Dicuil (9th century) on Triangular and Square Numbers

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Abstract

Dicuil was a ninth-century Irish monk who taught at the Carolingian school of Louis the Pious. He wrote a *Computus* or Astronomical Treatise in Latin in about 814-16 (previously untranslated), which contains a chapter on triangular and square numbers. Dicuil describes two methods for calculating triangular numbers: the simple method of summing the natural numbers, and the more complex method of multiplication, equivalent to the formula $n(n+1)/2$. He also states that a square number is equal to twice a triangular number minus the generating number, equivalent to $n^2 = [2n(n+1)/2] - n$. The multiplication formula for triangular numbers was first explicitly described in about the third century AD by the Greek authors Diophantus and Iamblichus. It was also known as a solution to other mathematical problems as early as 300 BC. It reappeared in the West in the sixteenth century. Dicuil thus fills a gap in our medieval knowledge.

Introduction

Dicuil was an Irish monk who was most famous for writing a book on geography in 825, *Liber de mensura orbis terrae*. The text is available in Latin with an English translation and commentary (Tierney 1967). The work is mainly a compilation from various sources, but contains original material on northern islands from the reports of monks who had visited the Faroe Isles and probably Iceland in the summer of 795.

Dicuil spent some time in France, and certainly spent the years 814-16 at the court of Louis the Pious, Charlemagne's successor. He was probably employed as a teacher of grammar. While there he composed a *Computus* or *Liber de astronomia* (Astronomical treatise) in Latin. A computus was originally a manual for calculating the date of Easter. The date was a source of controversy between the Celtic and Roman churches - consequently computistical studies were particularly advanced both in Britain and in Ireland (Walsh and Ó Cróinin 1988, 101-103). Later works expanded to become general scientific encyclopaedias. One of the most influential texts was *De ratione temporum* (The reckoning of time) written by the Northumbrian scholar Bede in 725 (Wallis 1999). Charlemagne held a congress of computists at Aachen in 809 which resulted in many more works on the subject (Tierney 1967, 12).

Dicuil wrote five books of computus, in mixed prose and verse. He presented the first book to King Louis as a gift in 814, on the occasion in May when annual presents were received by the monarch. It seems that he actually read aloud chapters 1-6, as he complains in a poem added later as chapter 8 (section 6) that Louis, clearly bored, paid no attention, nor even acknowledged it with a word. The chapters would not have been easy listening: the first five contain Dicuil's innovative method of calculating the date of Easter, and chapter 6 (the subject of this paper) expounds the relation between triangular and square numbers. Dicuil presented a second unappreciated book in 815. He wrote a further two between 815 and 816, which presumably he did not present. The work has remained untranslated and little known (Esposito 1914). It does not seem to have had a wide circulation even at the time, as it is preserved in only two manuscripts, both of the ninth century - one written at the abbey of Saint Amand and now at Valenciennes, and the other from the abbey of Saint Martin at Tours. In the Valenciennes manuscript, Dicuil's *Computus* (Books 1-4 only) is the last item in a collection of treatises by various authors on miscellaneous topics, most of which are not particularly related to computus. A twelfth century note added to the manuscript attributes the work to the more famous Alcuin. It is interesting that no contemporary names Dicuil (Manitius 1911, 648). The Tours manuscript consists of two parts: the first contains the *De arithmetica* and *Consolatio philosophiae* of Boethius, and the second contains Dicuil's *Computus* with two chapters added to the fourth book, and two more chapters constituting a fifth book (also the titles of two other works by Dicuil). The last two sheets of this manuscript are now in Paris (van de Vyver 1935, 31-33; Cordoliani 1960, 325; Tierney 1967, 13-17). The *Liber de astronomia* had been briefly noticed by Ernest Duemmler in 1897 in his survey of the manuscripts of poetic texts of the Carolingian Age in Valenciennes. It was first published by Esposito from the Valenciennes manuscript in 1907, with a revised edition in 1920.

Little is known about Mario Esposito, except that his Italian parents moved to Dublin where he was brought up and where he became an eminent scholar of Hiberno-Latin literature (Lapidge 1990, vii-x). Esposito himself tells us (1920-21) that he was only 19 years old when, in August 1907, he published Dicuil's *Computus*, thus implying that he was born in 1886-1887. He later moved to Italy, where he died in about 1961.

