Stochastic Hybrid Systems: Modelling and Verification

by

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Abstract

Hybrid systems now form a classical computational paradigm unifying discrete and continuous system aspects. The modelling, analysis and verification of these systems are very difficult. One way to reduce the complexity of hybrid system models is to consider randomization. The need for stochastic models has actually multiple motivations. Usually, when building models complete information is not available and we have to consider stochastic versions. Moreover, non-determinism and uncertainty are inherent to complex systems. The stochastic approach can be thought of as a way of quantifying non-determinism (by assigning a probability to each possible execution branch) and managing uncertainty. This is built upon to the - now classical - approach in algorithmics that provides polynomial complexity algorithms via randomization.

In this thesis we investigate the stochastic hybrid systems, focused on modelling and analysis. We propose a powerful unifying paradigm that combines analytical and formal methods. Its applications vary from air traffic control to communication networks and healthcare systems.

The stochastic hybrid system paradigm has an explosive development. This is because of its very powerful expressivity and the great variety of possible applications. Each hybrid system model can be randomized in different ways, giving rise to many classes of stochastic hybrid systems. Moreover, randomization can change profoundly the mathematical properties of discrete and continuous aspects and also can influence their interaction. Beyond the profound foundational and semantics issues, there is the possibility to combine and cross-fertilize techniques from analytic mathematics (like optimization, control, adaptivity, stability, existence and uniqueness of trajectories, sensitivity analysis) and formal methods (like bisimulation, specification, reachability analysis, model checking). These constitute the major motivations of our research. We investigate new models of stochastic hybrid systems and their associated problems. The main difference from the existing approaches is that we do not follow one way (based only on continuous or discrete mathematics), but their cross-fertilization. For stochastic hybrid systems we introduce concepts that have been defined only for discrete transition systems. Then, techniques that have been used in discrete automata now come in a new analytical fashion. This is partly explained by the fact that popular verification methods (like theorem proving) can hardly work even on probabilistic extensions of discrete systems. When the continuous dimension is added, the idea to use continuous mathematics methods for verification purposes comes in a natural way.

The concrete contribution of this thesis has four major milestones:

1. A new and a very general model for stochastic hybrid systems;

2. Stochastic reachability for stochastic hybrid systems is introduced together with a approximating method to compute reach set probabilities;

3. Bisimulation for stochastic hybrid systems is introduced and relationship with reachability analysis is investigated.

4. Considering the communication issue, we extend the modelling paradigm.

Thesis Supervisor: Savi Maharaj
Title: Dr.

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Chapter 1

Introduction

1.1 Hybrid systems

Continuous models have been a long-held standard in natural sciences, ranging from Newtonian mechanics to fluid dynamics. Their biggest advantage is their fidelity: since the models described detailed interactions in a physical system, accurate predictions and conclusions can be drawn based on these models.

Often, it is difficult to obtain a faithful continuous model of a system, or, it might be difficult to reason about a given complex continuous model. In these cases, the engineering practice consists in dividing the model into a finite set of behavioral modes whose dynamics are given by different sets of equations. Such a model is called hybrid, because it contains both continuous and discrete variables.

The term "hybrid" is used to characterize systems that combine *time-driven* and *event-driven* dynamics. The former are represented by differential (or difference) equations, while the latter may be described through various frameworks used for Discrete Event Systems, such as timed automata, queueing networks, or Petri nets (see [51]).

Hybrid systems can be thought of as interacting networks of digital and continuous systems. These systems typically contain variables or signals that take values from a continuous set and also variables that take values from a discrete, typically finite set. These continuous or discrete-valued variables depend on independent variables such as time, which may also be continuous or discrete. The evolution of the hybrid systems is given by equations of motion that generally

depend on all variables. In turn these equations contain mixtures of logic, discrete-valued dynamics, and continuous-variable dynamics. The continuous dynamics of such systems is the time-driven dynamics, whilst the discrete-variable dynamics of hybrid systems is the event-driven dynamics. The continuous and discrete dynamics coexist and interact with each other and because of this it is important to use models that accurately describe the dynamic behavior of such hybrid systems.

The modelling and verification techniques developed for discrete systems are not directly applicable because the state space is now uncountably infinite and from a given state, a hybrid system can make a transition such that the next state comes from an uncountable set of states.

The safety-critical nature of numerous hybrid systems, such as automated transportation systems, has encouraged the formal modelling and validation through deductive reasoning.

The first step is to chose *suitable models* for hybrid systems. Broadly speaking, two categories of modelling framework have been proposed to study hybrid systems: those that extend event-driven models to include time-driven dynamics; and those that extend the traditional time-driven models to include event-driven dynamics; for an overview, see [32].

Roughly speaking, the nature of these systems suggests that their models should be combinations of models of dynamic systems (continuous variables+differential equation systems) with models of discrete machines (automata+discrete data structures).
Hence, adequate models for hybrid systems are obtained by considering finite control location graphs supplied with discrete data structures (counters, stacks, etc) and real valued variables that change continuously at each control location. The transitions between control locations are conditioned by constraints on the values (or configurations) of the (discrete and / or continuous) variables and data structures of the system; the execution of these transitions updates or resets the discrete data structures and the continuous variables of the system.

Theoretical analysis deals with issues such as:

1. the analysis of potential capabilities of classes of systems, both as models of systems to be controlled and as controllers,

2. the derivation of necessary and sufficient conditions characterizing properties such as controllability,

3. the classification of systems under natural equivalence relations (changes of variables, action of "feedback group"), or

4. theorems guaranteeing existence and uniqueness of "internal" black box representations of given "external" behaviors.

In order to model hybrid systems, a number of formal models have been studied which capture the behavior of plants in various levels of abstraction:

1. *Timed Automata.* Timed automata are probably the most popular modeling formalism, but for many applications timed automata are not sufficiently expressive, because they cannot handle hybrid phenomena.

2. *Hybrid Automata.* Hybrid automata [93] are essentially an extension of state machines in which nodes model continuous input/output behavior and guarded transitions model discrete changes in system state. Whereas the reachability problem for timed automata is decidable, reachability is undecidable even for seemingly very minor extensions of this model.

3. *Other Hybrid Formalisms.* For instance, hybrid I/O automata [125], which specialize hybrid automata by an additional distinction between input and output. In this model, the continuous behavior of a system is described by a set of trajectories.

Hybrid systems have been proposed as a source of new models for capturing the mixed nature of real-world behaviours. The strong expressive power of hybrid systems makes them promising for their use in the challenging area of embedded control. Hybrid systems play the role of an interchange format, which allows the integration of tools and methods available for hybrid controller design.

1.2 Probabilistic Hybrid Models

Probabilistic hybrid models can be thought of as extensions of discrete models, such as hidden Markov models [72], Markov chains, semi-Markov processes [122], or dynamic Bayesian networks [58] to continuous dynamical models. In practice, the modelling of many phenomena requires

the integration of both probabilistic and hybrid (mixed discrete - continuous) aspects. Even though deterministic hybrid models can capture a wide range of behaviours encountered in practice, stochastic features are very important, because of the uncertainty inherent in most real world applications. As compared to more traditional hybrid systems, such as I/O automata, probabilistic hybrid models have properties crucial for reasoning under uncertainty, including probabilistic transitions between modes or noisy observations.

In Probabilistic Hybrid Automata (PHA) [101], a system is modeled by a hybrid automaton that has both discrete and continuous variables. This framework can be viewed as an extension of a hidden Markov model. The Hidden Markov Model is a finite set of states, each of which is associated with a probability distribution. Transitions among the states are governed by a set of probabilities called transition probabilities. In a particular state an outcome or observation can be generated, according to the associated probability distribution. It is only the outcome, not the state visible to an external observer and therefore states are hidden to the outside. that incorporates discrete and continuous inputs, stochastic continuous dynamics and autonomous mode transitions. Simpler versions of PHA can be found in [150]. These are variants of hybrid automata augmented with discrete probability distributions.

1.3 Stochastic Hybrid Systems

In the real world there exist complex systems whose hybrid dynamics can not be specified using only simple discrete probability distributions, but have to be described in terms of continuous time Markov process, diffusion processes, stochastic kernels, etc. Moreover, the mode transitions for these systems depend also on the continuous dynamics inside the modes. Then it is naturally to introduce the concept of Stochastic hybrid systems (SHS).

SHS are used as a paradigm for modelling embedded systems with safety critical performance requirements. Embedded systems of this type have to operate in an uncertain and often adversarial environment. Stochastic analysis and control of hybrid systems is therefore essential to study and improve the performance of embedded systems in the presence of uncertainty.

