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ON TREES AS STAR COMPLEMENTS IN REGULAR GRAPHS

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Abstract

Let G be a connected r-regular graph ($r > 3$) of order n with a tree of order t as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$. It is shown that $n \leq \frac{1}{2}(r+1)t - 2$. Equality holds when G is the complement of the Clebsch graph (with $\mu = 1$, $r = 5$, $t = 6$, $n = 16$).

Keywords: eigenvalue, regular graph, star complement, tree.

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1. Introduction

Let G be a graph of order n with an eigenvalue μ of multiplicity k. A *star set* for μ in G is a set X of k vertices such that $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ. Star sets and star complements exist for any eigenvalue of any graph (see [5, Chapter 5]). If G is regular, $\mu \notin \{-1, 0\}$ and $t = n - k$ then $t > 2$ and $k \leq \frac{1}{2}(t+1)(t-2)$ with equality if and only if G is an extremal strongly regular graph [1, Theorem 3.1]. The regular graphs with a star as a star complement are described in [9], and those with a generalized star as a star complement are investigated in [7], but the techniques used there do not extend to an arbitrary tree as a star complement.

Now suppose that G is r-regular, $r > 2$ and $\mu \notin \{-1, 0\}$. If $r = 3$, then $k \leq 2t$ with equality if and only if $\mu = 1$ and G is the Petersen graph [6, Theorem 1.1]. If $r > 3$, then $k < \frac{r-1}{r+1}n$, equivalently $2k < (r-1)t$ [8, Theorem 3.4], and in this case we let $2k = (r-1)t - a$ ($a \in \mathbb{N}$). In [8, Section 5] it is shown that if G has a tree as a star complement for μ, then $a \notin \{1, 2\}$. On the other hand, the complement of the Clebsch graph is an example in which $a = 4$ and $K_{1,5}$ is a star complement for the eigenvalue 1. To close the gap, we develop new tools which enable us to show that $a \neq 3$. Thus $n \leq \frac{1}{2}(r+1)t - 2$ whenever $r > 3$ and G has a tree as a star complement of order t for an eigenvalue different from -1 and 0.

It follows from the preliminary results in Section 2 that when $r > 3$ and $a = 3$ we have $t \leq 23$ and $n \leq 275$. This leaves an uncomfortably large number of possibilities for the parameters involved, but many are excluded by the new results in Section 3. These results also serve to reduce substantially the arguments required in the remaining cases, which are ruled out individually in Sections 4, 5 and 6.

2. Preliminaries

The fundamental properties of star sets and star complements are established in [5, Chapter 5]. We shall require the following results, where we write $u \sim v$ to mean that vertices u and v are adjacent. For any $U \subseteq V(G)$, we write G_U for the subgraph of G induced by U, and $\Delta_U(v)$ for the set $\{u \in U : u \sim v\}$. For the subgraph H of G it is convenient to write $\Delta_H(v)$ for $\Delta_{V(H)}(v)$.

Theorem 2.1 ([5], Theorem 5.1.7)**.** *Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X.*

(i) *Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \tag{1}$$

(ii) *If X is a star set for μ, then the eigenspace $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ $\left(\mathbf{x} \in \mathbb{R}^k\right)$.*

Let $H = G - X$, where X is a star set for μ. In the notation of Theorem 2.1, C is the adjacency matrix of H, while the columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$. We write $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}^\top(\mu I - C)^{-1}\mathbf{y}$ $\left(\mathbf{x}, \mathbf{y} \in \mathbb{R}^t\right)$, where $t = n - k$. Equation (1) shows that

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and we deduce the following from Theorem 2.1.

Lemma 2.2. *If X is a star set for μ, and $\mu \notin \{-1, 0\}$, then the neighbourhoods $\Delta_H(u)$ $(u \in X)$ are non-empty and distinct.*

We write $\mathbf{j}$ for an all-1 vector, its length determined by context. Recall that μ is a *main* eigenvalue of G if $\mathcal{E}(\mu)$ is not orthogonal to $\mathbf{j}$, and that in an r-regular graph, every eigenvalue other than r is non-main. The next observation follows from Theorem 2.1(ii).

Lemma 2.3. *If X is a star set for the non-main eigenvalue μ, then $\langle \mathbf{b}_u, \mathbf{j} \rangle = -1$ for all $u \in X$.*

Proof. See [5, Proposition 5.2.4]. ■

Lemma 2.4. *If G is r-regular of order n and $\mu < r$, then $\langle \mathbf{j}, \mathbf{j} \rangle = n/(\mu - r)$.*

Proof. See [8, Lemma 2.4]. ■

Lemma 2.5. *Let μ be an eigenvalue of the graph G. If G is connected, then G has a connected star complement for μ.*

Proof. See [5, Theorem 5.1.6]. ■

We write $E(G)$ for the edge-set of G, and for subsets U, V of $V(G)$ we write $E(U, V)$ for the set of edges between U and V. When $H = G - X$ it is convenient to write $\overline{X}$ for $V(H)$. The authors of [3] have determined all the graphs with a star set X for which $E(X, \overline{X})$ is a perfect matching, equivalently all the graphs for which $B = I$ in Equation (1). Their result is the following.

Theorem 2.6. *Let G be a graph with X as a star set for the eigenvalue μ. If $E(X, \overline{X})$ is a perfect matching, then one of the following holds.*

(a) *$G = K_2$ and $\mu = \pm 1$,*

(b) *$G = C_4$ and $\mu = 0$,*

(c) *G is the Petersen graph and $\mu = 1$.*

Now let G be a connected regular graph of degree $r > 3$ and order n, with an eigenvalue $\mu \notin \{-1, 0, r\}$ of multiplicity $k = \frac{1}{2}((r-1)t - a)$ $(a > 0)$. By Lemma 2.6 we may take H $(= G - X)$ to be a connected star complement for μ. Let $\{\mathbf{f}_1, \ldots, \mathbf{f}_t\}$ be the standard orthonormal basis of $\mathbb{R}^t$. Most of the following observations in this section are extracted from [8].

Let $Q = \{i \in X : |\Delta_H(i)| = 1\}$ and $R = X \setminus Q$. Let Q' be the set of vertices in $\overline{X}$ with a neighbour in Q, and let $R' = \overline{X} \setminus Q'$. By Lemma 2.2, $E(Q, Q')$ is a matching when $Q \neq \emptyset$. We take $X = \{1, \ldots, k\}$, $Q = \{1, \ldots, q\}$, $\overline{X} = \{1', \ldots, t'\}$ and $Q' = \{1', \ldots, q'\}$ with $\Delta_H(i) = \{i'\}$ $(i = 1, \ldots, q)$. Without loss of generality, $\mathbf{b}_i = \mathbf{f}_i$ $(i = 1, \ldots, q)$.

Lemma 2.7. *Suppose that S is a non-empty subset of $\overline{X}$ such that $\langle \mathbf{f}_i, \mathbf{j}\rangle = -1$ when $i' \in S$, and $\langle \mathbf{f}_i, \mathbf{j}\rangle = 0$ when $i' \notin S$. Then $\mu \geq 1$ and G_S is regular of degree $1+\mu$. In particular, $\mu \geq 1$ and H is not a tree.*

Proof. See [8, Lemma 3.3]. ∎

Lemma 2.8. *If $j \in R$, then $\mathbf{b}_j$ is not a linear combination of the vectors $\mathbf{b}_i$ ($i \in Q$). Thus each vertex in R is adjacent to a vertex in R'.*

Proof. If $\mathbf{b}_j = \Sigma_{i\in Q} a_i \mathbf{b}_i$, then $\Sigma_{i\in Q} a_i \geq 2$, and by Lemma 2.3 we have $-1 = \langle \mathbf{b}_j, \mathbf{j}\rangle = \Sigma_{i\in Q} a_i \langle \mathbf{b}_i, \mathbf{j}\rangle = -\Sigma_{i\in Q} a_i \leq -2$, a contradiction. ∎

By Lemma 2.2 we have $|\Delta_H(j)| \geq 2$ for all $j \in R$, and so

$$q + 2(k-q) \leq |E(X, \overline{X})| = rt - 2|E(H)| \leq (r-2)t + 2. \tag{3}$$

It follows that if $q = t - b$, then $b \leq a+2$. Moreover, Equations (2) and (3) together show that $2|E(H)| \leq 2t + a - b$. Note that $b \geq 1$ for otherwise $R = \emptyset$ by Lemma 3.1, and then Theorem 2.6 affords a contradiction. Since $t > 1$, no vertex of H is isolated in H, and so by Lemma 2.8, we have

$$k - q \leq |E(R, R')| \leq (r-1)(t-q),$$

equivalently $\frac{1}{2}(r-1)t - \frac{1}{2}a \leq t + (r-2)b$. Hence

$$t \leq 2b + \frac{a+2b}{r-3} \leq 5a + 8. \tag{4}$$

This inequality distinguishes the case $r > 3$ from the case $r = 3$: when $r > 3$ only finitely many graphs arise for prescribed a. Accordingly the general strategy, formulated in [8], is to eliminate successive values of a until we reach a sharp upper bound.