Esposito's opinion of Dicuil's *Computus* changed over the years. In 1907 he wrote '… the whole work is full of interesting and curious information, and it is certainly surprising that it has not yet been published'. However, the work was the subject of scathing comments by Manitius (1911), who found it disorganised, with chapters following arbitrarily on each other, the author proceeding 'in absurd fashion' and not following even his own proposed scheme, in short 'a work of no further interest'. He also found Esposito's text unsatisfactory. This negative view of the work influenced later commentators, including Esposito himself (in 1920), van de Vyver (1935, 30-31) and Cordoliani (1960, 332). These last two, however, did find some originality in the work, especially Cordoliani (332). The remarks of recent commentators tend to be more favourable: Walsh and Ó Cróinin (1988, 103) wrote 'Dicuil's *Liber* is as remarkable for its originality of content as for its structure'; and again, Ó Cróinin (2005) commented that Dicuil's *Computus* 'is remarkably original in a field where one does not expect to find such innovation in either content or form'. A very positive assessment, based on a close study of the text, has been offered by Bergmann (2011) who argues that Dicuil developed a reliable method of calculating the date of Easter using a 532-year cycle - though too late, because Bede's inferior methods had already been accepted by the Carolingian court. Wantjes (2013) was also positive: 'The first two decades of the ninth century saw a considerable number of original minds, the foremost among computists being the Irishman Dicuil'.

None of these authors (except Bergmann) make any mention of Book I, chapter 6, concerning number sequences. The Latin list of contents gives this chapter as *'De crescenti numero et per semet multiplicato'*. Literally this translates as 'On increasing numbers and numbers multiplied by themselves', but its meaning is 'On triangular numbers and square numbers'. This chapter is unusual in describing two methods of calculating triangular numbers – the obvious method of summation and the less familiar method of multiplication. Bergmann (2011) describes this chapter and the two formulae, and argues that it is all part of the mathematics needed for calculating the 532-year cycle. We give the text and a translation of that chapter later in this paper.

Background to triangular and square numbers

It is generally agreed that figurate numbers originated with the Pythagoreans in the sixth century BC (Heath 1921, 76-77; Kline 1972, 28-30). The Pythagoreans discussed the number of dots or pebbles that could form a geometrical configuration, such as a triangle, square, pentagon, hexagon et cetera. The simplest figures were equilateral triangles, giving the sequence 1, 3, 6, 10 et cetera (Figure 1). The square numbers were 1, 4, 9, 16 et cetera. The

Pythagoreans knew that a square number was equal to the triangular number of the same rank added to its predecessor (Figure 2). In modern notation, T_n is a triangular number of rank n; it is the sum of an arithmetic progression $(1+2+3+4 \ldots+n)$; and it also equals $n(n+1)/2$, while its predecessor (T_{n-1}) equals $(n-1)n/2$. The sum of these two numbers is n^2. It is not clear whether the Pythagoreans could prove the latter relationship (Kline 1972, 30), or whether they relied on diagrams. Diagrams can also be used to show that a triangular number of rank n is equal to $n(n+1)/2$: two triangular numbers of the same rank (n) can be added together to make a rectangle of sides n and $n+1$, so half the product of these numbers is equal to a single triangular number of rank n (Figure 3). Rectangular numbers (with sides n and $n+1$) are also known as oblong, pronic or heteromecic numbers, but their relation to square numbers was not commonly stated (Knorr 1975, 149-150). Unfortunately any relevant Pythagorean writings are lost, and assumptions about Pythagorean knowledge are based on the Greek works of later authors such as Theon of Smyrna (c.115-140 AD) and Nicomachus of Gerasa (c.100 AD). These authors do not include any statement describing the formula $n(n+1)/2$, and it has been argued that this formula may not have originated with the Pythagoreans but rather in the Near East (Høyrup 2008).
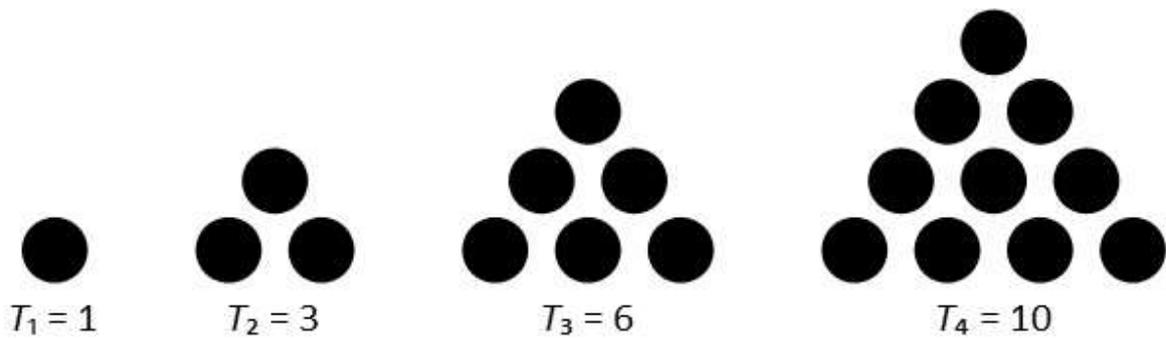

Figure 1. The first four triangular numbers (T_1 to T_4), showing that each is the sum of the natural numbers.
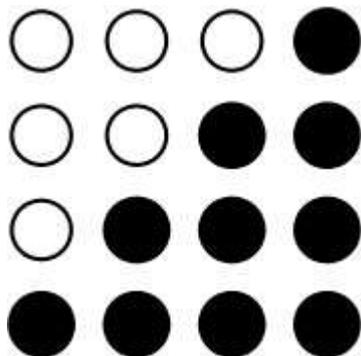