In the last three years, I have been involved in two EU projects: HYBRIDGE and COLUMBUS. In these projects, we have used hybrid systems for modelling embedded systems, involved

in Air Traffic Management (ATM), with safety critical performance requirements. Embedded systems of this type have to operate in an uncertain and often adversarial environment. Stochastic analysis and control of hybrid systems is essential to study and improve the performance of ATM systems in the presence of uncertainty.

The problem of safety analysis is addressed from the perspective of the current centralized ATM systems, where aircraft are prescribed to follow certain flight plans, and all flights are controlled by an Air Traffic Controller (ATC) from gate to gate. In the context of ATM, different safety relevant operation cases might occur as follows: vertical crossings; overtake manoeuvres in unmanaged airspace; ATC sector transitions; missed approaches (see [137] for a detailed presentation), aircraft-to-aircraft conflict and aircraft-to-airspace conflict. For example, the ATCs are responsible for maintaining a sufficiently large distance between aircraft to avoid dangerous situations and ultimately collisions, by issuing trajectory specifications to the pilots. Separation assurance forms a major part of the current ATC workload. If the level of automation in the ATM process increases, some of the separation assurance tasks can be transferred to the automated system. One approach for doing this is to rely on conflict detection and resolution (CDR) strategies to assist ATC. These strategies try to predict the trajectories of aircraft within managed airspace, analyse these trajectories in order to decide if there is a substantial possibility of loss of separation (conflict detection) and, if there is, issue advisories to the ATC and/ or pilots on how to resolve the problem (conflict resolution).

The model for predicting the aircraft future position should incorporate information on the aircraft flight plan, the aircraft dynamics, and the flight management systems. Each aircraft has to follow a flight plan, which typically consists of airways (straight lines between given way points traveled at constant speed). The actual aircraft motion might deviate from the planned motion because of different sources of uncertainty. We assume that wind is the main source of uncertainty on the actual aircraft dynamics. The hybrid nature of the model is due to the change in the dynamics when a way-point is reached. The stochastic component is due to the wind described by a random field, which is used to model the spatial perturbation to the aircraft motion due to the wind.

Intuitively, SHS can be thought of as traditional hybrid systems augmented with some stochastic features. These systems typically contain variables or signals that take values from a

continuous set and also variables that take values from a discrete (finite or countable) set. Differential equations or stochastic differential equations generally give the continuous dynamics of such systems. A Markov chain generally governs the discrete-variable dynamics of SHS. The stochastic features might be present in the continuous dynamics or in the discrete dynamics, or in both. The continuous and discrete dynamics coexist and interact with each other and because of this it is important to use models that accurately describe the dynamic behaviour of such hybrid systems.

We have named this class of systems 'stochastic' and not 'probabilistic' because their dynamics is stochastic, that means it can be perturbed by a noise process, or it can have mode jumps according with a stochastic kernel, which depend also on the continuous dynamics ('hybrid jumps'). This dynamics can be thought of as a 'deterministic dynamical system parametrised by the sample probability state', i.e. the realization of an SHS is given by the possible trajectories of a stochastic process.

1.4 Thesis contributions

The main contributions of this thesis consist of the following issues:

1. General models for SHS and distributed SHS: modelling, properties, etc. These models encompass almost all models for SHS proposed in the literature. This point will be briefly explained in Chapter 3. For a detailed comparison of the different classes of SHS models existing in the literature and how these models can be embedded in a very general class of models, the reader is referred to [137].

2. An incremental presentation of the different possible ways to tackle the SHS verification: from analytical approaches to the formal methods.

3. New approaches for stochastic reachability. In this thesis, we develop some methodologies for verifying temporal properties of stochastic hybrid systems. We define the reachability problem in the stochastic framework and we investigate some possible analytical and statistical solutions. Our framework does not require explicit computation of reachable states. Instead, we use Markovian properties of the SHS models to prove properties

as safety or reachability. Moreover, it is possible to treat safety verification of SHS by computing upper-bounds on the probability of reaching the unsafe states. The results, which we have obtained, have rather difficult mathematical expressions and make use of the Markov process theory. This drawback can be remedied if an abstraction theory for SHS would be available. This motivates our further work in the direction of defining bisimulation concepts for SHS (see next item).

4. New concepts of bisimulation for SHS. By introducing bisimulations preserving the reach set probabilities the solutions obtained for reachability become computationally feasible. As well, these new concepts of stochastic bisimulation can be viewed as the basis for further development of model checking for SHS.

1.5 Thesis layout

The thesis is structured as follows. In Chapter 2 we give a mathematical background which will be used in this thesis. As well, we briefly present the concepts of hybrid automata and probabilistic hybrid automata which have already been studied in the literature. This is motivated by the fact that we will develope further these concepts in this thesis to the stochastic case obtaining different models for stochastic hybrid systems. In Chapter 3 we introduce a very general model for SHS and we study its properties. The proofs of these properties require a strong background in the Markov process theory, but it is worth doing since these properties will be further employed in the verification process of this model. In Chapter 4, the model will be augmented with parallelism and communication features. In Chapter 5 we give a brief overview of the verification methods existing in the literature for deterministic hybrid systems. In Chapter 6, we study different verification methods for stochastic hybrid systems. We start with probabilistic hybrid system verification methods, which have been recorded in the literature. The lifting of these methods to the case of SHS is not straightforward. The passing from discrete probability distributions to continuous ones makes difficult the use of formal methods for verification of SHS. Then, in a first instance we formulate the problem of SHS verification as a stochastic reachability problem and we give analytical solutions. The main difficulty is that the computation of these analytical solutions is at least as hard as is the finding of these

solutions. In order to ease the methodology of solving reachability problem for SHS we develop, in Chapter 7, a new concept for SHS bisimulation. This will constitute the theoretical base for model-checking for SHS.

Chapter 2

Terminology and Models

2.1 Overview

This chapter provides a literature review of preliminary ideas and definitions that the reader must understand in order to appreciate this work.

2.2 Preliminaries

Throughout this thesis, we assume familiarity with the notation and concepts of ordinary differential equations (ODE), dynamical systems, stochastic differential equations (SDE), Markov processes, and diffusion processes.

2.2.1 Ordinary Differential Equations

In this thesis, the continuous dynamic behavior of some stochastic hybrid system models can be expressed using ordinary differential equations (ODE):

$$\frac{dx}{dt} = f(t, x) \tag{2.1}$$

where $x(t) \in X \subset \mathbb{R}^n$. Function $f : X \to \mathbb{R}^n$ is called *vector field* on $\mathbb{R}^n$.

A system of ODEs is called time-invariant if its vector field does not depend explicitly on time. It is known that given the initial value for x, i.e. $x(0) = x_0$ the solution of the (2.1) is

unique.

A *plant* or an ODE with *inputs and outputs* is given by

$$\begin{aligned} \dot{x}\,(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t)) \end{aligned}$$

where the number of components of the state vector, n, is called the order of the system. The input u and the output y have p and q components respectively, i.e. $u(t) \in U \subset \mathbb{R}^p$, $y(t) \subset \mathbb{R}^q$, $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^q$.

2.2.2 Dynamical Systems

A dynamical system, roughly speaking, is a set of objects which are allowed to change over time, while obeying a set of rules.

Examples of dynamical systems include a chemical plant, the population growth of a country, the behavior of a country economic structure, etc. Many complex dynamical systems have to be systematically analysed. A well-developed theory of dynamical systems is available in literature (see. for example, [12, 147]). Systems in this class are associated, in one way or another, with algebraic, difference or differential equations, which are used to represent the behavior of the dynamical system.

In engineering and mathematics, a dynamical system is a deterministic process in which a function's value changes over time according to a rule that is defined in terms of the function's current value. Generally, dynamical systems come in two flavors: discrete and continuous. A discrete dynamical system involves step-by-step state changes. A dynamical system is called discrete if time is measured in discrete steps; these are modeled as recursive relations. If time is measured continuously, the resulting continuous dynamical systems are expressed as ordinary differential equations. In contrast, a discrete dynamical system's behaviour is described using a transition relation or state space graph. For instance, a finite automaton is a discrete dynamical system.

2.2.3 Stochastic Processes

Stochastic processes concern sequences of events governed by probabilistic laws. In finance and economics problems, sequences of events take time, so we can think on random events parameterized by time. Formally, a stochastic process $M = \{x_t | t \in T\}$ is a collection of random variables. That is, for each t in the index set T, x_t is a random variable. Note that we fix a probability space $(\Omega, \mathcal{F}, P)$ (see below) and all random variables are defined on this probability space.

We often interpret t as time and call x_t the state of the process at time t. The index set T can be countable set and we have a discrete-time stochastic process, or non-countable continuous set and we have a continuous-time stochastic process. Any realization of M is named a sample path, which can be discrete or continuous.