For $i \in R$ let $|\Delta_H(i)| = 1 + g_i$ ($g_i \geq 1$) and $g = \Sigma_{i\in R} g_i$. Then $g \geq k - q$.

Lemma 2.9. *If H is a tree, then*
(i) $g = \frac{1}{2}((r-3)t + a + 4)$,
(ii) $g - (k-q) = a + 2 - b$.

Proof. The first assertion follows from the observation that

$$k + g = |E(X, \overline{X})| = rt - 2|E(H)| = (r-2)t + 2.$$

Part (ii) follows since $q = t - b$. ∎

In subsequent sections we deal with the case in which H is a tree and $a = 3$.

3. The Case $a = 3$, Preparations

In this section we take G be a connected regular graph of degree $r > 3$ and order n, with an eigenvalue $\mu \notin \{-1, 0\}$ of multiplicity $k = \frac{1}{2}((r-1)t - 3)$, and with a tree H $(= G - X)$ of order t as a star complement for μ. Note that r is even and t is odd. By Equation (4), we have $t \leq 23$ and so $n \leq \frac{1}{2}(t-1)(t+2) \leq 275$.

Lemma 3.1. (i) $r \leq t$, (ii) $t \geq 7$, (iii) $\mu = 1$.

Proof. The first assertion follows from the inequality $k \leq \frac{1}{2}(t+1)(t-2)$ [1]. For (ii), suppose by way of contradiction that $t \leq 5$. Then $r = 4$ and $t = 5$, whence $k = 6$. This contradicts [2, Theorem 2.7], which shows that $k \leq 2t - 5$ for quartic graphs. For (iii), we note first that μ is an integer, for otherwise μ has an algebraic conjugate which is a second eigenvalue of multiplicity k. Then $2k < n$, whence $(r-3)\,t < 3$, a contradiction. Now suppose that the eigenvalues different from μ are $\mu_1, \ldots, \mu_t$, with mean d. Then $k\mu + td = 0$ and $k\mu^2 + \Sigma_{i=1}^{t} \mu_i^2 = nr$. Accordingly we have

$$\sum_{i=1}^{t} (\mu_i - d)^2 = \sum_{i=1}^{t} \mu_i^2 - td^2 = (t+k)r - \mu^2 \left(k + t \left(\frac{k^2}{t^2} \right) \right) = (t+k) \left(r - \frac{\mu^2 k}{t} \right).$$

It follows that $k \leq rt/\mu^2$. If $\mu \neq 1$, then $k \leq rt/4$ and we obtain the contradiction $(r-2)\,t \leq 6$. ∎

The next result turns out to be more powerful than Lemma 2.7 for eliminating various configurations. Here we write y_i for $\langle \mathbf{f}_i, \mathbf{j} \rangle$ $(i = 1, \ldots, t)$. Note that by Lemma 2.3, $y_i = -1$ $(i = 1, \ldots, q)$.

Lemma 3.2. *Suppose that* $Z = \{i' \in \overline{X} : y_i \notin \mathbb{Z}\} \neq \emptyset$ *and* $q > 0$. *Then*

(i) G_Z *is not connected,*

(ii) $|Z| \notin \{1, 2, 3\}$.

Proof. We note first that if $\mathbf{y} = (y_1, \ldots, y_t)^\top$, then $(I - C)^{-1}\mathbf{j} = \mathbf{y}$, whence

$$y_i = 1 + \sum_{j \sim i} y_j \ (i = 1, \ldots, t). \tag{5}$$

For (i), suppose by way of contradiction that G_Z is connected. Note that $|Z| < t$ because $y_1 = -1$. Therefore $\overline{X} \setminus Z$ contains a vertex i' adjacent to Z. Since $y_i \in \mathbb{Z}$, Equation (5) shows that $\Delta_H(i')$ contains at least two vertices in Z. Since G_Z is connected, H contains a cycle, a contradiction.

For (ii), suppose first that $i' \in Z$. Then Equation (5) shows that i' has a neighbour in Z, and so $|Z| \geq 2$. Moreover, if $Z = \{i', j'\}$, then $i' \sim j'$ and we have a contradiction from (i). If $Z = \{h', i', j'\}$, then without loss of generality, $h' \nsim j'$ since H has no 3-cycle. Each of h', j' has a neighbour in Z and so necessarily $h' \sim i' \sim j'$; then again we have a contradiction from (i). ∎

Theorem 3.3. *If $r-1$ does not divide n, then $b \geq 4$.*

Proof. From Lemma 2.4, we have $n = (1-r)\langle \mathbf{j}, \mathbf{j}\rangle = (1-r)\Sigma_{i=1}^{t} y_i$. Hence if $r-1$ does not divide n, then $Z \neq \emptyset$. We may suppose that $q > 0$ for otherwise $b = t \geq 7$ by Lemma 3.1(ii). Hence $|Z| \geq 4$ by Lemma 3.3(ii). Since $y_i = -1$ $(i = 1, \dots, q)$, we have $Z \subseteq R'$, and so $b \geq 4$. ∎

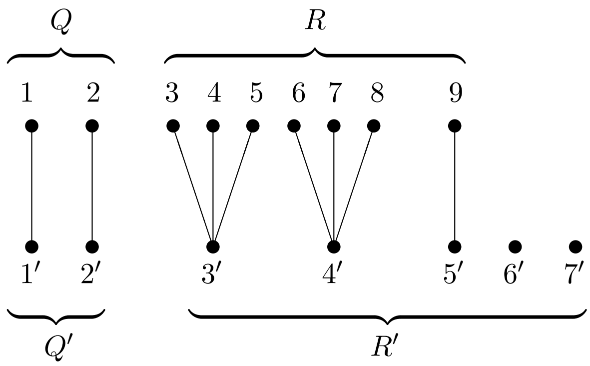

Figure 1. The case $b = 5$, $(t, r, n) = (7, 4, 16)$, $r_3 r_4 r_5 r_6 r_7 = 33100$.

In view of Lemma 2.8 we may partition R as $R_{q+1} \dot\cup \cdots \dot\cup R_t$, where each vertex of R_i is adjacent to i' $(i = q+1, \dots, t)$. Let $r_i = |R_i|$ $(i = q+1, \dots, t)$. Unless otherwise stated we order $\overline{X}$ so that $r_{q+1} \geq \cdots \geq r_t$. Then $r - 1 \geq r_{q+1}$, and we order X so that $u < v$ whenever $u \in R_i$ and $v \in R_j$ with $i < j$. We write $r_{q+1} r_{q+2} \cdots r_t$ for the partition $k - q = r_{q+1} + r_{q+2} + \cdots + r_t$ and say that R *admits* the partition $r_{q+1} r_{q+2} \cdots r_t$. Figure 1 illustrates the labelling of vertices in a (quartic) graph of order 16 in which R admits the partition 33100. Such diagrams can be used to follow the detailed arguments in subsequent sections. Note that any partition $R_{q+1} \dot\cup \cdots \dot\cup R_t$ determines $k - q$ edges in $E(R, R')$, and g further edges are required to complete $E(R, \overline{X})$.