Figure 2. A square of side n dots can be split into two triangles, one with sides of n dots and the other with sides of n-1 dots. This is shown for n=4.
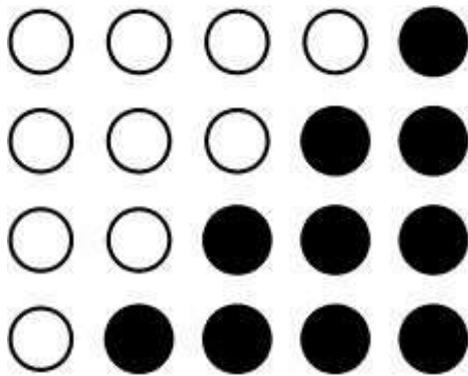
Figure 3. Two triangles with sides of n dots can be joined to form a rectangle with sides of *n* and *n+1* dots. This is shown for *n=4*.

Theon of Smyrna wrote *Mathematics useful for the understanding of Plato* (Lawlor & Lawlor 1978). Theon describes triangular numbers and gives the formula by addition (Part 1, xix). He also states that the sum of two consecutive triangular numbers is a square number (Part 1, xxviii), but without giving a formula. Nicomachus wrote *Arithmetike isagoge* (Introduction to arithmetic) (D'Ooge 1926). In Book II, chapter 8, he describes triangular numbers but gives only the formula by addition, with diagrams showing the number of dots in the triangles. He again states that the sum of two consecutive triangular numbers is a square number (Book II, chapter 12). Iamblichus (c.245-325) makes a similar statement in his commentary on Nicomachus (Heath 1921, 114; Pistelli, 1975, 75, lines 25-27).

Another well-known Greek author of about the same period as Theon and Nicomachus was Plutarch (c.50 - c.120 AD). In his *Moralia: Platonic questions* 1003f (Loeb Classical Library 1976) he states correctly that 8 times a triangular number plus 1 is a square number. This implies some knowledge of the multiplication formula, even though Plutarch does not state that formula. The relationship can be demonstrated by various diagrams, and the multiplication formula can also be used as follows:

8. $\frac{1}{2} n (n+1) +1 = 4n (n+1) + 1 = 4n^2 + 4n + 1 = (2n+1)^2$

The first classical author who explicitly states the multiplication formula is Diophantus of Alexandria, who may have lived anywhere between 150 BC and 280 AD, but most probably in the third century AD. He describes both formulae in his Greek tract on *Polygonal numbers*. We quote the Greek text and a Latin translation from Tannery (1974, 470-473):

Καὶ ἀπεδείχθη τὸ παρὰ Ὑψικλεῖ ἐν ὅρῳ λεγόμενον, ὅτι, ' ἐὰν ὦσιν ἀριθμοὶ ἀπὸ μονάδος ἐν ἴσῃ ὑπεροχῇ ὁποσοιοῦν, μονάδος μενούσης τῆς ὑπεροχῆς, ὁ σύμπας ἐστιν <τρίγωνος>.....Ὅθεν ἐπεὶ οἱ

τρίγωνοι μονάδος ὄυσης τῆς ὑπεροχῆς γίνονται, καὶ πλευραὶ αὐτῶν εἰσιν ὁ μέγιστοι τῶν ἐκτιθεμένων καὶ ὁ ὑπὸ τοῦ μεγίστου τῶν ἐκτιθεμένων καὶ τοῦ μονάδι μείζονος ἀυτοῦ, διπλασίων ἐστι τοῦ σημαινομένου τριγώνου.'

Demonstratum quoque est quod ab Hypsicle in definitione dictum fuit, nempe: 'Si sint numeri ab unitate in aequali differentia quotlibet, et unitas remaneat differentia, omnium summa erit triangulus…. Unde quoniam fiunt trianguli si differentia sit unitas, et illorum latera sunt maximi expositorum, et productus maximi expositorum et numeri unitate maioris est duplus indicati trianguli.'

Diophantus first quotes Hypsicles of Alexandria (c.120 BC), then says (in the translation of Heath 1910, 252) 'If there are as many numbers as we please beginning from 1 and increasing by the same common difference, then, when the common difference is 1, the sum of all terms is a triangular number …. Hence, since we have triangles when the common difference is 1, the sides of the triangles will be the greatest term in each case, and the product of the greatest term and the greatest term increased by 1 is double the triangle.' The first sentence gives the triangular formula by addition, and the second gives the multiplication formula.

Iamblichus (c.245-325) also describes both formulae in his Greek commentary on Nicomachus. Like Diophantus, he first describes the method of adding the natural numbers to generate triangular numbers, and then adds the multiplication formula. He writes: 'Πάλιν δὲ καὶ αὐτῶν τῶν καθ' αὐτοὺς τῶν ἀνομοίων τὰ ἡμίσυ τοὺς ἀπὸ μονάδος εὐτάκτους τριγώνους ποιήσει.' 'Again, half of those same [the natural numbers] multiplied by the successive [odd/even] numbers will give the triangular number sequence from unity' (Pistelli 1975, 86, lines 14-16. Our translation).