Although in most applications the index set is simply a set of time instants t_k, for the case of technical uncertainty it is not true. Imagine a sequential investment in information process to determine the volume of a oil reserve. In the sequence of information revelation random variables the index set is a sequence of investments in information (a set of events, each event being one investment in information). That is, they are event-driven processes (evolves only if a new investment in information is performed) and not time-driven processes as most stochastic processes, which evolve with the pure passage of time.

We assume that the reader knows the basic ideas of measure theory and probability as expounded, for example in [123] or the appendix of [65]. In the following, we briefly recall some definitions:

- A σ-algebra (or σ-field) on a set X is a family of subsets of X which includes X itself and which is closed under complementation and countable unions. A set X equipped with a σ-algebra (denoted by $\sum_X$) is called a *measurable space* and is denoted by $(X, \sum_X)$. Given a topological space $(X, \mathcal{T})$, we can define the σ-algebra, often written $\mathcal{B}$ or $\mathcal{B}(X)$, generated by the open sets (or, equivalently, by the closed sets). This is usually called Borel σ-algebra.

- A function $f : (X, \sum_X) \to (Y, \sum_Y)$ between measurable spaces is said to be measurable if $\forall B \in \sum_Y . f^{-1}(B) \in \sum_X$.

A probability space $(\Omega, \mathcal{F}, P)$ is a measurable space $(\Omega, \mathcal{F})$ provided with a probability measure $P : \mathcal{F} \to [0, 1]$ (i.e. P is a measure such that $P(\Omega) = 1$). A filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a probability space $(\Omega, \mathcal{F}, P)$ together with a filtration $(\mathcal{F}_t | t \in T)$, a filtration being any family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

A process M is adapted to a filtration $\mathcal{F}_t$ if for each t, x_t is an $\mathcal{F}_t$ random variable. In this case, we say that the family $\{\mathcal{F}_t\}$ is an *admissible filtration.* An admissible filtration $\{\mathcal{F}_t\}$ is *right continuous* if $\mathcal{F}_t = \mathcal{F}_{t+} = \cap\{\mathcal{F}_{t'} | t' > t\}$.
Given an admissible filtration $\{\mathcal{F}_t\}$, a $[0, \infty]$-valued function τ on Ω is called an $\{\mathcal{F}_t\}$-*stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t$, $\forall t \geq 0$.

Lévy processes

Lévy processes are stochastic processes with stationary independent increments and continuous in probability. Stationary increment property means that the probability distribution for the changes in the stochastic variable x, depends only on the time interval length. Independent increments means that for all time instant t, the increments are independent. The two most basic types of Lévy processes are Wiener processes and Poisson processes.

Martingales

A process $M = \{x_t | t \in \mathbb{R}_+\}$ is a martingale if, given all the information until the time t (expressed by x_t), the expected value of x in the future instant $t + s$ is x_t. In math notation:

$$\mathbb{E}_t[x_{t+s}] = x_t$$

where the subscript in the expected value operator $\mathbb{E}_t$ denotes that the expectation is conditional to the information available at time t. Note that a martingale has constant expectation, i.e.

$$\mathbb{E}x_t = \mathbb{E}x_0, \forall t \in \mathbb{R}_+.$$

Markov Processes

Markov processes have the following property: given that its current state is known, the probability of any future event of the process is not altered by additional knowledge concerning its past behaviour. Formally, a stochastic dynamical system satisfies the Markov property (formulated by A.A. Markov in 1906) if the probable (future) state of the system at any time $t > s$ is independent of the (past) behavior of the system at times $t < s$, given the present state at time s. This is the stochastic analogue of an important property shared with solutions of initial value problems involving ODEs, and so stochastic processes satisfying this property arise naturally.

One can consider a stochastic process taking values in a measurable space $(X, \mathcal{B})$, called the *state space*. If X is a Hausdorff (or separated) topological space we use to denote by $\mathcal{B}(X)$ or $\mathcal{B}$ its Borel σ-algebra.

Usually, the stochastic processes which appear in this work take values in open subsets of different Euclidean spaces. Then these spaces might have nice topological properties. Concretely, we deal with Markov processes whose the state space can be:

- A *Polish space* is the topological space underlying a complete, separable metric space, i.e. it has a countable dense subset.
- a *Borel space*, i.e. a topological space, which is homeomorphic (i.e. it can be "identified" as a topological space) to a Borel subset of a complete separable metric space.
- a *Lusin space,* i.e. a topological space, which is homeomorphic with a Borel subset of a compact metric space.
- an *analytic space*, i.e. a topological space which is the image of a Polish space under a continuous function from one Polish space to another.

The classical theory of Markov processes is typically carried out in the setting of Polish spaces rather than on abstract measure spaces. The analytic spaces generalize Polish spaces. In Chapter 3, we work with Borel spaces, which seem to be the most appropriate for SHS. In Chapter 7, we define bisimulation of Markov processes defined on analytic spaces (i.e. in a very general framework).

Formally, a stochastic process $M = \{x_t | t \in \mathbb{R}_+\}$ with the state $(X, \mathcal{B})$, defined on a filtered probability space is a Markov process if for any times t, s with $t \geq s$ and any bounded measurable function $f : X \to \mathbb{R}$ the following equality holds

$$\mathbb{E}[f(x_t)|\mathcal{F}_s] = \mathbb{E}[f(x_t)|x_s]$$

where $\mathbb{E}$ is the expectation with respect to P. This says that the only information relevant to evaluating the behaviour of the process beyond time s is the value of the current state, x_s. It implies in particular that M is adapted to $\mathcal{F}_t$.

Wiener Process and White Noise

A Wiener process is also a special case of a strong diffusion process that is a particular class of a continuous time Markov process.

A continuous time Wiener process (also called Brownian motion) is a stochastic process with three properties:

1. It is a Markov process. This means that all the past information is considered in the current value, so that future values of the process depends only on its current value not on past values. The future values are not affected by the past values history. In finance, this is consistent with the efficient market hypothesis (that the current prices reflect all relevant information).

2. It has independent increments. Change in one time interval is independent of any other time interval (nonoverlapping).

3. The changes of value over any finite time interval are normally distributed. So, it has stationary increments, besides the property of independent increments. Therefore, it is a particular *Lévy process*.

Formally, the standard m-dimensional Wiener process $W(t) = \{W_1(t), ..., W_m(t)\}$, defined for $t \geq 0$, has $\mathbb{R}^m$ as its state space and is a stochastic process whose components $W_j(t)$, $j = 1, ..., m$, are independent scalar standard Wiener processes, i.e. each W_j is a scalar process

with independent, stationary, and normal distributed increments $W_j(t) - W_j(s)$ and satisfying $W_j(0) = 0$ with probability 1.

Diffusion Processes

A Markov process (x_t) with $t \in [t_0, T]$, state space $\mathbb{R}^n$, and continuous sample paths with probability 1 is called a diffusion process if its transition probability $p(s, x, t, B)$ is smooth, i.e. it satisfies the following three conditions for every $s \in [t_0, T]$, $x \in \mathbb{R}^n$, and $\varepsilon > 0$:

- $\lim_{t \searrow s} \frac{1}{t-s} \int_{|y-x|>\varepsilon} p(s, x, t, dy) = 0$
- $\lim_{t \searrow s} \frac{1}{t-s} \int_{|y-x|\leq\varepsilon} (y - x) p(s, x, t, dy) = a(s, x)$
- $\lim_{t \searrow s} \frac{1}{t-s} \int_{|y-x|\leq\varepsilon} (y - x)(y - x)^T p(s, x, t, dy) = B(s, x)$

where $a(s, x)$ and $B(s, x)$ represent well-defined $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$- valued functions respectively. These functions, called the coefficients of the diffusion process, are referred as follows: a is called the *drift vector* and B the *diffusion matrix.*

The reason for this terminology is that processes of this kind were first encountered in physics in studying diffusion phenomena.

2.2.4 Markov Process Characterizations

In what follows, $(\Omega, \mathcal{F}, P)$ will denote a probability space, X an analytic space, $\mathcal{B}$ its Borel σ-algebra, T an interval of the real line. For each $t \in T$, $x_t : (\Omega, \mathcal{F}, P) \to (X, \mathcal{B})$ a measurable function. Usually, $T = [0, \infty)$.