On occasion it is convenient to consider an *extremal* partition of R, where successive R_i are chosen as follows: R_{q+1} is a neighbourhood $\Delta_X(i')$ $(i' \in R')$ of largest size, and for $q + 2 \leq i \leq t$, R_i is a set $\Delta_X(j') \cap (R \setminus (R_{q+1} \dot\cup \cdots \dot\cup R_{i-1}))$ $(j' \in R')$ of largest size. (This construction determines an ordering of R'.) An extremal partition has the properties (i) $r - 1 \geq r_{q+1} \geq \cdots \geq r_t \geq 0$, (ii) if $j < t$ and $R_j \subseteq \Delta_X(i')$, then i' is not adjacent to any vertex in $R_{j+1} \dot\cup \cdots \dot\cup R_t$. For the most part we do not require a partition to be extremal because we need the flexibility to substitute edges and re-order vertices.

Theorem 3.4. *Suppose that the partition $R_{q+1} \dot\cup \cdots \dot\cup R_t$ is extremal.*

(i) *If $R_t \neq \emptyset$, then $y_i \in \mathbb{Z}$ for all $i \in \overline{X}$.*

(ii) *If $g = k - q$ (equivalently $b = a + 2$), then $R_t = \emptyset$.*

Proof. (i) We have $y_j = -1$ for all $j' \in Q'$. We show that $y_j \in \mathbb{Z}$ ($j = q{+}1, \dots, t$) by induction on $i = t + 1 - j$. If $i = 1$, then consider $u \in R_t$. We have $-1 = \langle \mathbf{b}_u, \mathbf{j} \rangle = \Sigma\{y_h : h' \in \Delta_H(u)\}$. Since $\Delta_H(k) \setminus \{t'\} \subseteq Q'$, we have $-1 = -g_k + y_t$, whence $y_t \in \mathbb{Z}$. Now suppose that $i > 1$ and all of $y_{j+1}, \dots, y_t$ are integers. If $u \in R_j$, then $\Delta_H(u) \setminus \{j'\} \subseteq Q' \dot\cup \{(j+1)', \dots, t'\}$. Since $y_h \in \mathbb{Z}$ for every h' in this set, again the equality $-1 = \langle \mathbf{b}_u, \mathbf{j} \rangle$ shows that $y_j \in \mathbb{Z}$.

(ii) When $g = k - q$, we have $g_i = 1$ for all $i \in R$. We suppose by way of contradiction that $R_t \neq \emptyset$; now we can refine the argument for (i). If $u \in R_t$, then $\langle \mathbf{b}_u, \mathbf{j} \rangle = y_h + y_t$ for some $h' \in Q'$, and so $y_t = 0$. Now suppose that $j < t$ and each of $y_{j+1}, \dots, y_t$ is -1 or 0. If $u \in R_j$, then $\langle \mathbf{b}_u, \mathbf{j} \rangle = y_h + y_j$ for some $h' \in Q' \dot\cup \{(j+1)', \dots, t'\}$. Then y_h is -1 or 0, and so the same is true of y_j. It follows by induction that $y_j \in \{-1, 0\}$ for all $j' \in \overline{X}$, and so Lemma 2.7 affords a contradiction ∎

4. The Case $a = 3$, the Details When $b \leq 3$

Throughout Sections 4, 5 and 6, G denotes a connected regular graph of degree $r > 3$ and order n, with an eigenvalue $\mu \notin \{-1, 0\}$ of multiplicity $k = \frac{1}{2}((r-1)t-3)$, and with a tree H ($= G - X$) of order t as a star complement for μ. We use the notation of Section 3 with $a = 3$; then $b \leq 5$. By Lemma 3.1 we have $\mu = 1$, $r \leq t$ and $t \geq 7$, where r is even and t is odd. By Lemma 2.9, we have $g = \frac{1}{2}((r-3)t+7)$ and $g = (k - q) + 5 - b$. We make implicit use of Equation (4) and Lemmas 2.2, 2.3, 2.4. We write $u \overset{*}{\sim} v$ to mean that $u \sim v$ without loss of generality, and $u \sim Q'$ to mean that u is adjacent to a vertex in Q'. In this section we eliminate the cases $b = 1, 2, 3$. If $b = 1$, then $(t, r, n) = (7, 4, 16)$, and so Theorem 3.3 affords a contradiction.

If $b = 2$, then the possibilities for (t, r, n) are $(11, 4, 26), (9, 4, 21), (7, 4, 16)$. Theorem 3.3 eliminates the first and last of these, so we let $(t, r, n) = (9, 4, 21)$. Then $g = 8$, $q = 7$, $k = 12$ and by Lemma 2.4 we have $\langle \mathbf{j}, \mathbf{j} \rangle = -7$. Here $R_8 = \{8, 9, 10\}$ and $R_9 = \{11, 12\}$. Since $8'$ is not isolated in H, both $\Delta_H(11) \setminus \{9'\}$ and $\Delta_H(12) \setminus \{9'\}$ are contained in Q'. Hence either (a) $g_{11} = g_{12} = 1$ and $y_9 = 0$ or (b) $g_{11} = g_{12} = 2$ and $y_9 = 1$. In case (a), $y_8 = 0$ since $\Sigma_{i=1}^{9} y_i = -7$, and without loss of generality both $\Delta_H(8) \setminus \{8'\}$ and $\Delta_H(9) \setminus \{8'\}$ are contained in Q'. Hence $g_8 = g_9 = 1$, and then $g_{10} = 4$ because $g = 8$. Now $\langle \mathbf{b}_{10}, \mathbf{j} \rangle \neq -1$, a contradiction. In case (b), $y_8 = -1$ and so $\langle \mathbf{b}_i, \mathbf{j} \rangle \in \{-g_i{-}1, -g_i{+}1\}$ ($i = 8, 9, 10$). It follows that $g_i = 2$ ($1 = 8, 9, 10$), giving the contradiction $g = 10$.

If $b = 3$, then the possibilities for (t, r, n) are $(15, 4, 36), (13, 4, 31), (11, 4, 26), (9, 6, 30), (9, 4, 21), (7, 6, 23), (7, 4, 16)$. Those with $t \neq 9$ or 15 are eliminated by Theorem 3.3. If $(t, r, n) = (15, 4, 36)$, then $|R_{13}| = |R_{14}| = |R_{15}| = 3$ and each of R_{13}, R_{14}, R_{15} has a vertex $i \sim Q'$ such that $g_i = 1$. Hence $y_{13} = y_{14} = y_{15} = 0$

and Lemma 2.7 affords a contradiction. An entirely analogous argument disposed of the case $(t, r, n) = (9, 6, 30)$.

Now suppose that $(t, r, n) = (9, 4, 21)$; then $g = 8$, $q = 6$, $k = 12$, $\langle \mathbf{j}, \mathbf{j} \rangle = -7$ and $r_7r_8r_9 \in \{330, 321, 222\}$. If $r_7r_8r_9 = 330$, then without loss of generality either (a) $g_7 = g_8 = g_{10} = g_{11} = 1$ or (b) $g_7 = g_{10} = g_{11} = g_{12} = 1$ and $g_8 = g_9 = 2$. In case (a), without loss of generality each of the vertices 7, 10 has a neighbour in Q' and so $y_7 = y_8 = 0$. Since $\Sigma_{i=1}^{9} y_i = -7$ we have $y_9 = -1$, and so Lemma 2.7 affords a contradiction. In case (b), we have the same contradiction if vertex 7 has a neighbour in Q'. Accordingly $7 \sim 9'$ and so $y_7 + y_9 = -1$. Then $y_8 = 0$. If $8 \sim 9'$, then $-1 = \langle \mathbf{b}_8, \mathbf{j} \rangle = -1 + y_7 + y_9$, a contradiction. Hence $8 \not\sim 9'$ and $-1 = -2 + y_7$; therefore $y_7 = 1$ and $y_9 = -2$. Now $7'$ is an endvertex in H, and if $\Delta_H(7') = \{j'\}$, then $1 = 1 + y_j$ by Equation (5). Hence $j' = 8'$ (also an endvertex of H), and so H is not connected, a contradiction.

