s \in S$, $J \times \{1\}$ is the set of relative left identities of $(s, g) \in S \rtimes G$.

(v) If S is left cancellative and $U = \{1\}$, then the inverse hull of $S \rtimes G$ is fundamental if and only if G acts on S faithfully.

Proof

Let S, U and G be as above.

(i) Suppose S is left cancellative and $(a, g)(b, h) = (a, g)(c, k)$. Then $ag(b) = ag(c)$ and $gh = gk$, so $h = k$ and $g(b) = g(c)$, so $b = c$.

Suppose $S \rtimes G$ is left cancellative and $ab = ac$. Then $(a, 1), (b, 1) = (a, 1)\ (c, 1)$, so $(b, 1) = (c, 1)$, so $b = c$.

(ii) Suppose R is a right ideal and $x \in R\backslash\{\theta\}$. Then for $g, h \in G$ and $y \in S$, $(x, g)(y, h) = (xg(y), gh)$, so $xg(y) \in R$. Thus $R \to ((R\backslash\{\theta\}) \times G) \cup \{\theta\}$ is a semilattice morphism for the stated semilattices. It is a monomorphism; all that remains to be proved is that it is a surjection for it to be an isomorphism. Let T be a right ideal of $S \rtimes G$, and let $R = \{\theta\} \cup \{s \in S : (s, g) \in T$ for some $g \in G\}$. Then if $(s, g) \in T$, then $(s, g)(1, g^{-1}h) = (s, h)$, so $T = ((R\backslash\{\theta\}) \times G) \cup \{\theta\}$. Suppose $s \in R$. Then if $t \in S$, $(s, 1)(t, 1) = (st, 1) \in T$, so $st \in R$. Thus the morphism is a surjection, and thus an isomorphism.

If R is a principal right ideal, so is $((R\backslash\{\theta\}) \times G) \cup \{\theta\}$, and if $((R\backslash\{\theta\}) \times G) \cup \{\theta\}$ is generated by (s, g), R is generated by s.

(iii) $(1, 1)$ is the identity of $S \rtimes G$. Suppose $(s, g)(t, h) = (t, h)(s, g) = (1, 1)$. Then $h = g^{-1}$, and $sg(t) = th(s) = 1$. Then $1 = g(1) = g(t)s$, so s is invertible. If $(u, g) \in U \rtimes G$, $(u, g)^{-1} = (g^{-1}(u^{-1}), g^{-1})$.

(iv) Suppose $(t, h)(s, g) = (s, g)$. Then $hg = g$, so $h = 1$. But then $s = th(s) = ts$.

(v) We shall use proposition 4.12. Let $s \in S$. Suppose $(t, 1)(S \rtimes G) = (s, 1)(S \rtimes G)$. Then there exist $u, v \in S$ such that $su = t$ and $tv = s$, so $suv = s$ and $tvu = t$, so $uv = vu = 1$, so $s = t$. Now $(1, g)(s, 1)(S \rtimes G)$ $(1, g)(s, 1)(S \rtimes G) = (g(s), g)(S \rtimes G) = (g(s), 1)(S \rtimes G)$. Then if $(1, g)R = R$ for every principal right ideal of $S \rtimes G$, $g(s) = s$ for all $s \in S$, so the inverse hull of $S \rtimes G$ is fundamental if and only if G acts on S faithfully. $\square$

Corollary 4.22

Let S be a 0-bisimple monoid whose subgroups are trivial and whose semilattice is strongly disjunctive. Then if U is its left unit semigroup and G acts faithfully on U then the inverse hull of $U \rtimes G$ is a fundamental 0-bisimple monoid satisfying property (A) and having strongly disjunctive semilattice.

Proof

By Theorem 4.11 and theorem 4.21 (i), $U \rtimes G$ is a left cancellative monoid. Let Σ be its inverse hull. By theorem 4.21 (ii) Σ is 0-simple with strongly disjunctive semilattice. Σ satisfies property (A) by proposition 4.14 and theorem 4.21 (iii) and (iv). Σ is fundamental by proposition 4.12 and theorem 4.21(v).