Let $\mathcal{B}^b(X)$ be the lattice of bounded positive measurable functions on X. This is a Banach space under the norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Operator Methods

Operator methods begin with a local characterization of the Markov process dynamics. This local specification takes the form of an *infinitesimal generator.* The infinitesimal generator is itself an operator mapping test functions into other functions. From the infinitesimal generator,

one can construct a family (semigroup) of conditional expectation operators. The operators exploit the time-invariant Markov structure. Each operator in this family is indexed by the forecast horizon, the interval of time between the information set used for prediction and the object that is being predicted. Operator methods allow us to ascertain global, and in particular, long-run implications from the local or infinitesimal evolution.

The stochastic analysis identifies concepts (like infinitesimal generator, semigroup of operators, resolvent of operators) that characterize in an abstract sense the evolutions of a Markov process. Under standard assumptions, all these concepts are equivalent, in the sense that given one concept then all the others can be constructed from it. For a detailed presentation of these notions and the connections between them, the reader can consult, for example [126].

There are several different but essentially equivalent ways to parameterize continuous time Markov processes. In this section we briefly describe four possible parametrizations.

Transition functions

A function $p : (X \times \mathcal{B})$ is a *transition probability* or a *stochastic kernel* if $p(x, \cdot)$ is a probability measure in $\mathcal{B}$, and $p(\cdot, B)$ is measurable, for each $(x, B) \in (X \times \mathcal{B})$.

A *transition function* is a family $p_{s,t}$, with $(s,t) \in T^2$, $s < t$ that satisfies for each $s < t < u$ the Chapman-Kolmogorov equation:

$$p_{s,u}(x, B) = \int p_{t,u}(y, B) p_{s,t}(x, dy)$$

A transition function is time homogeneous if $p_{s,t} = p_{s',t'}$ whenever $t - s = t' - s'$. In this case we write p_{t-s} instead of $p_{s,t}$.

Let $\mathcal{M}_t \subset \mathcal{F}$ an admissible filtration for a stochastic process M. M is Markov with the transition function $p_{s,t}$ if for each non-negative Borel measurable $f : X \to \mathbb{R}$ and each $(s,t) \in T^2$, $s < t$,

$$\mathbb{E}[f(x_t)|\mathcal{M}_s] = \int f(y) p_{s,t}(x_s, dy).$$

Given a transition function $p_{s,t}$ on $(X, \mathcal{B})$ and a probability measure μ on $(X, \mathcal{B})$, there exists a unique probability measure P_μ on $(X^{[0,\infty)}, \mathcal{B}^{[0,\infty)})$, such that the coordinate process is Markov with respect to $\sigma(X_u | u \leq t)$, with transition function $p_{s,t}$ and the distribution of x_0 given by μ [133].

Suppose we are given a homogeneous transition function p_t and an initial probability $\mu = \delta_x$, where δ_x denotes the Dirac measure at $x \in X$ defined by $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$. We can then construct a Markov process $M = \{x_t | t \in \mathbb{R}_+\}$ having p_t as its transition function and $P(x_0 = x) = 1$. It is however possible that the same construction $\mu = \delta_{x'}$ for some $x' \neq x$ would lead to a process defined on some other probability space. A *Markov family* is a collection $(\Omega, \mathcal{F}, \mathcal{F}_t, x_t, (P_x)_{x \in X})$, where $(\Omega, \mathcal{F})$ is a measurable space, $(\mathcal{F}_t)$ a filtration, $(x_t | t \in \mathbb{R}_+)$ a family of X-valued random variables such that x_t is $\mathcal{F}_t$-measurable for each t, and, for each $x \in X$, P_x is a probability measure on $(\Omega, \mathcal{F})$ such that (x_t) is a Markov process on $(\Omega, \mathcal{F}, P_x)$ with transition function p_t and initial distribution δ_x, i.e. $P_x(x_0 = x) = 1$. We will write $\mathbb{E}_x$ for the expectation w.r.t. P_x. Note that the measure P_x and the transition function p_t are related by

$$p_t(x, A) = P_x(x_t \in A). \tag{2.2}$$

In a Markov family, only the measure P_x depends on the initial point $x \in X$; all the other ingredients are the same for every x. This provides yet another way of expressing the Markov property: because the transition function is the same for every P_x we easily see that for $f \in \mathcal{B}^b(X)$ and $s, t \geq 0$ we have

$$\mathbb{E}_x[f(x_{t+s}) | \mathcal{F}_s] = \mathbb{E}_{x_s}[f(x_t)]$$

Thus the behaviour of the process beyond time s is just another process started at x_s.

From now on we will generally consider Markov families rather than Markov processes, and use the notation $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, (P_x)_{x \in X})$ for a Markov process.

Semigroup of operators

Let p_t be a homogeneous transition function. For each t define conditional expectation operator by

$$P_t f(x) = \int f(y) p_t(x, dy) = \mathbb{E}_x f(x_t), \forall x \in X \tag{2.3}$$

where $\mathbb{E}_x$ is the expectation w.r.t. P_x. The Chapman-Kolmogorov equation guarantees that the linear operators P_t satisfy

$$P_{t+s} = P_t P_s.$$

This suggests another parameterization for Markov processes: the semigroup of (conditional

expectation) operators $\mathcal{P} = (P_t)_{t>0}$. The *operator semigroup* associated to M, maps $\mathcal{B}^b(X)$ into itself. The semigroup $\mathcal{P} = (P_t)_{t>0}$ can be thought of as an *abstraction* of M, since that from $\mathcal{P}$ one can recover the initial process [25]. This kind of abstraction can be related with the concept of abstract control system from [156], but in our case due to the stochastic features of the model, the domain of the abstraction is not longer the state space X, but $\mathcal{B}^b(X)$.

A one-parameter family of linear operators in a Banach subspace of $\mathcal{B}^b(X)$, $\{P_t | t \geq 0\}$ is called a *strongly continuous contraction semigroup* if (a) $P_0 = I$ (the identity), (b) $P_{t+s} = P_t P_s$ for all $t, s \geq 0$, (c) $\lim_{t \to 0} P_t f = f$ and (d) $||P_t|| \leq 1$. Let $\mathcal{B}_0$ be the subset of $\mathcal{B}_b(X)$ consisting of those bounded, measurable functions f for which $\lim_{t \searrow 0} ||P_t f - f|| = 0$. The semigroup is *strongly continuous* on $\mathcal{B}_0$. $\mathcal{B}_0$ is a closed linear subspace of $\mathcal{B}_b(X)$.

The *resolvent of operators* $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ associated with the semigroup $\mathcal{P}$ is given by formula

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt.$$

Let denote by V the initial operator V_0 of $\mathcal{V}$, which is known as the *kernel operator* of Markov process M.

A function f is *excessive* (w.r.t. the semigroup (P_t) or the resolvent (V^α)) if it is measurable, non-negative and $P_t f \leq f$ for all $t \geq 0$ and $P_t f \nearrow f$ as $t \searrow 0$. Let denote by $\mathcal{E}_M$ the set of all excessive functions associated to M. The strong Markov property can be characterized in terms of excessive functions [133].

Let Δ be the cemetery point for X, which is an adjoined point to X, $X_\Delta = X \cup \{\Delta\}$. The existence of Δ is assumed in order to have a probabilistic interpretation of $P_x(x_t \in X) < 1$, i.e. at some 'termination time' $\zeta(\omega)$ the process M escapes to and is trapped at Δ.

Infinitesimal generators

Associated with the semigroup (P_t) is its *strong generator* which, loosely speaking, is the derivative of P_t at $t = 0$. Let $D(L) \subset \mathcal{B}_b(X)$ be the set of functions f for which the following limit exists

$$\lim_{t \searrow 0} \frac{1}{t}(P_t f - f) \tag{2.4}$$

and denote this limit Lf. The limit refers to convergence in the norm $\|\cdot\|$, i.e. for $f \in D(L)$ we

have

$$\lim_{t \searrow 0} ||\frac{1}{t}(P_t f - f) - Lf|| = 0.$$

Specifying the domain $D(L)$ is an essential part of specifying the operator L.

If (P_t) is a strongly continuous contraction semigroup then $D(L)$ is dense. In addition, L is closed, that is if $f_n \in D(L)$ converges to f and Lf_n converges to g then $f \in D(L)$ and $Lf = g$. If (P_t) is a strongly continuous contraction semigroup we can reconstruct P_t using its infinitesimal generator L (e.g. [73] Prop. 2.7 of Chapter 2). This suggests using L to parameterize the Markov process. The Hille-Yosida theorem (e.g. [73], Th.2.6 of Chapter 1) gives necessary and sufficient conditions for a linear operator to be the generator of a strongly continuous positive contraction semigroup. Necessary and sufficient conditions to insure that the semigroup can be interpreted as a semigroup of conditional expectations are also known (e.g. [73], Th. 2.2 of Chapter 4).