If $r_7r_8r_9 = 321$, then $R_7 = \{7, 8, 9\}$, $R_8 = \{10, 11\}$ and $R_9 = \{12\}$. We assume that $12 \not\sim 8'$ because otherwise R admits 330. If $g_{12} = 3$ then $y_9 = 2$ and $g_i = 1$ $(i = 7, 8, 9, 10, 11)$. Each of R_7,R_8 has a neighbour in Q' and so $y_7 = y_8 = 0$. Then $\langle \mathbf{j}, \mathbf{j} \rangle = -4$, a contradiction. If $g_{12} = 2$, then $y_9 = 1$ and we may take $g_7 = g_8 = g_{10} = 1$. In this case, if $7 \not\sim Q'$ and $8 \not\sim Q'$, then $7 \overset{*}{\sim} 8'$ and $8 \sim 9'$; then $y_7 = -2$, $y_8 = 1$, $\langle \mathbf{j}, \mathbf{j} \rangle = -6$, a contradiction. Accordingly $7 \overset{*}{\sim} Q'$, and so $y_7 = 0$, $y_8 = -2$, $10 \sim 9'$, $g_{11} = 2$. Then $\langle \mathbf{b}_{11}, \mathbf{j} \rangle \in \{-2, -4\}$, a contradiction. Hence $g_{12} = 1$ and $y_9 = 0$. Suppose first that at least two of g_7, g_8, g_9 are equal to 1, say $g_7 = g_8 = 1$. If $7 \not\sim Q'$ and $8 \not\sim Q'$, then $7 \overset{*}{\sim} 8'$ and $8 \sim 9'$, whence $y_7 = -1$, $y_8 = 0$ and Lemma 2.7 affords a contradiction. Consequently, $7 \overset{*}{\sim} Q'$, $y_7 = 0$, $y_8 = -1$ and we invoke Lemma 2.7 again. It follows that we may take $g_7 = 1$ and $g_8 = g_9 = 2$. Then $g_{10} = g_{11} = g_{12} = 1$, $10 \overset{*}{\sim} 9'$, $y_8 = 0$, $y_7 = -1$ and we appeal to Lemma 2.7 once more.

If $r_7r_8r_9 = 222$ then we may assume that $g_7 = g_8 = g_9 = 1$ and that R does not admit 321. Then each of the vertices 7, 8, 9 is adjacent to Q' and so $y_7 = y_8 = 0$. Hence $y_9 = -1$ and we obtain a contradiction from Lemma 2.7.

We conclude that $b \geq 4$.

5. The Case $a = 3$, the Details When $b = 4$

Here $g = k - q + 1$ by Lemma 2.9(ii). It follows from Lemma 3.2 that if $r - 1$ does not divide n, then $y_j \notin \mathbb{Z}$ for all $j' \in R'$, and in this case, by Theorem 3.4(i), it suffices to eliminate all partitions with $r_t = 0$. The possibilities for (t, r, n) are (19, 4, 46), (17, 4, 41), (15, 4, 36), (13, 4, 31), (11, 6, 37), (11, 4, 26), (9, 8, 39), (9, 6, 30), (9, 4, 21), (7, 6, 23), (7, 4,16).

If $(t, r, n) = (19, 4, 46)$ or $(17, 4, 41)$, then we find readily that $y_j \in \{-1, 0\}$ for all $j \in \overline{X}$, a contradiction.

If $(t,r,n) = (15,4,36)$, then $q = 11$, $k-q = 10$, $g = 11$, $\langle \mathbf{j},\mathbf{j}\rangle = -12$ and either (a) $r_{12}r_{13}r_{14}r_{15} = 3331$ or (b) $r_{12}r_{13}r_{14}r_{15} = 3322$ and R does not admit 3331. In both cases, again $y_j \in \{-1,0\}$ for all $j \in \overline{X}$.

If $(t,r,n) = (13,4,31)$, then $q = 9$, $k-q = 9$, $g = 10$, $\langle \mathbf{j},\mathbf{j}\rangle \notin \mathbb{Z}$ and we may assume that $r_{10}r_{11}r_{12}r_{13} = 3330$. In this case, each of R_{10}, R_{11}, R_{12} has a vertex $i \sim Q'$ with $g_i = 1$. Hence $y_{10} = y_{11} = y_{12} = 0$, a contradiction.

If $(t,r,n) = (11,6,37)$, then $r_8r_9r_{10}r_{11} = 5554$ and we find that $y_j = 0$ for all $j' \in R'$. Then Lemma 2.7 affords a contradiction.

If $(t,r,n) = (11,4,26)$, then $q = 7$, $k-q = 8$, $g-9$, $\langle \mathbf{j},\mathbf{j}\rangle \notin \mathbb{Z}$ and we may assume that $r_8r_9r_{10}r_{11} = 3320$. Without loss of generality $g_8 = g_9 = g_{10} = 1$. Since $y_8 \neq 0$, $\Delta_H(i) = \{7', j'\}$ where $j' \in \{10'11'\}$, and so we have two vertices in R_8 with the same H-neighbourhood.

If $(t,r,n) = (9,8,39)$, then $q = 5$, $k-q = 25$, $g = 26$ and $\langle \mathbf{j},\mathbf{j}\rangle \notin \mathbb{Z}$. Note that $r_7 = 7$ and so R_7 has a least 6 vertices with an H-neighbourhood of size 2. At least three of these vertices have an H-neighbourhood of the form $\{j', 6'\}$ $(j' \in Q')$, and so $y_6 = 0$, a contradiction.

If $(t,r,n) = (9,6,30)$, then $q = 5$, $k-q = 16$, $g = 17$ and $\langle \mathbf{j},\mathbf{j}\rangle = -6$. Note that $r_6 \in \{4,5\}$. If $r_6 = 5$, then $r_6r_7r_8r_9 \in \{5551, 5542, 5533, 5443\}$ and for $h \in \{6,7,8\}$, R_h has a vertex i such that $\Delta_H(i) = \{j', h'\}$ $(j' \in Q')$. Then $y_6 = y_7 = y_8 = 0$, $y_9 = -1$, and we have a contradiction from Lemma 2.7. It follows that $r_6 = r_7 = r_9 = r_9 = 4$, and we may assume that $|\Delta_R(j')| = 4$ for all $j' \in R'$. Then $\Delta_H(i) \setminus \{j'\} \subseteq Q'$ for all $i \in \Delta_R(j')$ $(j = 6,7,8,9)$. Hence $y_6 = y_7 = y_8 = y_9 = 0$ and $\langle \mathbf{j},\mathbf{j}\rangle = -5$, a contradiction.

If $(t,r,n) = (9,4,21)$, then $q = 5$, $k-q = 7$, $g = 8$, $\langle \mathbf{j},\mathbf{j}\rangle = -7$ and $r_6r_7r_8r_9 \in \{3310, 3220, 3211, 2221\}$. Suppose first that $r_6r_7r_8r_9 = 3310$. Without loss of generality, $g_6 = g_7 = g_8 = g_9 = g_{10} = 1$. Then $6 \overset{*}{\sim} Q'$ and so $y_6 = 0$. If $g_{11} = 1$, then similarly $y_7 = 0$, while $g_{12} = 2$. Now $\langle \mathbf{b}_{12},\mathbf{j}\rangle = -1 + y_8 + y_9$ or $-2 + y_8$. In the first case, we have the contradiction $\langle \mathbf{j},\mathbf{j}\rangle = -5$; in the second case, $y_8 = 1, y_9 = -3$, $\deg_H(8') = 3$ and the equation $y_8 = 1 + \Sigma_{j' \sim 8'} y_j$ cannot be satisfied. Hence $g_{11} = 2$, $g_{12} = 1$. If $y_7 \neq 0$, then $9 \overset{*}{\sim} 8'$, $10 \sim 9'$ and so $y_7 + y_8 = y_7 + y_9 = -1$ while $y_7 + y_8 + y_9 = -2$, a contradiction. Hence $y_7 = 0$, $y_8 + y_9 = -2$, $12 \not\sim 9'$, $12 \sim Q'$, $y_8 = 0$; then $y_9 = -2$ and $\Delta_H(11) = \{j', 7', 8'\}$ for some $j' \in Q'$. By Equation (5) we have $6' \sim Q'$, $7 \sim Q'$ and so $8' \not\sim 6'$, $8' \not\sim 7'$. Again the equation $y_8 = 1 + \Sigma_{j' \sim 8'} y_j$ cannot be satisfied.