There seem to be no later authors - until Dicuil in the ninth century - who describe the multiplication formula in the context of triangular numbers, and whose work is extant.

The knowledge of Greek in Western Europe in the ninth century

The question arises whether Dicuil could have used the Greek texts described above. Two issues need to be considered: whether such sources were available in northern France in the ninth century and whether Dicuil could read them. The barbarian invasions of the fourth to fifth centuries into western Europe caused both the destruction of libraries and schools and the disruption of the tradition of late Greco-Roman culture, taking with it the ability to read and understand Greek. The Church maintained and disseminated as best it could what it thought necessary - mainly biblical and religious texts - and there were also isolated survivals of secular culture. The main revival of classical knowledge occurred in the seventh and eighth centuries when there was a considerable influx of texts in both Greek and Latin into western Europe from Italy, though a trickle of texts from Italy had never ceased entirely. What was available in any one place was however a miscellaneous collection, due to chance circumstances. Sweeney (in Bolgar 1971, 34) speaks of 'the isolation and inaccessibility of

texts in various monasteries and cathedral libraries'. We happen to know of some Greek texts available in the Carolingian schools (such as Dionysius the Areopagite, Plato's *Timaeus*), but there may well have been others.

Opinion on the question of Irish knowledge of Greek has seen some remarkable variation. Starting from nothing in the fourth century, by the seventh and eighth centuries the Irish were famed for their learning (Bede, *Ecclesiastical History*, book III, chapter 27, in McClure and Collins, 1994), and in the nineteenth century there developed an exaggerated belief in their expertise and indeed fluency in both Greek and Latin. As a reaction to this, an alternative view developed that Irish knowledge of Greek was extensive but superficial, restricted mainly to copying manuscripts and making lists of Greek words without any understanding of the grammar and structure of the language (Esposito, 1912; Bolgar, 1963, 93-4, 122; and to some extent Bischoff 1967; Herren 2010; Moran 2012). Even the more generous assessment of Moran (2012) allows only 'at best a very basic reading ability'. It is assumed that the Irish learnt most of their Greek after they arrived on the continent. Only Howlett (1988) makes a case for a real understanding of the Greek language by scholars in Ireland, which was why they were welcomed on the continent. Whether their knowledge was extensive or basic, it was more than the Continentals possessed, and certainly enough for them to make good use of any new resources they found there. Such Greek as was known under the Franks was an Irish monopoly (Dillon and Chadwick, 2000, 195). So Dicuil may have been able to read the relevant Greek texts, if they were available, especially as he was already a skilled computist.

Dicuil's sources

Regrettably, Dicuil did not identify his sources for his *Computus*. In section 2 of the chapter under discussion he mentions the 'philosophers' in relation to triangular numbers, but he copied this from Isidore of Seville (Esposito 1920-21). According to Esposito, Dicuil would have had access to many encyclopaedic, astronomical and computistical works in Latin at the Frankish court, including those of Pliny the Elder (d.79 AD), Victorius of Aquitaine (fifth century), Dionysius Exiguus (sixth century), Isidore of Seville (c.600), Bede (c.672-735), Alcuin (c.735-804) and many authors now lost to us. None of the authors named above mentions triangular numbers, except Isidore (Barney *et al.* 2006) who lists them among the planar numbers in his *Etymologies* (Book III, chapter 7, section 4), but does not go into details.

Several texts from the computus congress of 809 made use of an anonymous seventh century Irish tract, *De ratione conputandi*, but Dicuil did not use that tract (Walsh and Ó Cróinín, 1988, 103). Neither did he use Bede's *De ratione temporum* (Bergmann, 1993). He did refer to works often attributed to Alcuin - *De cursu et saltu lunae* and *De bissexto* – but these works were probably not by Alcuin (Lohrmann, 1993; Springsfeld, 2002, 64-79). Dicuil's use of these works was only to criticise them (Bergmann, 1993). Dicuil preferred his native Irish dating system to the English method, and claimed in Book 1, chapter 5, section 2, that his

system was based on the rules of the Greeks and Romans (Cordoliani, 1960, 334 and footnote 38).

Dicuil must have been familiar with the book by Boethius (c.500) *De institutione arithmetica* (On arithmetic) (Masi 1983, Oosthout and Schilling 1999), which was widely available in Western Europe. Boethius wrote in Latin, but based his book on earlier Greek authors such as Theon of Smyrna and Nicomachus. The *Introduction to arithmetic* by Nicomachus was translated into Latin by Apuleius in the second century (Midonick 1968, 15-16), and Boethius probably had access to this. Boethius discusses figurate numbers in Book II. He states that triangular numbers are the basis of all other figurate numbers (chapters 6 and 18). Unity is the mother of all numbers, and not a real triangular number: 3 is the first triangular number, and has 2 as a side; 6 is the second and has 3 as a side; the sides grow by unity (chapters 7 and 8). The sequence of triangular numbers grows by addition of the natural numbers (chapter 9). Later chapters deal with squared numbers and other figurate numbers. He describes rectangular or heteromecic numbers (chapter 26), and says that the sum of two consecutive triangular numbers is a square number (chapters 18 and 28); but he does not give the multiplication formula, or state that a rectangular number is twice a triangular number. However, these formulae can readily be deduced from this book, so Dicuil probably benefitted from reading Boethius.