Proposition 1 (Martingale property) *[62] For $f \in D(L)$ we define the real-valued process $(C_t^f)_{t \geq 0}$ by*

$$C_t^f = f(x_t) - f(x_0) - \int_0^t Lf(x_s)ds. \tag{2.5}$$

Then for any $x \in X$, the process $(C_t^f)_{t \geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$.

There may be other functions f, not in $D(L)$, for which something akin to (3.27) is still true. In this way we get the notion of *extended generator* of the process.

Let $D(\widehat{L})$ denote the set of measurable functions $f : X \to \mathbb{R}$ with the following property: there exists a measurable function $h : X \to \mathbb{R}$ such that the function $t \to h(x_t)$ is integrable $P_x - a.s.$ for each $x \in X$ and the process

$$C_t^f = f(x_t) - f(x_0) - \int_0^t h(x_s)ds$$

is a local martingale. Then we write $h = \widehat{L}f$ and call $(\widehat{L}, D(\widehat{L}))$ the extended generator of the process (x_t).

Quadratic forms

Suppose $D = L^2(X, \mu)$ (the space of square integrable μ-measurable extended real valued

functions on X) where we have the natural inner product

$$< f, g >= \int f(x)g(x)d\mu(x).$$

If $f \in \mathcal{D}(L)$ and $g \in L^2(\mu)$ then we may define the (closed) form

$$\mathcal{E}(f, g) = - < Lf, g >$$

This leads to another way of parameterizing Markov processes. Instead of writing down a generator one starts with a form. As in the case of a generator it is typically not easy to fully characterize the domain of the form. For this reason one starts by defining a form on a smaller space and showing that it can be extended to a closed form in subset of $L^2(\mu)$. When the Markov process can be initialized to be stationary, the measure μ is typically this stationary distribution. More generally, μ does not have to be a finite measure.

This approach to Markov processes was pioneered by Beurling and Deny [22] and Fukushima [76] for symmetric Markov Processes. In this case both the operator L and the form $\mathcal{E}$ are symmetric. A stationary symmetric Markov process is time-reversible. If time were reversed, the transition operators would remain the same. On the other hand, multivariate standard Brownian motion is a symmetric nonstationary Markov process that is not time reversible. The literature on modelling Markov processes with forms has been extended to the non-symmetric case by Ma an Rockner [126].

2.2.5 Strong Markov Processes

In this thesis, we make use some standard notions in the Markov process theory as: underlying probability space, natural filtration, translation operator, Wiener probabilities, admissible filtration, stopping time, strong Markov property [25]. To ease the reading of this thesis we briefly recall in the following some useful definitions from the Markov process theory.

Suppose that $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P_x)$, is a Markov process. We denote the state space of M by $(X, \mathcal{B})$. Here, $(\Omega, \mathcal{F}, P_x)$ denotes the sample probability space for each process with initial start point x. The family of σ-algebras $\{\mathcal{F}_t^0\}$ denotes the *natural filtration*, i.e. $\mathcal{F}_t^0 = \sigma\{x_s, s \leq t\}$, i.e. the smallest σ-algebra in $\mathcal{F}$ with respect to all the random variables $x_s, s \in [0, t]$ are

measurable. Let us take $\mathcal{F}^0_\infty = \vee_t \mathcal{F}^0_t$. The trajectories of M are modelled by a family of X-valued random variables (x_t), which, as functions of time, have some continuity properties (as the càdlàg property, i.e. right continuous with left limits). This means that, for each $t > 0$ the function $x_t : (\Omega, \mathcal{F}) \to (X, \mathcal{B})$ is a $\mathcal{F}^0/\mathcal{B}$-measurable function for all $t \geq 0$. Then $\mathcal{F}^0_t$ is the *minimum admissible filtration.* The shift operator or *translation operator* $\theta_t : \Omega \to \Omega$, for all $t \geq 0$, has the following property

$$x_s \circ \theta_t = x_{t+s},\ t, s \geq 0.$$

The *(Wiener) probability* $P_x : (\Omega, \mathcal{F}) \to [0, 1]$ is a probability measure such that $P_x(x_t \in A)$ is $\mathcal{B}$-measurable in $x \in X$, for each $t \in [0, \infty)$ and $A \in \mathcal{B}$, and $P_x(x_0 = x) = 1$.
If μ is a probability measure on $(X, \mathcal{B})$, written as $\mu \in \mathcal{P}(X)$, then we can define

$$P_\mu(\Lambda) = \int_{X_\Delta} P_x(\Lambda)\mu(dx), \Lambda \in \mathcal{F}^0.$$

We then denote by $\mathcal{F}$ (resp. $\mathcal{F}_t$) the completion of $\mathcal{F}^0_\infty$ (resp. $\mathcal{F}^0_t$) with respect to all P_μ, probability measure on $(X, \mathcal{B})$. This has the advantage that the class of null sets is the same for every $t \in \mathbb{R}_+$. From now on, we will refer to the family $\{\mathcal{F}_t\}_t$ as the *natural filtration* of M.

For an admissible filtration $\{\mathcal{M}_t\}$, we say that M is *strong Markov* with respect to $\{\mathcal{M}_t\}$ if $\{\mathcal{M}_t\}$ is right continuous and

$$P_\mu(x_{\tau+t} \in E | \mathcal{M}_\tau) = P_{x_\tau}(x_t \in E);\ P_\mu - a.s.$$

$\mu \in \mathcal{P}(X)$, $E \in \mathcal{B}$, $t \geq 0$, for any $\{\mathcal{M}_t\}$-stopping time τ.

2.2.6 Stochastic Differential Equations

Ordinary differential equations, which have the general form (2.1) provide simple deterministic descriptions of the laws of motion of physical systems. The solution $x(t)$ of an initial value problem consisting of (2.1) together with the initial value

$$x(t_0) = x_0 \tag{2.6}$$

represents the state of such a system at time $t > t_0$, given that the state (2.6) was attained at time t_0. If random aspects in the physical system are to be considered, a number of modifications can be made in the formulation of the initial value problem (2.1), (2.6). The initial point x_0 may be replaced by a random variable X_0; the deterministic function $f(t,x)$ may be replaced by a random function $F(t,X,Y)$, where $Y = Y(t)$ designates a random input process uncoupled with the solution variable X; or, in the latter case, Y may represent the random coefficients of a linear or nonlinear operator whose form is specified by F. The first of these three possibilities for randomizing (2.1), (2.6) is exemplified by the motion of a space vehicle whose state consisting of position and momentum components changes according to a deterministic law but whose initial values may be subject to some uncertainty. An ac electric power circuit with state described by voltages and phase angles of nodes whose rates of change are forced by noise is a particular case of the second type of randomness. An example of the third type arises by considering intrinsic birth-death rates and interaction rates as stochastic processes in differential equation models of multispecies population evolution; a random initial value problem where the stochastic input is coupled with the solution results. The term random differential equation is reserved for the last of these three types, while stochastic differential equation refers to equations of the second type, which are driven by noise and interpreted mathematically as Ito equations.

The ultimate goal of the analysis of any random initial value problem

$$\frac{dX(t)}{dt} = F(t,X(t),Y(t)) \tag{2.7}$$

$$X(t_0) = X_0 \tag{2.8}$$

generally, is to obtain the distribution of the solution process $X(t)$ in terms of the distributions of X_0, $Y(t)$, and the statistical and deterministic properties of F, determining the sample path structure of $X(t)$ is even more ambitious. When randomness enters the problem only through the initial condition, a situation sometimes called "cryptodeterministic", the solution is a deterministic transformation of the random variable X_0. The mean square theory for the random initial value problem, in this instance, is a direct analogue of the ordinary differential equations (ODE) theory. For ODE, closed-form expressions for solutions are often unobtainable, and so one must be satisfied with numerical approximations or less than complete qualitative

information about solutions. It is unreasonable to expect otherwise for random or stochastic differential equation case.

In general the function F from (2.7) is given by

$$F(t, X(t), Y(t)) = f(t, X(t)) + g(t, X(t))Y(t) \tag{2.9}$$

with $Y(t)$ representing a Gaussian white noise process. The definition of the stochastic integral in the corresponding integral equation shows that such an equation is at best mathematically ambiguous. More precisely, usually the interest is to interpret (2.7), (2.9) as the Ito equation [9]

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) \tag{2.10}$$

where $W(t)$ denotes a Wiener or Brownian motion process; in (2.10) f and g are deterministic functions, but with slight modifications, the theory extends to explicitly random functions. Stochastic differential equations (SDE) were introduced by K. Ito in 1942, and the basic theory was developed independently by Ito and I. Gihman during the 1940s. Applications to control problems in electrical engineering motivated by the need for more sophisticated models spurred further work on these equations in the 1950s and 1960s. In the last period applications have been extended to many other areas including population dynamics in biology.