Now suppose that $r_6r_7r_8r_9 = 3220$. We may assume that $g_6 = g_7 = g_9 = g_{10} = g_{11} = 1$ and that R does not admit 3310. Then $9 \overset{*}{\sim} Q'$ and $y_7 = 0$. If $g_8 = 1$, then $g_{12} = 2$; also $y_6 = 0$ for otherwise $6 \overset{*}{\sim} 7'$, $7 \overset{*}{\sim} 8'$, $8 \sim 9'$ and R admits 3310. Similarly $11 \not\sim 7$, while $11 \not\sim 9'$ because $y_8 + y_9 = -2$. Hence $11 \sim Q'$, $y_8 = 0$, $y_9 = -2$ and $\langle \mathbf{b}_{12},\mathbf{j}\rangle \in \{-2,-3\}$, a contradiction. Hence $g_8 = 2$ and $g_{12} = 1$, $11 \overset{*}{\sim} Q'$ and $y_8 = 0$, $y_6 + y_9 = -2$, $6 \not\sim 9'$. By Lemma 2.7, $y_6 \neq -1$ and so $6 \not\sim 7'$, $6 \not\sim 8'$. Hence $6 \sim Q'$, $y_6 = 0$, $y_9 = -2$ and again $\langle \mathbf{b}_8,\mathbf{j}\rangle \in \{-2,-3\}$.

Now suppose that $r_6r_7r_8r_9 = 3211$. We may assume that $g_6 = g_7 = g_9 = g_{11} = 1$ and that R does not admit 3310 or 3220; in particular, $11 \sim Q'$ and so $y_8 = 0$. If $g_8 = 1$, then $y_6 = 0$ for otherwise $6 \overset{*}{\sim} 7'$, $7 \overset{*}{\sim} 8'$, $8 \sim 9'$ and $y_6 + y_7 = y_6 + y_8 = y_6 + y_9 = -1$, while $y_6 + y_7 + y_8 + y_9 = -2$; then $y_6 = y_7 = y_8 = y_9 = -\frac{1}{2}$ and $\langle \mathbf{b}_{11}, \mathbf{j}\rangle = -\frac{3}{2}$, a contradiction. If $g_{12} = 1$, then $g_{10} = 2$, $12 \sim Q'$ and $y_9 = 0$; then $y_7 = -2$ and $\langle \mathbf{b}_{10}, \mathbf{j}\rangle \neq -1$. Therefore $g_{12} = 2$, $g_{10} = 1$, $10 \overset{*}{\sim} Q'$, $y_7 = 0$, $y_9 = -2$ and $\langle \mathbf{b}_{12}, \mathbf{j}\rangle \neq -1$. Hence $g_8 = 2$, $g_{12} = 1$, $9 \overset{*}{\sim} Q'$, $12 \sim Q'$ and so $y_7 = y_9 = 0$; then $y_6 = -2$ and $\langle \mathbf{b}_8, \mathbf{j}\rangle \neq -1$.

Lastly suppose that $r_6r_7r_8r_9 = 2221$. We may assume that $g_6 = g_7 = g_8 = g_9 = g_{10} = 1$ and that R does not admit 3310, 3220 or 3211. If $g_{11} = 1$, then $6 \overset{*}{\sim} Q'$, $8 \overset{*}{\sim} Q'$, $10 \overset{*}{\sim} Q'$ and so $y_6 = y_7 = y_8 = 0$, $y_9 = -2$ and $\langle \mathbf{b}_{12}, \mathbf{j}\rangle \neq -1$. Hence $g_{11} = 2$ and $g_{12} = 1$. Now $12 \sim Q'$, $6 \overset{*}{\sim} Q'$, $8 \overset{*}{\sim} Q'$ and so $y_6 = y_7 = y_9 = 0$, $y_8 = -2$ and $\langle \mathbf{b}_{11}, \mathbf{j}\rangle \neq -1$.

If $(t, r, n) = (7, 6, 23)$, then $q = 3$, $k - q = 13$, $g = 14$ and $\langle \mathbf{j}, \mathbf{j}\rangle \notin \mathbb{Z}$. If $r_4 = 5$, then R_4 has a vertex i, with a neighbour in Q', such that $g_i = 1$; then $y_4 = 0$, a contradiction. Hence $r_4 = 4$ and we may assume that $|\Delta_R(j')| \leq 4$ for all $j' \in R'$. If $r_5 = 4$, then R_5 has two vertices with the same H-neighbourhood (since $y_5 \neq 0$), and so $r_4r_5r_6r_7 = 4333$. We may assume that $g_i = 1$ for all $i \notin \{7, 13\}$, and since $y_4 \neq 0$ we have $4 \overset{*}{\sim} 5'$, $5 \overset{*}{\sim} 6'$, $6 \sim 7'$. Then $y_4 + y_5 = y_4 + y_6 = y_4 + y_7 = -1$. Since $8 \not\sim 4'$, $9 \not\sim 4'$ and $y_5 \neq 0$, we have $8 \overset{*}{\sim} 6'$ and $9 \sim 7'$; then $y_5 + y_6 = y_5 + y_7 = -1$. It follows that $y_4 = y_5 = y_6 = y_7 = -\frac{1}{2}$ and so $\langle \mathbf{j}, \mathbf{j}\rangle = -5$, a contradiction.

If $(t, r, n) = (7, 4, 16)$, then $q = 3$, $k - q = 6$, $g = 7$, $\langle \mathbf{j}, \mathbf{j}\rangle \notin \mathbb{Z}$, and $r_4r_5r_6r_7 \in \{3300, 3210, 3111, 2220, 2211\}$. If $r_4r_5r_6r_7 = 3300$, then we may assume that $g_4 = g_5 = g_6 = 1$. Since $y_4 \neq 0$, R_4 has two vertices with the same H-neighbourhood.

Suppose that $r_4r_5r_6r_7 = 3210$. We may assume that $g_4 = g_5 = g_7 = 1$ and that R does not admit 3300; in particular, $9 \not\sim 5'$. If $g_6 = 1$, then $4 \overset{*}{\sim} 5'$, $5 \overset{*}{\sim} 6'$, $6 \sim 7'$ and $y_4 + y_5 = y_4 + y_6 = y_4 + y_7 = -1$. Now $9 \not\sim 7'$ for otherwise $y_6 + y_7 = -1$ or 0 and $\langle \mathbf{j}, \mathbf{j}\rangle \in \mathbb{Z}$. Hence $9 \sim Q'$ and $y_6 = 0$ or 1, a contradiction. Hence $g_6 = 2$ and $g_i = 1$ for all $i \neq 6$; then $7 \overset{*}{\sim} 6'$, $8 \overset{*}{\sim} 7'$ and $y_5 + y_6 = y_5 + y_7 = -1$. Since $y_6 \neq 0$, we have $\Delta_H(9) = \{6', 7'\}$ and so $y_6 + y_7 = -1$. Now, considering the three possibilities for $\{\Delta_H(4), \Delta_H(5)\}$, we find that $y_4 = y_5 = y_6 = y_7 = -\frac{1}{2}$, a contradiction as before.

Suppose that $r_4r_5r_6r_7 = 3111$. We may assume that $g_7 = 1$ and that R does not admit 3210. Then $7 \sim Q'$ and $y_5 = 0$, a contradiction.

Suppose that $r_4r_5r_6r_7 = 2220$. We may assume that $g_4 = g_5 = g_6 = g_7 = g_8 = 1$ and that R does not admit 3210. Then R has 5 vertices adjacent to $7'$, a contradiction.

Lastly suppose that $r_4r_5r_6r_7 = 2211$. We may assume that $g_4 = g_5 = g_6 = g_8 = 1$ and that $|\Delta_R(j')| \leq 2$ for all $j' \in R'$. But, then $4 \overset{*}{\sim} 6'$, $5 \sim 7'$, $8 \sim 7'$ and we have the contradiction $|\Delta_R(7')| = 3$.

We conclude that $b = 5$.