REFERENCES

1. B.A. Barnes, Representations of the ℓ^1-algebra of an Inverse Semigroup, Trans. Amer. Math. Soc. 218 (1976), 361-396

2. J. Boidol, H. Leptin, J. Schürman & D. Vahle, Räume Primitiver Ideal von Gruppenalgebren, Math. Ann. 236 (1) (1978), 1-13

3. R.A. Bonic, Symmetry in Group Algebras of Discrete Groups, Pacific J. Math. 11 (1961), 73-94

4. F.F. Bonsall & J. Duncan, Complete Normed Algebras (Berlin: Springer, 1973)

5. P. Civin & B. Yood, Involutions on Banach Algebras, Pacific J. Math. 9 (1959), 415-436

6. A.H. Clifford & G.B. Preston, Algebraic Theory of Semigroups, Volume II (Providence, R.I. : Amer. Math. Soc., 1967)

7. J. Dixmier, Les C*-algèbras et leurs représentations (Paris: Gauthier Villars, 1969)

8. R.S. Doran, A Generalisation of a Theorem of Civin and Yood on Banach *-algebras, Bull. London Math. Soc. 4 (1972) 25-26

9. J. Duncan & I. Namioka, Amenability of Inverse Semigroups and Their Semigroup Algebras, Proc. Roy. Soc. Edinburgh Sect. A 80 (1978) 309-321

10. J.M.G. Fell, The Dual Spaces of C*-algebras, Trans. Amer. Math. Soc. 94 (1960) 365-403

11. J.B. Fountain, R.W. Ramsay & J.H. Williamson, Functions of Measures on Compact Groups, Proc. Roy. Irish Acad. Sect. A 76 No. 23 (1976) 235-251

12. A. Guichardet, Tensor Products of C*-algebras, Soviet Math. Dokl. 6 (1965) 210-213

13. R. Godement, Les Fonctions de Type Positif et la Théorie des Groupes, Trans. Amer. Math. Soc. 63 (1948) 1-84

14. E. Hewitt & H.S. Zuckermann, The ℓ^1-algebra of a Commutative Semigroup, Trans. Amer. Math. Soc. 83 (1956), 70-97

15. J.M. Howie, An Introduction to Semigroup Theory, (London: Academic Press, 1976)

16. T. Husain & S.A. Warsi, Positive Functionals on BP*-algebras, Periodica Mathematica Hungarica 8 (1)(1977) 15-28

17. B.E. Johnson, An Introduction to the Theory of Centralizers, Proc. London Math. Soc. (3) 14 (1964) 299-320

18. H. Leptin, Verallgemeinerte L^1-algebren und Projektive Darstellungen Lokal Compakter Gruppen I, Inv. Math. 3 (1967) 257-281

19. H. Leptin, On Group Algebras of Nilpotent Lie Groups, Studia Math. 47 (1) (1973) 37-49

20. H. Leptin, Ideal Theory in Group Algebras of Locally Compact Groups, Inv. Math. 31 (3) (1976) 259-278

21. H. Leptin, Symmetric in Banachschen Algebren, Arch. Math. 27(4) (1976) 394-400

22. D.B. McAlister, Groups, Semilattices and Inverse Semigroups II, Trans. Amer. Math. Soc. 196 (1974) 351-370

23. W.D. Munn, Simple Contracted Semigroup Algebras, Proceedings of the Conference on Semigroups in Honour of Alfred H. Clifford (Tulane Univ. New Orleans, La., 1978) pp.35-43, (Tulane Univ. New Orleans, La., 1979)

24. T.W. Palmer, The Reducing Ideal is a Radical, Pacific J. Math. 43 (1972) 207-219

25. A.L.T. Paterson, Weak Containment and Clifford Semigroups, Proc. Roy. Soc. Edinburgh Sect. A 81 (1978), 23-30

26. C.E. Rickart, General Theory of Banach Algebras (Princeton: Van Nostrand, 1960)

27. Z. Sebestyén, Every C*-seminorm is Submultiplicative, Periodica Mathematica Hungarica 10 (1) (1979) 1-8

28. M. Takesaki, Theory of Operator Algebras (New York: Springer-Verlag, 1979)

29. J. Wichmann, The Symmetric Radical of an Algebra with Involution, Arch. Math. 30 (1) (1978) 83-88

30. J.R. Wordingham, The Left-regular *-representation of an Inverse Semigroup, Proc. Amer. Math. Soc. to appear