2.3 Jump Diffusion Processes

The State Process

In this section, we give a short presentation of controlled Markov Jump-Diffusion-Processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}$, that satisfies the usual assumptions, i.e. $\mathbb{P}$-completeness and right-continuity with left hand limits.

Let denote $X \subseteq \mathbb{R}^k$ the state space and $U \subseteq \mathbb{R}^l$ the metric and compact control space.

The evolution of the controlled state process in is governed by an SDE of the following type

$$dx_t(\theta_t) = b(t, x_t, \theta_t) + \sigma(t, x_t, \theta_t)dW_t + \int_\Gamma q(t, x_{t-}, \rho, \theta_t)\widetilde{N}(dt, d\rho) \tag{2.11}$$

where (W_t) denotes a k-dimensional Brownian Motion and

$$\widetilde{N}(\omega, t, A) := N(\omega, t, A) - h(t, A)$$

is a compensated homogeneous $\mathcal{F}_t$-Poisson random measure [73] on $\mathbb{R} \times \mathbb{R}^k$ with deterministic compensator

$$h(t, A) = \lambda t \times Q(A)$$

for any $(t, A) \in \mathcal{B} \times \mathcal{B}^k$ and fixed $\lambda \in \mathbb{R}$.

The bounded measurable function

$$q(t, x, \rho, \theta) : \mathbb{R} \times X \times \mathbb{R}^k \times U \to \mathbb{R}^k$$

computes the state- and control-dependent jump-size. The Lévy-measure $Q(x)$ is assumed to have a compact support $\Gamma \subset \mathbb{R}^k$. If the closure of of $A \in \mathcal{B}^k$ does not contain point 0, then $N(t, A) < \infty$ with probability 1.

The meaning of the measure $N(\cdot, t, A)$ is as follows :

Observe the random variable

$$\mu(t) = \int_{\mathbb{R}^k} uN(t, du)$$

Then $N(\cdot, t, A)$ is equal to the number of jumps of the process $\mu(t)$ with values in the set A, i.e. the number of instances of time s, $s < t$ such that $\mu(s) - \mu(s^-) \in A$.

Alternative Definition of the Jump Term

The process

$$\widetilde{N}_t := \int_0^t \int_\Gamma N(ds, d\rho) = \int_\Gamma N(t, d\rho)$$

is a Poisson counting-process with paramete λ and arrival times $T_i = \inf\{t | \widetilde{N}_t = i\}$. The inter-occurence times $\{T_{i+1} - T_i\}$ are exponentially distributed with mean value $1/\lambda$. Furthermore

$$\rho_i = \int_{T_{i-1}}^{T_i} \int_\Gamma \rho N(dt, d\rho)$$

is a sequence of i.i.d. random variables with distribution Q and $\{T_{i+1} - T_i, \rho_i, i < \infty\}$ are mutually independent. Hence, the Poisson-measure can be written

$$N(\omega, dt, d\rho) = \sum_{T_i(\omega)} \delta_{(T_i(\omega),\rho_i(\omega))}(dt, d\rho)$$

where δ denotes the Dirac-measure. Then

$$\int_{[0,t]\times\Gamma} q(s, x_{s-}, \rho, \theta_s) N(ds, d\rho) = \sum_{T_i \le t} q(T_i, x_{T_i^-}, \rho_i, \theta_{T_i}).$$

Existence and uniqueness of a solutions to the state-process SDE

To ensure the existence of the (stochastic) integrals and the existence and uniqueness of a solution of (2.11) we need the following conditions, to which we will refer as the *standard assumptions* in the remaining sequel.

For any $\theta \in U$ we assume the following conditions: (*standard assumptions*)

There exist constants $C, L \in \mathbb{R}$ such that

$$\int_0^T |b(s, x(\cdot), \theta)|^2 dt < \infty \tag{2.12}$$

$$\int_0^T \int_\Gamma |q(t, x(\cdot), \rho, \theta)|^2 Q(d\rho) dt < \infty \tag{2.13}$$

$$|b(t, x, \theta)|^2 + |\sigma(t, x, \theta)|^2 + \lambda \int_\Gamma |q(t, x, \rho, \theta)|^2 Q(dp) \le C(1 + |x|^2) \tag{2.14}$$

$$|b(t, x_1, \theta) - b(t, x_2, \theta)| \le L|x_1 - x_2| \tag{2.15}$$

$$|\sigma(t, x_1, \theta) - \sigma(t, x_2, \theta)|^2 + \lambda \int_\Gamma |q(t, x_1, \rho, \theta) - q(t, x_2, \rho, \theta)|^2 Q(d\rho) \le L^2 |x_1 - x_2|^2 \tag{2.16}$$

Theorem 2 (Existence and uniqueness) *If the functions $b(t, x, \theta)$, $\sigma(t, x, \theta)$ and $q(t, x, \rho, \theta)$ are linearly bounded by the constant C and satisfy a uniform Lipschitz-condition with the constant L, i.e. they satisfy the standard assumptions, then the SDE (2.11) has for every $\theta \in U$ a unique solution $x_t \in \Phi$, where $\Phi := \Phi^k(\mathcal{F}_t, [0, T])$ denotes the class of random processes $\phi(t)$, $t \in [0, T]$ adapted to the filtration $\mathcal{F}_t$ with values in $\mathbb{R}^k$ and sample paths, that are continuous from the right and have left-hand limits.*

If for any $t_1 \in [0,T)$ and $x \in x$ equation (2.11) possesses a unique solution $(x_s^{t_1 x})_{s\in[t_1,T]}$, satisfying the initial condition $x_{t_1}^{t_1 x} = x$, then the family $\{x_s^{t_1 x}, s \in [t_1,T], (t_1,x) \in [0,T) \times X\}$ is a Markov Process with the transition kernel

$$P(t,x,s,B) = \mathbb{P}\{x_s^{t_1 x} \in B\},\ B \in \mathcal{B}(X).$$

For any transition kernel we have the following requirements

- $P(t,x,s,\cdot)$ is a probability measure for all fixed $(t,x,s) \in [0,T] \times X \times [0,T]$
- $P(t,\cdot,s,B)$ is $\mathcal{B}(X)$-measurable for all fixed $(t,s,B) \in [0,T] \times [o,T] \times \mathcal{B}(X)$
- $P(t,x,t,\{x\}) = 1$
- The Chapman-Kolmogorov-Equation

$$P(t,x,s,B) = \int_X P(r,y,s,B)P(t,x,r,dy)$$

holds for $t < r < s \in [0,T]$.

2.4 Hybrid Automata

Now we will introduce the formal definition of a hybrid system as given in [5].

A hybrid system is a sextuple $H = (Loc, Var, Lab, Edg, Act, Inv)$ where:

1. Loc is a finite set of vertices called *locations*;
2. Var is a finite set of real variables. There is a function, called a *valuation*, $\nu : Var \to \mathbb{R}$ which assigns values to the variables: if $x \in Var$, then $\nu(x) \in \mathbb{R}$ is the value assigned to x. V denotes the set of possible valuations of ν. We define a *state of the hybrid system* H to be a pair (l,ν) where $l \in Loc$ and $\nu \in V$. Denote the set of possible states by $\sum$.
3. Lab is a finite set of synchronization labels that contains the shutter label $\tau \in Lab$. The meaning of the shutter label is described below.

4. Edg is a finite set of edges called *transitions*. Each transition is a quadtuple $e = (l, a, \mu, l')$ where $l \in Loc$ is a *source location*, $l' \in Loc$ is a *target location*, $a \in Lab$ is a *synchronization label*, and $\mu \in V^2$ is a transition relation. For every location $l \in Loc$, there is a set $Con_l \subset Var$ of *controlled variables* and a stutter transition (l, τ, Id_{Con}, l) where

$$(\nu, \nu') \in Id_{Con} \Leftrightarrow \forall x \in Var, x \in Con_l \Rightarrow \nu(x) = \nu'(x).$$

A transition e is enabled in a state (l, ν) if for some valuation $\nu' \in V$, $(\nu, \nu') \in \mu$. The state (l', ν') is then a *transition successor* of (l, ν).

5. Act is a labeling function that assigns a set of *activities* to each location $l \in Loc$. An activity is a function from $\mathbb{R}_+$ to V. Each activity must be time-invariant: for every $l \in Loc$, $f \in Act(l)$, and $t \in \mathbb{R}_+$, we require that $(f + t)(t') = f(t + t')$ for all $t \in \mathbb{R}_+$. For every $l \in Loc$, $f \in Act(l)$, and $x \in Var$, we define $f^x : \mathbb{R}_+ \to \mathbb{R}$ so that $f^x(t) = f(t)(x)$.