6. The Case $a = 3$, the Details When $b = 5$

Here $g = k - q$ and so $g_i = 1$ for all $i \in R$. By Lemma 3.2, if $\langle \mathbf{j}, \mathbf{j} \rangle \notin \mathbb{Z}$, then at most one of $y_{q+1}, \ldots, y_t$ is an integer. By Theorem 3.4(ii), it suffices to eliminate all partitions for which $r_t = 0$. The possible values of (t, r, n) are (23, 4, 56), (21, 4, 51), (19, 4, 46), (17, 4, 41), (15, 4, 36), (13, 6, 44), (13, 4, 31), (11,10, 59), (11, 8, 48), (11, 6, 37), (11, 4 26), (9, 8, 39), (9, 6, 30), (9, 4, 21), (7, 6, 23), (7,4,16). The cases (23, 4, 56), (21, 4, 51), (19, 4, 46), (13, 6, 44), (11,10, 59) are eliminated by Lemma 2.7.

If $(t, r, n) = (17, 4, 41)$, then $q = 12$, $k - q = 12$, $\langle \mathbf{j}, \mathbf{j} \rangle \notin \mathbb{Z}$ and $r_{13}r_{14}r_{15}r_{16} = 33330$. Each of $R_{13}, R_{14}, R_{15}, R_{16}$ has a vertex adjacent to Q' and so $y_{13} = y_{14} = y_{15} = y_{16} = 0$, a contradiction.

If $(t, r, n) = (15, 4, 36)$, then $q = 10$, $k - q = 11$, $\langle \mathbf{j}, \mathbf{j} \rangle = -12$ and $r_{11}r_{12}r_{13}r_{14}r_{15} = 33320$. Each of $R_{11}, R_{12}, R_{13}, R_{14}$ has a vertex adjacent to Q' and so $y_{11} = y_{12} = y_{13} = y_{14} = 0$, $y_{15} = -2$. Moreover, $\deg_H(14') = 2$. By Equation (5), each of $11', 12', 13'$ is adjacent to Q', and so the equation $y_{14} = 1 + \Sigma_{j' \sim 14'} y_j$ cannot be satisfied.

If $(t, r, n) = (13, 4, 31)$, then $q = 8$, $k - q = 10$, $\langle \mathbf{j}, \mathbf{j} \rangle \notin \mathbb{Z}$ and $r_9r_{10}r_{11}r_{12}r_{13} \in \{33310, 33220\}$. If $r_9r_{10}r_{11}r_{12}r_{13} = 33310$, then each of R_9, R_{10}, R_{11} has a vertex adjacent to Q' and so $y_9 = y_{10} = y_{11} = 0$, a contradiction. If $r_9r_{10}r_{11}r_{12}r_{13} = 33220$ and R does not admit 33310, then similarly $y_{11} = y_{12} = 0$, a contradiction.

If $(t, r, n) = (11, 8, 48)$, then $q = 6$, $k - q = 31$ and no partition has $r_t = 0$, contradicting Theorem 3.4(ii).

If $(t, r, n) = (11, 6, 37)$, then $q = 6$, $k - q = 20$, $\langle \mathbf{j}, \mathbf{j} \rangle \notin \mathbb{Z}$ and $r_7r_8r_9r_{10}r_{11} = 55550$. In this case, we have $y_7 = y_8 = y_9 = y_{10} = 0$, a contradiction.

If $(t, r, n) = (11, 4, 26)$, then $q = 6$, $k - q = 9$, $\langle \mathbf{j}, \mathbf{j} \rangle \notin \mathbb{Z}$ and $r_7r_8r_9r_{10}r_{11}$ is one of 33300, 33210, 32220. If $r_7r_8r_9r_{10}r_{11} = 33300$, then $y_7 = y_8 = y_9 = 0$, a contradiction. Suppose that $r_7r_8r_9r_{10}r_{11} = 33210$ and that R does not admit 33300; then $15 \not\sim 9'$. Also, $15 \not\sim Q'$ for otherwise $y_{10} = 0$, $y_7 \neq 0$, $y_8 \neq 0$, $7 \overset{*}{\sim} 9'$, $10 \overset{*}{\sim} 9'$ and $9'$ is isolated in H. Hence $15 \sim 11'$ and $y_{10} + y_{11} = -1$. If $y_8 \neq 0$, then $10 \overset{*}{\sim} 9'$, $11 \overset{*}{\sim} 10'$, $12 \overset{*}{\sim} 11'$, $7 \overset{*}{\sim} Q'$ and so $y_8 + y_9 = y_8 + y_{10} = y_9 + y_{11} = -1$, $y_7 = 0$. Then $y_8 = y_9 = y_{10} = y_{11} = -\frac{1}{2}$ and $\langle \mathbf{j}, \mathbf{j} \rangle = -8$, a contradiction. If $r_7r_8r_9r_{10}r_{11} = 32220$ and R does not admit 33300 or 33210, then each of R_8, R_9, R_{10} has a vertex adjacent to Q', and so $y_8 = y_9 = y_{10} = 0$, a contradiction.

If $(t, r, n) = (9, 6, 30)$, then $q = 4$, $k - q = 17$, $\langle \mathbf{j}, \mathbf{j} \rangle = -6$ and $r_5r_6r_7r_8r_9$ is one of 55520, 55430, 54440. In all cases, each of R_5, R_6, R_7, R_8 has a vertex with a neighbour in Q', and so $y_5 = y_6 = y_7 = y_8 = 0, y_9 = -2$. In particular, no vertex in R_5 is adjacent to $8'$ or $9'$. Hence two vertices in R_5 have the same H-neighbourhood, a contradiction.

If $(t, r, n) = (9, 4, 21)$, then $q = 4$, $k - q = 8$, $\langle \mathbf{j}, \mathbf{j} \rangle = -7$ and $r_5r_6r_7r_8r_9$ is one of 33200, 33110, 32210, 22220. Suppose that $r_5r_6r_7r_8r_9 = 33200$. If $y_6 \neq 0$, then $8 \overset{*}{\sim} 7'$, $9 \overset{*}{\sim} 8'$ and $10 \sim 9'$, whence $y_6 + y_7 = y_6 + y_8 = y_6 + y_9 = -1$. Also, $5 \overset{*}{\sim} Q'$ and so $y_5 = 0$. If $y_7 \neq 0$, then $y_7 + y_8 = y_7 + y_9 = -1$ and we have $y_6 = y_7 = y_8 = y_9 = -\frac{1}{2}$; then $y_5 = -1$ and so $\langle \mathbf{b}_5, \mathbf{j} \rangle \neq -1$. Hence $y_7 = y_8 = y_9 = 0$ and $y_6 = -1$, $y_5 = -2$ and again $\langle \mathbf{b}_5, \mathbf{j} \rangle \neq -1$. Accordingly $y_6 = 0$, and similarly $y_5 = 0$. If $y_7 \neq 0$, then again $11 \overset{*}{\sim} 8'$, $12 \sim 9'$ and we find that $y_7 = 1$ and $y_8 = y_9 = -2$. Now $\deg_H(7') = 2$ and the equation $y_7 = 1 + \Sigma_{j' \sim 7'} y_j$ shows that $\Delta_H(7') = \{5', 6'\}$. Hence the vertices $5', 6', 7'$ induce a component of H, a contradiction. Therefore $y_5 = y_6 = y_7 = 0$ and we have $y_8 + y_9 = -3$, while no vertex in $R_5 \dot{\cup} R_6$ is adjacent to $7'$. In this situation, we use Equation (5) to reconstruct H as far as necessary to obtain a contradiction.

For $j' \in \overline{X}$, let $d_j = \deg_H(j')$, with $d_1 \geq d_2 \geq d_3 \geq d_4$. If there is no edge from X to $\{8', 9'\}$, then $d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = 1$, $d_7 = 2$, $d_8 = d_9 = 4$ and H consists of two stars of order 5 intersecting in $\{7'\}$. Now Equation (5) shows that $y_8 + y_9 = -1$, a contradiction. If there is an edge from X to $\{8', 9'\}$, say from X to $8'$, then $y_8 = -1$, and so $y_9 = -2$, $d_9 = 4$, $d_8 \in \{1, 2, 3\}$. Note that if $j' \in Q'$ and $d_j = 1$, then $j' \sim 9'$ by Equation (5).