Cassiodorus (c.485-585) wrote *Institutions of divine and secular learning, and on the soul* (Halporn and Vessey, 2004) in which he briefly mentioned triangular numbers (Book 2, Part 4, Section 6), but not the multiplication formula. This book was not widely used or cited by Carolingian authors (Halporn and Vessey, 2004, p.82). Butzer (1993) concludes that the major mathematical contributions of the Carolingian period were: the *Pseudo-Boethian Geometry 1*, perhaps produced in Corbie (c.780-800); the *Libellus annalis* of 793, a precursor of the *Computus* of 809, probably written by Alcuin; the Seven Book Computus, assembled at Aachen 809/10; and the *Propositiones ad acuendos iuvenes*, the oldest collection of mathematical exercises written in Latin, probably by Alcuin. We shall now turn our attention to the last of these, because it contains an example of relevance to the multiplication formula.

Other sources for the multiplication formula

A different tradition from the mathematical textbook lay in collections of practical problems and riddles (Taub 2017). The multiplication formula can be used for the sum of the natural numbers in other contexts besides triangular numbers. One example occurs in *Propositiones ad acuendos juvenes* (Problems to sharpen the young), an early collection of recreational mathematical problems (Folkerts 1978; Hadley and Singmaster 1992; Folkerts and Gericke 1993). Problem 42 is of particular interest. It is entitled *Propositio de scala habente gradus centum* (Problem concerning a staircase with a hundred steps) It runs:

'Est scala una habens gradus C. In primo gradu sedebat columba una, in secondo duae, in tertio tres, in quarto IIII, in quinto V. Sic in omni gradu usque ad centesimum. Dicat, qui potest, quot columbae in totum fuerunt.

Solutio. Numerabis autem sic: A primo gradu, in quo una sedet, tolle illam, et iunge ad illas XCVIIII, quae in nonagesimo nono gradu consistunt, et erunt C. Sic secundum ad nonagesimum octavum, et invenies similiter C. Sic per singulos gradus unum de superioribus gradibus et alium de inferioribus hoc ordine coniunge, et reperies semper in binis gradibus C. Quinquagesimus autem gradus solus et absolutus est non habens parem. Similiter et centesimus solus remanebit. Iunge ergo omnes simul, et invenies columbas MMMMML. (Latin from Folkerts and Gericke 1993)

'A staircase has 100 steps. On the first step stands a pigeon; on the second two; on the third three; on the fourth 4; on the fifth 5. And so on on every step to the hundredth. How many pigeons are there altogether?

Solution. We count them as follows. Take the single one on the first step and add it to the 99 on the ninety-ninth step, making 100. Taking the second with the ninety-eighth likewise gives 100. So for each step, one of the higher steps combined with one of the lower steps, in this manner, will always give 100 for the two steps. However, the fiftieth step is alone, not having a pair. Similarly the hundredth remains alone. Join all together and get 5050 pigeons.' (Trans, Hadley and Singmaster 1992)

The arithmetic here is 50 times 100, plus 50; or in algebraic terms $(n/2)n + n/2$, which is equivalent to $(n^2 + n)/2$ or $n(n+1)/2$.

Hadley and Singmaster add the comment: 'A method of summing arithmetic progression was known to the later Greeks, but does not appear in Euclid. It seems to have been known to the Egyptians and the Babylonians. Alcuin is the oldest text we know that presents the problem in this kind of fanciful setting.'

A version of the multiplication formula was indeed known to the Egyptians, as early as 300 BC (Gillings 1978; Høyrup 2008). A papyrus in the British Museum contains mathematical problems, including the sum of the arithmetical progression formed by the natural numbers. The scribe gives an example for the sum up to 10, and instructs us to take 10 times 10 (100), add 10 (110), and then take half (55). This is equivalent to the formula $(n^2 + n)/2$, which gives the same result as Alcuin's formula.

The multiplication formula appears again in the story that Carl Friedrich Gauss (1777-1855) reinvented it at primary school when asked to add all the numbers from 1 to 100, and quickly produced the answer 5050. There are many variations of the story (Hayes 2006). The incident might be taken to imply that the formula was not well known at the time, since Gauss's teacher was unaware of it and wanted to keep the pupils occupied for a long time; but it is possible that the teacher knew the formula, and wanted to see if any pupil could find a quick solution. Similarly, Wertheimer (1945, 89-97) questioned children on this type of problem, and found that some boys aged 11 and 12 were able to generate versions of the multiplication rule. Understanding the formula does not require genius, and it was probably reinvented many times. Possibly Dicuil invented it independently of other sources.