6. Inv is a labeling function that assigns an invariant $Inv(l) \subset V$ to each location $l \in Loc$.

The hybrid system H is *time-deterministic* if for every location $l \in Loc$ and every valuation $\nu \in V$, there is at most one activity $f \in Act(l)$ with $f(0) = \nu$.

2.5 Probabilistic Hybrid Automata

Formally, a PHA is defined as a tuple $< M, \mathbf{x}_c, \mathbf{y}_c, \mathcal{U}, F_c, G_c, \mathcal{T} >$ [101], where:

- The finite set M denotes the modes $m_i \in M$ of the automaton;
- $\mathbf{x}_c$ and $\mathbf{y}_c$ denotes the set of independent continuous state-variables and output variables respectively. The set of input variables, $\mathcal{U} = \mathbf{u}_c \cup \mathbf{u}_d \cup \mathbf{v}_c$, is divided into continuous control variables $\mathbf{u}_c$, continuous exogenous variables $\mathbf{v}_c$, and discrete control variables $\mathbf{u}_d$. Components of continuous variables range over different $\mathbb{R}^n$, whereas components of discrete variables range over finite domains D.
- The set F_c and G_c associate with each mode $m_i \in M$ functions f_{ci} and g_{ci} that govern the continuous dynamics exhibited at mode m_i by (in terms of discrete-time difference

equations and algebraic equations)

$$\mathbf{x}_{c,(k+1)} = f_{ci}(\mathbf{x}_{c,(k)}, \mathbf{u}_{c(k)}, \mathbf{v}_{c(k)})$$

$$\mathbf{y}_{c(k)} = g_{ci}(\mathbf{x}_{c(k)}, \mathbf{v}_{c(k)})$$

- $\mathcal{T}$ specifies for each mode $m_{i(k)}$ a set of transition functions $\mathcal{T}_i = \{\tau_{i1}, ..., \tau_{in}\}$. Each transition τ_{ij} has an associated guard condition $C_{ij}(x_{c,(k)}, u_{d,(k)})$ and specifies the probability distribution over target modes $m_{l(k+1)}$ together with an assignment for $x_{c,(k+1)}$.

Chapter 3

Stochastic Hybrid Systems

3.1 Overview

In the face of the growing complexity of control systems, stochastic modelling has attained a crucial role. Indeed, stochastic techniques for modelling control and hybrid systems have attracted the attention of many researchers and constitute one of the most important issues in contemporary research.

Hybrid systems have been extensively studied in the past decade, both concerning their theoretical framework, as well as relating to the increasing number of applications they are employed for. However, the subfield of stochastic hybrid systems is fairly young. There has been considerable current interest in stochastic hybrid systems due to their ability to represent such systems as maneuvering aircraft [110], switching communication networks [98]. Different issues related to stochastic hybrid systems have found applications to insurance pricing [61], capacity expansion models for the power industry [64], flexible manufacturing and fault tolerant control [80, 81], etc.

A considerable amount of research has been directed towards this topic, both in the direction of extending the theory of deterministic hybrid systems [106], as well as discovering new applications unique to the probabilistic framework.

3.1.1 Objectives of the chapter

This chapter has three objectives:

1. Introduce a very general framework for modelling stochastic hybrid processes: General Stochastic Hybrid System, abbreviated with GSHS.

2. Develop a theoretical construction for mixing Markov processes which preserves the Markov property. The result of this mixing operation will be called *Markov string.*

3. Show how GSHS can be embedded in the Markov string constructions and hence deduce the basic properties of GSHS as Markov property, strong Markov property

A GSHS might be thought of a 'conventional' hybrid system enriched with three uncertainty characteristics:

1. the continuous-time dynamics are driven by stochastic differential equations (SDE) rather then classical ODE,

2. a jump takes place when the continuous state hits the mode boundary or according to a transition rate

3. the post jump locations are randomly chosen according to a stochastic kernel.

Intuitively, GSHS can be described as an interleaving between a finite or countable family of diffusion processes and a jump process. Our goal is to prove that GSHS is indeed a 'good model'. This means that we need to investigate the stochastic properties of this model. A natural property we were looking for is the Markov property. Analysing the form of the GSHS executions (paths or trajectories), the first observation is that these are, in fact, 'concatenations' of the diffusion component paths. The continuity inherited from the diffusion trajectories is perturbed by the jumps between the diffusion components.

This observation leads to the investigation of a general mechanism for mixing Markov processes that preserves the Markov property. Given a finite or countable family of Markov processes with reasonably good properties, this machinery will allow us to get a new Markov process whose paths are obtained by 'sticking' together the component paths. Roughly speaking, Markov strings are sequences of Markov processes. The jump structure of a Markov string is completely described by a renewal kernel given a priori and a family of terminal times associated with the initial processes. We require that the Markov string have finitely many jumps in

finite time. Under these assumptions we prove that the Markov strings, as stochastic processes, enjoy useful properties like the strong Markov property and the càdlàg property.

We then return to GSHS and show how GSHS can be embedded in the framework of Markov strings. The class of GSHS inherits the strong Markov and càdlàg properties from Markov strings. We have to stress that the strong Markov property is a fundamental hypothesis in defining of the stochastic reachability.

Finally, we develop the expression of the infinitesimal generator associated to GSHS.

3.1.2 Related work

A well-known and very powerful class of continuous time stochastic processes with stochastic jumps (for the discrete state and also for the continuous state) is the piecewise-deterministic Markov processes (PDMP), introduced in [62], and applied to hybrid system modelling in [40]. The other modelling approaches are those presented in [106] (stochastic hybrid systems, abbreviated SHSM), [19] (stochastic hybrid models abbreviated SHM), [81, 82] (switching diffusion processes, abbreviated SDP), [27] (general switching diffusion processes abbreviated GSDP), see, also, [137] for quick presentation and comparisons. A very general formal model for stochastic hybrid systems is proposed in [41], which extends the model from [106], where the deterministic differential equations for the continuous flow are replaced by their stochastic counterparts, and the reset maps are generalized to (state-dependent) distributions that define the probability density of the state after a discrete transition. In this model transitions are always triggered by deterministic conditions (guards) on the state. Another model for stochastic hybrid processes are switching jump-diffusion developed in [23]. Switching jump-diffusion processes (SJDP) have jumps which: i) happen simultaneously with mode switching, and ii) depend on the mode after switching. Jumps satisfying both i) and ii) are referred to as hybrid jumps. There are two types of hybrid jumps: a forced jump that happens at an instant of hitting some boundary, and a Poisson type of jump that happens at a sudden instant. It is important to point out that the term of hybrid jump has been introduced for the first time in ?? and the mathematical basis for this has been developed in [24].

GSHS generalize PDMP allowing a stochastic evolution (diffusion process) between two consecutive jumps, while for PDMP the inter-jump motion is deterministic, according to a

vector field. As well, GSHS might be thought of as a kind of extended SHSM for which the transitions between modes are triggered by some stochastic event (boundary hitting time and transition rate). Moreover, GSHS generalise SDP also permitting that the continuous state to have discontinuities when the process jumps from one diffusion to another. The table 3.1.2 shows the how GSHS generalize different models of stochastic hybrid systems.

	PDMP	SHSM	SDP	SJDP	GSHS
Stochastic Continuous Evolution	non	√	√	√	√
Forced Transitions	√	√	non	√	√
Spontaneous Transitions	√	non	√	√	√

Table 3.1.2

For a detailed comparison of the first three models, the reader is referred to [137].

It can be shown also that the class of SJDP models can be considered as a subclass of GSHS whose stochastic kernel, which gives the post jump locations, is chosen in an appropriate way such that the change of the discrete state and continuous state at a jump depend on the post jump location (continuous and discrete).

3.2 General Stochastic Hybrid Systems

3.2.1 Informal Discussion

General Stochastic Hybrid Systems (GSHS) are a class of non-linear stochastic continuous-time hybrid dynamical systems. GSHS are characterized by a hybrid state defined by two components: the continuous state and the discrete state. The continuous and the discrete parts of the state variable have their own natural dynamics, but the main point is to capture the interaction between them.