If $d_8 = 3$, then $d_2 = d_3 = d_4 = 1$ and so $2', 3', 4'$, are adjacent to $9'$. If the fourth neighbour of $9'$ is j', then j' is not an endvertex of H, and $y_j = 0$ by Equation (5). Hence $j' = 7'$ and by Equation (5), $0 = y_7 = 1 + (-2) + y_h$, whence $y_h = 1$, a contradiction.

If $d_8 = 2$, then we may repeat the preceding argument when $d_2 = d_3 = d_4 = 1$. Hence $d_1 = d_2 = 2$, $d_3 = d_4 = 1$ and as before $3', 4'$ are adjacent to $9'$. If $9' \sim 5'$, then $\Delta_H(9') = \{3', 4', 5', j'\}$, where $d_j > 1$ and $y_j = -1$. If $j' = 1'$, then $\Delta_H(1') = \{9', 1'\}$, where $d_h > 1$ and $y_h = 0$; hence $h' = 7'$ and $\Delta_H(7') = \{1', l'\}$, where $d_l > 1$ and $y_l = 0$. No such l exists and so $j' \neq 1'$; similarly $j' \neq 2'$. Therefore $\Delta_H(9') = \{3', 4', 5', 8'\}$. A similar argument shows that $8' \sim 7'$. Now $\Delta_H(7') = \{8', i'\}$, where $0 = y_7 = 1 + (-1) + y_i$; then $y_i = 0$ and so i' is an endvertex of H, a contradiction. Hence $9' \not\sim 5'$, and similarly $9' \not\sim 4'$; therefore $9' \sim 7'$. By Equation (5) we have $\Delta_(7') = \{9', j'\}$ where $0 = y_7 = 1 + (-2) + y_j$, and so $y_j = 1$, a contradiction.

If $d_8 = 1$, then $d_4 = 1$ and by Equation (5), $8' \sim 9'$, $4' \sim 9'$. Let $\Delta_H(9') = \{4', 8', i', j'\}$, where $y_i = 0$, $y_j = -1$. Then i' is an endvertex of H for otherwise $i' = 7'$, $\Delta_H(7') = \{9', h'\}$ where $y_h = 1$. Hence $i' = 5'$ without loss of generality, and $d_j = 2$ or 3. If $d_j = 3$, then by Equation (5), $\Delta_H(j') = \{6', 7', 9'\}$, $\Delta_H(7') = \{j', h'\}$ where $y_h = 0$. This is a contradiction because $\overline{X}$ has only three vertices h' such that $y_h = 0$. Hence $d_j = 2$, $\Delta_H(j') = \{9', h'\}$ where $y_h = 0$ and h' is not an endvertex of H. Thus $h' = 7'$ and $\Delta_H(7') = \{j', l'\}$ where $y_l = 0$ and l' is not an endvertex of H. No choice remains for l and so we have eliminated the partition 33200.

Now suppose that $r_5 r_6 r_7 r_8 r_9 = 33110$ and that R does not admit 33200. Then $11 \not\sim 8'$ and $12 \not\sim 7'$. If $y_6 \neq 0$, then $8 \overset{*}{\sim} 7'$, $8 \overset{*}{\sim} 8'$, $10 \sim 9'$ and so

$y_6+y_7=y_6+y_8=y_6+y_9=-1$. If also $11\sim 9'$, then $y_8+y_9=-1$, $y_6=y_7=y_8=y_9=-\frac{1}{2}$, $y_5=-1$, $\langle\mathbf{b}_5,\mathbf{j}\rangle\neq -1$. Hence $11\not\sim 9'$, and similarly $12\not\sim 9'$. Since $\Delta_H(11)\neq\Delta_H(12)$, we may assume that $11\sim Q'$; then $y_7=0=y_8=y_9$, $y_6=-1$, $y_5=-2$ and again $\langle\mathbf{b}_5,\mathbf{j}\rangle\neq -1$. Therefore $y_6=0$, and similarly $y_5=0$. If $12\sim 9'$, then $11\not\sim 9'$, $11\sim Q'$ and $y_7=0$. Since also $y_8+y_9=-1$, we find that $\langle\mathbf{j},\mathbf{j}\rangle=-5$, a contradiction. Hence $12\not\sim 9'$, $12\sim Q'$ and $y_8=0$. Similarly $11\sim Q'$ and $y_7=0$. Hence $y_9=-3$ and by Equation (5), $\Delta_H(9')=Q'$. On the other hand, each vertex of R is adjacent to Q' and so $|E(Q',X)|=12$, whence $d_1=d_2=d_3=d_4=1$. Accordingly $5\not\sim Q'$ and the equation $y_5=1+\Sigma_{j'\sim 5'}y_j$ cannot be satisfied.

Now suppose that $r_5r_6r_7r_8r_9=32210$ and that R does not admit 33200 or 33110. If $y_6\neq 0$ and $y_7\neq 0$, then $8\overset{*}{\sim}8'$, $9\sim 9'$, $10\overset{*}{\sim}8'$, $11\sim 9'$ and we find that R does admit 33110. Accordingly we may assume that $y_6=0$. If $y_7\neq 0$, then $10\overset{*}{\sim}8'$, $11\sim 9'$; moreover $12\not\sim 9'$ for otherwise $y_7+y_8=y_7+y_9=y_8+y_9=-1$ whence $y_7=y_8=y_9=-\frac{1}{2}$, $y_5=-\frac{3}{2}$ and $\langle\mathbf{b}_5,\mathbf{j}\rangle\neq -1$. Hence $12\sim Q'$ and $y_8=0$, $y_7=-1$, $y_9=0$, $y_5=-2$ and again $\langle\mathbf{b}_5,\mathbf{j}\rangle\neq -1$. Hence $y_6=y_7=0$. If $y_5\neq 0$, then $5\overset{*}{\sim}6'$, $y_5=-1$, $y_8+y_9=-2$ and so $12\not\sim 9'$, $12\sim Q'$, $y_8=0$, $y_9=-2$. From Equation (5) we find in turn that $5'\sim 9'$, $6'\sim Q'$ and $7'\sim Q'$. We also have $6\overset{*}{\sim}7'$, $7\sim 8'$ and $d_8\leq 2$. If $d_8=2$, then by Equation (5), $8'\sim 7'$ and $8'\sim Q'$, while if $d_8=1$, then $8'\sim 7'$. In either case, each of $6',7',8'$ is adjacent to Q' and so the equation $y_9=1+\Sigma_{j'\sim 9'}y_j$ cannot be satisfied. Therefore $y_5=y_6=y_7=0$, $y_8+y_9=-3$, $12\not\sim 9'$, $12\sim Q'$, $y_8=0$, $y_9=-3$. Then each vertex of X is adjacent to Q' and so $d_1=d_2=d_3=d_4=1$. Now the equation $y_1=1+\Sigma_{j'\sim 1'}y_j$ cannot be satisfied.

Lastly suppose that $r_5r_6r_7r_8r_9=22220$ and that R does not admit 33200, 33110 or 32210. Then each of R_5,R_6,R_7,R_8 has a vertex adjacent to Q' and so $y_5=y_6=y_7=y_8=0$, $y_9=-3$. Then $9'\not\sim X$ and each vertex of X is adjacent to Q'. Therefore $d_1=d_2=d_3=d_4=1$. By Equation (5), $\Delta_H(9')=Q'$ and so the subgraph induced by $5',6',7',8'$ is 2-regular. Hence H has a cycle, a contradiction.