We now give Dicuil's text followed by an English translation.

Latin text of Book 1, chapter 6.

We are presenting Esposito's text as published in 1907, with the emendations later suggested by him (1920-21).

1. En iterum poteris bina argumenta videre,
Si placet auriculis nova iura haec sumere vestris;
Id crescens numerus per sese ac multiplicatus,
 Ut per se semper monstrentur utrique vicissim.

2. Si vis numerum, quem geometrico trigonum iure philosophi nominant, quem saepe dicimus crescentem numerum, apte per argumentum magis facile cognoscere, e contra quam per simplicem communis rationem sermonis quemcumque volueris numerum per sequentem continuatim, uel eundem sequentem eodem modo per praecedentem debes multiplicare. Praedictis binis numeris alternatim per se multiplicatis, relicta altera, dimidiam solam partem teneto, quam semper Triangulum numeri prioris habebis. Nam in crescentis numeri ratione semper quantum dicis, tantum addere servabis super omnes praecedentes ab unario congregatos pariter numeros. Si enim v sexies multiplicaveris, xxx habebis, expulsa dimidia eiusdem numeri parte, xv remanere videntur. Quos iure crescentis numeri quinarius possidet. Quando dico unarium, unum solum habebo. Quando binarium, iuncto cum unario simul, iii. Quando ternarium cum binis antecedentibus summulis, vi. Quando quaternarium, x. Quando quinarium, xv plene esse videbo. Sic semper in aliis cunctis haec regula immobiliter stabit.

3. Si rursum per trigonum vicissim scire multiplicatum per semet ipsum numerum desideres, trigono per binarium semper crescente, de illa dupliciter multiplicatione numerum, quo novissime suppletus fuerit ille triangulus, subtrahe, post haec per se ipsum multiplicatus totus tantum remanebit numerus. Nam si ex triangulo senarii, qui per binarium ductus xlii efficit, senarium expellas, factos per senarium numeros solum modo hoc est xxxvi remanere non dubitabis. Qui compotus per senarium multiplicatum per se ipsum procreatur. Cum sine hac regula trigonus in multitudinem nimis crescens difficulter agnoscitur. Per hanc igitur rationem ingeniosus quisque eum sine labore reperiet. Ita inter utrumque numerum per haec duo argumenta alter alterum monstrabit.

4. Si cupias, breviter hoc ius per metra profabor,
Quod prius historicae narravi famine prosae.
Namque iuvat merito mutatio saepe ciborum,
Praesertim regum mensis dum multa parantur.
Propter ea regi pauper convivia feci,
Ut vidua Heliae dans caenam aliena parabat.
Postremum ecce prior si multiplicaverit in se
Dimidia numerus crescens in parte locatur.
Quinque quater facti viginti rite creabunt,
Dimidiam partem, denos qui semper habebunt,
Crescenti in numero retinent quos quattuor illi.
Cernitur en numerus crescens per multiplicatum.

Si post consumptum primum vis scire secundum
Hic argumentum, quae contemplare sequuntur.
Crescentem numerum si iam duplicaveris, ille,
Qui per se crescit pulsa genitrice manebit.
Crescentem numerum retinet quinarius omnem.
Quinque ter educti bis qui triginta locabunt,
Ex quis si fuerint disiuncte quinque relicti,
In iurum norma viginti quinque manebunt.
Quem numerum penitus per semet quinque creabunt.
Sic inter numeros per dictos famine binos,
Alter et alterius quod monstrat iura videtur
Proditor alterius, velut alter uterque vicissim
Esset, cum nec hic illius vult damna videre.

5.   Successor Caroli, felix Hloduice, valeto,
Dicuil haec ego quae feci argumenta videto,
Post octingentos post septennosque bis annos
Conceptu domini haec in mense sequente peregi.
Namque cito adventum speravi cernere vestrum,
Dum mensis Maius septem bis lumina sumit,
Cum vobis tribuant dites iam munera digna.
Tradere tum volui quamvis mea uilia dona.
Nam vidua attribuens templo sua bina minuta,
Iam maiora dedit quam plurima dona potentum.
Sic ego quod potui vobis donare cupivi.
Ecce fere stabulis armenta gregesque feruntur.
Corpora tarda boum dissolvens fessus arator.

Translation of Book 1, chapter 6.

Below is our translation of the above text. We have used Arabic numerals to represent Roman numerals in the text.

1.   Well now, again, you can see a pair of demonstrations,
If you are prepared to receive these new rules into your ears;
Namely, how both types, the 'increasing number' and the one multiplied by itself,
Are always revealed by each other in turn.

2.   If, in order to determine a number of the sort that philosophers, following the established practice of geometry, call 'triangular' but which is generally spoken of as an 'increasing' number, you want to use a neat and actually easier abstract argumentation in contrast to the simple method employing the terms of everyday speech, you must multiply any number you like by the one immediately following in sequence, or likewise multiply that following number by the preceding one. When the two aforementioned numbers have been

multiplied together in turn, abandon one half of the total sum and keep the other, and this number will always be the 'triangular' of the lower number. In the 'increasing number' method, whatever number you name, you will always have to add that actual number to the combined sum of all the previous numbers starting from unity. Now, if you multiply 5 by six, you will get 30, and if half that number is subtracted, 15 is seen to remain. This number [15] is the result that five produces by the 'increasing number' rule: I say the word 'one', and I will have just one; I say 'two' and add it to one, getting 3; when 'three' is added to the two preceding little sums, I will have 6; when 'four' is added, I have 10; and 'five' added gives the full 15. And this rule will hold good for all the other numbers.

3.   Again, if you should want to use the 'triangle' in turn to find a number multiplied by itself, always double the triangular number, and from that number produced by doubling, subtract the last number that was added to the triangular number; and, after that, all that will remain is that last number multiplied by itself. For if you double the triangle relating to six [that is, 21], you get 42, and if you subtract six, you may be sure that the number that remains is one made up solely of sixes, that is 36, which computation is achieved through multiplying six by itself.

Without this rule, a triangular number of any considerable magnitude is difficult to ascertain, but by using this procedure any person of intelligence may easily discover it.

And so each of these two numbers will by means of these two procedures demonstrate the other in turn.

 4.   If you wish, I will show this system briefly through metre,
Which previously I set out through the language of narrative prose.
For a change of diet is often with good reason found helpful,
Especially when many things are prepared for the tables of kings.
For that reason, I, poor as I am, have made a feast for a king,
Just as the widow, in providing a meal for Elijah, made something different.
Behold, if the preceding number multiplies the following number by itself,
Its 'increasing number' is found to be half the total.
Four times five will duly create twenty,
Which will always have ten as its half part,
And that number [10] the number four governs by the 'increasing number' process.
So, the 'increasing number' is discovered through multiplication.
If, after absorbing the first demonstration, you want now to know the second,
Consider what follows.
If you have already doubled an 'increasing number',
Subtract the number that generated it, and you will have that number increased via itself.
The number five is a pattern for any 'increasing number'.
Three times five [the triangle 15] extended twice will put thirty in place,
From which if five is separated off and abandoned,
According to the rules of the system, twenty-five will remain,
Which number five can generate entirely by itself [that is, 5 by 5].
So between the two sorts of numbers treated in my utterance,

Because the one demonstrates the workings of the other,
It appears the producer of the other, as if each were the other in turn,
While neither does this one want to see the loss of the other.

5.   Hail happy Louis, successor of Charlemagne,
See these arguments which I, Dicuil, have made.
After eight hundred and fourteen years
From the Conception of the Lord [March 25], in the next month I wrote them,
For I expected soon to be a spectator at your Entry.
When the month of May takes twice seven days,
At the time when the wealthy give you worthy gifts as tribute,
Then I wished to deliver my paltry offerings, such as they are.
For the widow assigning her two mites to the temple
Now gave more than the many gifts of the powerful.
Even so I desired to give what I could to you.
See, the cattle and the flocks are being brought to their stables,
The weary ploughman releasing the slow bodies of the oxen.

Comments on the text

Section 1. Dicuil states that he will discuss the relation between 'increasing numbers' and square numbers. 'Increasing numbers' here refers to triangular numbers.

Section 2. This contains Dicuil's most valuable contribution, where he describes two methods of calculating triangular numbers. He first describes the multiplication method, equivalent to the formula $n(n+1)/2$, and then describes the method of summation of the natural numbers. He makes the point that the multiplication method is both neater and quicker than adding up the numbers verbally in a series of small sums. He uses the number 5 as an example, as he does in Section 4. Several authors use 5 as a typical number.

Section 3. This concerns square numbers, where Dicuil gives a laborious method of calculating them using triangular numbers: he implies $2n(n+1)/2 - n = n^2$. The point of this is to show that square numbers can be built up from two identical triangular numbers less one column of dots (as shown in Figure 3). Dicuil is unusual in stating this, as most earlier authors state only that a square is the sum of two consecutive triangular numbers (Figure 2).
        Dicuil explains how triangular numbers can be used to calculate square numbers, and claims that the reverse is true. It is not obvious how square numbers could assist in the calculation of triangular numbers, though diagrams such as Figures 2 and 3 can be used as a visual demonstration of the relationship. Bergmann (2011, 249) describes Dicuil's argument as 'banal'.

Section 4. At the beginning of this section Dicuil changes from prose to verse, and he first suggests that the change to verse is pleasant, just as in a banquet one enjoys a variety of different flavours. This is the only extended verse section in Book I, chapters 1-6. The second

idea is that a plain dish is a welcome change at a rich feast - that is, he is offering an intellectual banquet. Also his offering is unique. All this is elevated by a biblical parallel – the widow of Zarephath: the widow's supply of oil and meal was miraculously renewed every day, enabling her to feed herself, her son and the prophet Elijah on simple bannocks during the famine (1 Kings 17: 7-16). The implication is that Dicuil's plain and different dish is as significant as the biblical widow's dish.