If $(t,r,n)=(7,6,23)$, then $q=2$, $k-q=14$ and $\langle\mathbf{j},\mathbf{j}\rangle=-\frac{23}{5}$. Let $R_3\dot{\cup}R_4\dot{\cup}R_5\dot{\cup}R_6\dot{\cup}R_7$ be an extremal partiton of R, and suppose by way of contradiction that $|R_3|\geq 4$. If R_3 contains a vertex adjacent to Q', then $y_3=0$, $5\overset{*}{\sim}4'$ and $y_4=-1$, contradicting Lemma 3.2. Hence $3\overset{*}{\sim}4'$, $4\overset{*}{\sim}5'$, $5\overset{*}{\sim}6'$, $6\sim 7'$ and we have $y_3+y_4=y_3+y_5=y_3+y_6=y_3+y_7=-1$. Since $R_3\neq R$ we may consider a vertex $v\in R_4$. If $v\sim Q'$, then $y_j\in\{-1,0\}$ for all $j'\in\overline{X}$, contradicting Lemma 2.7. Therefore $v\sim R'$ and $y_3=y_4=y_5=y_6=y_7=-\frac{1}{2}$, whence $\langle\mathbf{j},\mathbf{j}\rangle=-\frac{11}{2}$, a contradiction. It follows that $|R_3|=|R_4|=|R_5|=R_6|=3$, and $|R_7|=2$. Now Theorem 3.4(ii) affords a contradiction.

If $(t,r,n)=(7,4,16)$, then $q=2$, $k-q=7$, $\langle\mathbf{j},\mathbf{j}\rangle=-\frac{16}{3}$ and $r_3r_4r_5r_6r_7$ is one of 33100, 32200, 32110, 22210. Suppose that $r_3r_4r_5r_6r_7=33100$ (Figure 1).

We may assume that $y_4 \neq 0$, and then $6 \sim 5'$, $7 \overset{*}{\sim} 6'$, $8 \sim 7'$ and $y_4 + y_5 = y_4 + y_6 = y_4 + y_7 = -1$. If also $y_3 \neq 0$, then similarly $4 \sim 5'$, $5 \overset{*}{\sim} 6'$, $5 \sim 7'$ and $y_3 + y_5 = y_3 + y_6 = y_3 + y_7 = -1$. In this case, if $y_5 \neq 0$, then $9 \overset{*}{\sim} 6'$, $y_3 = y_4 = y_5 = y_6 = y_7 = -\frac{1}{2}$ and $\langle \mathbf{j}, \mathbf{j} \rangle = -\frac{9}{2}$, a contradiction. Therefore $y_5 = 0$ and we have $y_j \in \{-1.0\}$ for all $j' \in \overline{X}$, contradicting Lemma 2.7. Hence $y_3 = 0$, and so $y_5 \neq 0$. Now $9 \sim 6'$, $y_5 + y_6 = -1$, $y_4 = y_5 = y_6 = y_7 = -\frac{1}{2}$ and $\langle \mathbf{j}, \mathbf{j} \rangle = -4$, a contradiction.

Suppose that $r_3r_4r_5r_6r_7 = 32200$ and that R does not admit 33100. If R_3 contains a vertex adjacent to Q', then $y_3 = 0$ and $y_j = -1$ for some $j > 3$, a contradiction. Hence without loss of generality there are just two possibilities: (a) $3 \sim 5'$, $4 \sim 6'$, $5 \sim 7'$, (b) $3 \sim 4'$, $4 \sim 5'$, $5 \sim 6'$. In case (a), $y_3 + y_5 = y_3 + y_6 = y_3 + y_7 = -1$, and if $y_4 \neq 0$, then $6 \overset{*}{\sim} 6'$, $7 \sim 7'$, $y_4 + y_6 = y_4 + y_7 = -1$, $y_3 = y_4 = y_5 = y_6 = y_7 = -\frac{1}{2}$, whence $\langle \mathbf{j}, \mathbf{j} \rangle = -\frac{9}{2}$, a contradiction. Hence $y_4 = 0$ and so $y_j \notin \mathbb{Z}$ $(j = 4, 5, 6, 7)$. Since R does not admit 33100, we have $8 \not\sim 4'$ and $9 \not\sim 4'$, and so $6 \overset{*}{\sim} 6'$, $9 \sim 7'$. Then $y_5 + y_6 = y_5 + y_7 = -1$, $y_3 = y_5 = y_6 = y_7 = -\frac{1}{2}$ and so $\langle \mathbf{j}, \mathbf{j} \rangle = -4$, a contradiction. In case (b), $y_3 + y_4 = y_3 + y_5 = y_3 + y_6 = -1$. Now $y_5 \neq 0$ for otherwise $y_3 = -1$, contradicting Lemma 3.2. Hence $8 \overset{*}{\sim} 6'$, $9 \sim 7'$, $y_5 + y_6 = y_5 + y_7 = -1$ and $\langle \mathbf{j}, \mathbf{j} \rangle = -\frac{9}{2}$, a contradiction.

Suppose that $r_3r_4r_5r_6r_7 = 32110$ and that R does not admit 33100 or 32200. If $y_4 \neq 0$, then $6 \overset{*}{\sim} 6'$ and $7 \sim 7'$ because R does not admit 32200. Moreover $8 \not\sim 4'$ and $8 \not\sim 6'$ because R does not admit 33100, and $8 \not\sim 7'$ because R does not admit 32200. Hence $8 \sim Q'$ and $y_5 = 0$, $y_j \notin \mathbb{Z}$ $(j = 3, 4, 6, 7)$ and $9 \not\sim Q'$. Also, $9 \not\sim 4'$ since R does not admit 33100, and $9 \not\sim 5'$ since R does not admit 32200. Therefore $9 \sim 7'$, $y_6 + y_7 = -1 = y_4 + y_6 = y_4 + y_7$, whence $y_4 = y_6 = y_7 = -\frac{1}{2}$. Since $y_3 \neq 0$, we see on considering $\Delta_H(3), \Delta_H(4), \Delta_H(5)$, that we may assume that $3 \sim 5'$ or $3 \sim 6'$. In the former case, $y_3 + y_5 = -1$ and so $y_3 = -1$, contradicting Lemma 3.2. In the latter case, $y_3 + y_6 = -1$, $y_3 = y_4 = y_6 = y_7 = -\frac{1}{2}$ and $\langle \mathbf{j}, \mathbf{j} \rangle = -4$, a contradiction. We conclude that $y_4 = 0$, and so y_5 and y_6 are not integers. Hence $\Delta_H(8) \cup \Delta_H(9) \subseteq \{5, 6, 7\}$, contrary to the assumption that R does not admit 32200.

Lastly suppose that $r_3r_4r_5r_6r_7 = 22210$ and that R does not admit 33100, 32200 or 32110. By Lemma 3.2, we may assume that y_4 and y_5 are non-zero. Then $5 \overset{*}{\sim} 6'$, $6 \sim 7'$, $7 \overset{*}{\sim} 6'$, $8 \sim 7'$. Then R admits 32210, contrary to the assumption.

We have proved that if $r > 3$, then $a \neq 3$. From [8], we know that $a \neq 1$ or $a \neq 2$, and so $2k \leq (r-1)t - 4$, equivalently we have the following.

Theorem 6.1. *Let G be an r-regular graph of order n with a tree of order t as a star complement for an eigenvalue different from -1 and 0. If $r > 3$, then $n \leq \frac{1}{2}(r+1)t - 2$.*

Equality holds in Theorem 5.1 when G is the complement of the Clebsch graph, i.e., the strongly regular graph Cl_5 with parameters $(16, 5, 0, 2)$ and spectrum $-3^{(5)}, 1^{(10)}, 5$. Here $K_{1,5}$ is a star complement for the eigenvalue 1.

7. Concluding Remarks

We mention two possible generalizations, the first concerning the case $a = 4$. Here $t \leq 28$, and with sufficient time and patience the same methods can be used to determine whether Cl_5 is the sole example that arises when $a = 4$.

Secondly suppose that the (connected) star complement H is not a tree. If $a \leq 4$, then $2|E(H)| \leq 2t + 3$ and so H is unicyclic or bicyclic. If H is unicyclic, then $b \leq a$, and if H is bicyclic, then $b \leq a - 2$. In particular, if $a = 3$ and H is bicyclic, then $b = 1$ and Equation (4) shows that $t = 7$. In this case, the possibilities for H can be identified from the list in [4], and then the possibilities for G can be determined by traditional methods using the star complement technique [5, p.145]. A similar remark applies when H is a tree of order 7, in which case one would need to consider each of the nine trees without 1 as an eigenvalue.

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