Dicuil inserted numerous verse passages into his *computus*, especially in Book 2. Many are just short summaries or introductions to a new section, but others present *computus* material in verse form. There was a long Greco-Roman tradition of writing in continuous verse on scientific or at least technical subjects, and most commentators have praised Dicuil for his obvious expertise and ingenuity in presenting his material in verse form. What is noteworthy in the *Computus* is the mixture of passages of prose and verse. The immediate precedent for this is to be found in the fifth-century Martianus Capella's *De nuptiis Philologiae et Mercurii* and Boethius' fifth-sixth century *Consolatio philosophiae*, two works which had recently been re-discovered and popularised by Alcuin. It is interesting that here in chapter 6, the verse passage does not continue the argument or comment on it, but actually translates the preceding prose passage. Lines 7-12 repeat the multiplication formula. Lines 15-16 repeat the derivation of square numbers from twice a triangular number minus the generating number. In line 18, 'Three times five extended twice' is a complicated way of expressing 15 times 2; but the reason for the complication is not merely that the ordinary word for 15 (*quindecim*) does not easily fit the metre, but that Dicuil enjoys playing with numbers. Dicuil clearly expected Louis to be impressed by his cleverness and enjoy the verse - but, alas, in vain.

The last three lines of this section are a complicated rephrasing of the last line of section 3.

Section 5. As he tells us in this section, Dicuil presented Book I to Louis on May 14 in the year 814. This was the annual occasion when persons of any importance were expected to present valuable gifts to the king. Dicuil offered a humble literary gift. He was not the first to do so – another *exul Hibernicus* or 'Irish ex-patriate', had offered a poem to Charlemagne (van de Vyver 1935, 44). Dicuil seems to have written his gift for Louis in April ready for presentation in May. For the translation of *adventus* as 'Entry' rather than 'arrival', see van de Vyver (1935, 27-28): Louis is not merely 'arriving', as he has been in Aachen since the end of February. This is his ceremonial entry for the gift-giving ceremony. (Esposito's punctuation has been emended here.)

Again we have a Biblical parallel. Dicuil compares his meagre offering to the biblical widow's mite (*Mark* 12: 41-44; *Luke* 21: 1-4), suggesting that it is his poor best. It is, however, quite common for medieval writers to denigrate their own poor efforts whilst actually being rather proud of them. Dicuil's justification for writing is that he is presenting Louis with a gift that is both different and valuable in its own way, as the biblical analogies suggest. He probably believed that the multiplication formula and the relation between triangular and rectangular numbers was a useful but little-known contribution.

In the last two lines of section 5 we find an example of the common trope whereby the fall of night or the end of a day's work is used to signal the closure of the writer's topic. For

example, Vergil wrote (37-29 BC) 'It is now time to unyoke the horses' steaming necks' (*Georgics*, II, lines 541-2). Dicuil uses a similar phrase at the end of his *De mensura orbis terrae*: 'At night on ending their labours the oxen are granted rest' (Tierney 1967, 103). Dicuil would expect his hearers or readers to pick up these references from their schooldays. This closure marks the end of Dicuil's reading to Louis on this occasion.

The revival of mathematics in the West

The death of Boethius marked the end of mathematics in the Western Roman Empire. The Carolingian renaissance of the ninth century brought only temporary relief (Ifrah 1998). There was a new revival in the eleventh and twelfth centuries when Greek and Sanskrit texts that had been translated into Arabic were retranslated into Latin, mainly at Toledo in Spain. The Hindu-Arabic numerals 1-9 were introduced in 976 by Gerbert of Aurillac, later followed by the symbol 0 for zero. The + and − signs first appeared in print in 1489 and the equals sign in 1557 (Higgins 2008, 24). In the seventeenth century Descartes developed the use of letters for algebraic formulae, improving on the early work of Diophantus and others. Mathematics benefitted.

The formulae for triangular numbers reappeared in the sixteenth century. Francisco Maurolico's *Arithmeticorum libri duo* of 1575 contains both formulae. About 1618 Thomas Harriot composed his *Magisteria magna* on triangular numbers and other sequences, and he gives the multiplication formula in algebraic form (Beery and Stedall 2009). By the time of Pascal in the seventeenth century the relation between figurate numbers and binomial numbers was understood, and many complex formulae were developed (Edwards 1987, 20-21).

Conclusions

Dicuil was unusual in describing the multiplication formula for triangular number sequences in the ninth century, and in stating that a square number is equal to twice a triangular number minus the generating number. He seems to have been almost the only Western scholar to have stated the multiplication formula between the third and sixteenth centuries, though that formula was known as a solution to other mathematical problems. Dicuil may have copied the formula from sources he does not credit, or he may have re-invented it independently. Unfortunately his *Computus* remained unknown until published by Esposito in 1907. We would therefore support Esposito's first opinion that Dicuil's *Computus* contains some interesting and curious information.

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