The time t is measured continuously. The state of the system is represented by a continuous variable x and a discrete variable i. The continuous variable evolves in some "cells" X^i (open sets in the Euclidean space) and the discrete variable belongs to a countable set Q. The intrinsic difference between the discrete and continuous variables, consists of the way that they evolve through time. The continuous state evolves according to an SDE whose vector field and drift factor depend on the hybrid state. The discrete dynamics produces transitions in both

(continuous and discrete) state variables x, i. Switching between two discrete states is governed by a probability law or occurs when the continuous state hits the boundary of its state space. Whenever a switching occurs, the hybrid state is reset instantly to a new state according to a probability law which depends itself on the past hybrid state. Transitions, which occur when the continuous state hits the boundary of the state space are called forced transitions, and those which occur probabilistically according to a state dependent rate are called spontaneous transitions. Thus, a sample trajectory has the form $(q_t, x_t, t \geq 0)$, where $(x_t, t \geq 0)$ is piecewise continuous and $q_t \in Q$ is piecewise constant. Let $(0 \leq T_1 < T_2 < ... < T_i < T_{i+1} < ...)$ be the sequence of jump times.

It is easy to show that GSHS include, as special cases, many classes of stochastic hybrid processes found in the literature PDMP, SHSM, etc.

In the following we make use of some standard notions from the Markov process theory as: underlying probability space, natural filtration, translation operator, Wiener probabilities, admissible filtration, stopping time, strong Markov property [25]. The basic definitions from the Markov process theory are summarized in the Chapter 2.

3.2.2 The Mathematical Model

If X is a Hausdorff topological space we use to denote by $\mathcal{B}(X)$ or $\mathcal{B}$ its Borel σ-algebra (the σ-algebra generated by all open sets). A topological space, which is homeomorphic to a Borel subset of a complete separable metric space is called Borel space. A topological space, which is homeomorphic with a Borel subset of a compact metric space is called Lusin space.

State space. Let Q be a countable set of discrete states, and let $d : Q \to \mathbb{N}$ and $\mathcal{X} : Q \to \mathbb{R}^{d(.)}$ be two maps assigning to each discrete state $i \in Q$ an open subset X^i of $\mathbb{R}^{d(i)}$. We call the set

$$X(Q, d, \mathcal{X}) = \bigcup_{i \in Q} \{i\} \times X^i$$

the hybrid state space of the GSHS and $\mathbf{x} = (i, x^i) \in X(Q, d, \mathcal{X})$ the hybrid state. The closure of the hybrid state space will be

$$\overline{X} = X \cup \partial X$$

where

$$\partial X = \bigcup_{i \in Q} \{i\} \times \partial X^i.$$

It is clear that, for each $i \in Q$, the state space X^i is a Borel space. It is possible to define a metric ρ on X such that $\rho(\mathbf{x}_n, \mathbf{x}) \to 0$ as $n \to \infty$ with $\mathbf{x}_n = (i_n, x_n^{i_n})$, $\mathbf{x} = (i, x^i)$ if and only if there exists m such that $i_n = i$ for all $n \geq m$ and $x_{m+k}^i \to x^i$ as $k \to \infty$. The metric ρ restricted to any component X^i is equivalent to the usual Euclidean metric [62]. Each $\{i\} \times X^i$, being a Borel space, will be homeomorphic to a measurable subset of the Hilbert cube, $\mathcal{H}$ (Urysohn's theorem, Prop. 7.2 [21]). It is known that $\mathcal{H}$ is the product of countable many copies of $[0, 1]$. The definition of X shows that X is, as well, homeomorphic to a measurable subset of $\mathcal{H}$. Then $(X, \mathcal{B}(X))$ is a Borel space. Moreover, X is a Lusin space because it is a locally compact Hausdorff space with countable base (see [62] and the references therein).

Continuous and discrete dynamics. In each mode X^i, the continuous evolution is driven by the following stochastic differential equation (SDE)

$$dx_t^i = b(i, x_t^i)dt + \sigma(i, x_t^i)dW_t, \tag{3.1}$$

where $(W_t, t \geq 0)$ is the m-dimensional standard Wiener process in a complete probability space. The discrete component remains constant, i.e.

$$q_t = i.$$

Assumption 1 (Continuous evolution) *Suppose that* $b : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)}$, $\sigma : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot) \times m}$, $m \in \mathbb{N}$, *are bounded and Lipschitz continuous in* x.

This assumption ensures, for any $i \in Q$, the existence and uniqueness (Theorem 6.2.2. in [9]) of the solution for the above SDE.

In this way, when i runs in Q, the equation (3.1) defines a family of diffusion processes $M^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}_t^i, x_t^i, \theta_t^i, P^i)$, $i \in Q$ with the state spaces $\mathbb{R}^{d(i)}$, $i \in Q$. For each $i \in Q$, the elements $\mathcal{F}^i$, $\mathcal{F}_t^i$, θ_t^i, P^i, $P_{x^i}^i$ have the usual meaning as in the Markov process theory.

The jump (switching) mechanism between the diffusions is governed by two functions: the

jump rate λ and the transition measure R. The jump rate $\lambda : X \to \mathbb{R}_+$ is a measurable bounded function and the transition measure R maps X into the set $\mathcal{P}(X)$ of probability measures on $(X, \mathcal{B}(X))$. Alternatively, one can consider the transition measure $R : \overline{X} \times \mathcal{B}(X) \to [0,1]$ as a reset probability kernel.

Assumption 2 (Discrete transitions) *(i) for all $A \in \mathcal{B}(X)$, $R(\cdot, A)$ is measurable;*
(ii) for all $\mathbf{x} \in \overline{X}$ the function $R(\mathbf{x}, \cdot)$ is a probability measure.
(iii) $\lambda : X \to \mathbb{R}_+$ is a measurable function such that $t \to \lambda(i, x_t^i(\omega^i))$ is integrable on $[0, \varepsilon(\omega^i))$, for some $\varepsilon(\omega^i) > 0$, for each $\omega^i \in \Omega^i$.

Since $\overline{X}$ is a Borel space, then $\overline{X}$ is homeomorphic to a subset of the Hilbert cube, $\mathcal{H}$. Therefore, its space of probabilities is homeomorphic to the space of probabilities of the corresponding subset of $\mathcal{H}$ (Lemma 7.10 [21]). There exists a measurable function $F : \mathcal{H} \times \overline{X} \to X$ such that $R(\mathbf{x}, A) = \mathfrak{p}F^{-1}(A)$, $A \in \mathcal{B}(X)$, where $\mathfrak{p}$ is the probability measure on $\mathcal{H}$ associated to $R(\mathbf{x}, \cdot)$ and $F^{-1}(A) = \{\boldsymbol{\omega} \in \mathcal{H} | F(\boldsymbol{\omega}, \mathbf{x}) \in A\}$. The measurability of such a function is guaranteed by the measurability properties of the transition measure R.

Construction. We construct an GSHS as a *Markov 'sequence' H*, which admits (M^i) as subprocesses. The sample path of the stochastic process $(\mathbf{x}_t)_{t>0}$ with values in X, starting from a fixed initial point $\mathbf{x}_0 = (i_0, x_0^{i_0}) \in X$ is defined in a similar manner as PDMP [62].

Let ω^i be a trajectory which starts in (i, x^i). Let $t_*(\omega^i)$ be the first hitting time of ∂X^i of the process (x_t^i). Let us define the following right continuous multiplicative functional

$$F(t, \omega^i) = I_{(t<t_*(\omega^i))} \exp[-\int_0^t \lambda(i, x_s^i(\omega^i))ds]. \tag{3.2}$$

This function will be the survivor function for the stopping time S^i associated to the diffusion (x_t^i), which will be employed in the construction of our model. This means that "killing" of the process (x_t^i) is done according to the multiplicative functional $F(t, \cdot)$. The stopping time S^i can be thought of as the minimum of two other stopping times:

1. first hitting time of boundary, i.e. $t_*|_{\Omega^i}$;

2. the stopping time $S^{i\prime}$ given by the following continuous multiplicative functional (which plays the role of the survivor function)

$$M(t,\omega^i) = \exp(-\int_0^t \lambda(i, x_s^i(\omega^i)))ds.$$

The stopping time $S^{i\prime}$ can be defined as

$$S^{i\prime}(\omega^i) = \sup\{t|\Lambda_t^i(\omega^i) \leq m^i(\omega^i)\},$$

where Λ_t^i is the following additive functional associated to the diffusion (x_t^i)

$$\Lambda_t^i(\omega^i) = \int_0^t \lambda(i, x_s^i(\omega^i)))ds$$

and m^i is an $\mathbb{R}_+$-valued random variable on Ω^i, which is exponentially distributed with the survivor function $P_{x^i}^i[m^i > t] = e^{-t}$. Then

$$P_{x^i}^i[S^{i\prime} > t] = P_{x^i}^i[\Lambda_t^i \leq m^i]. \tag{3.3}$$

We set $\omega = \omega^{i_0}$ and the first jump time of the process is $T_1(\omega) = T_1(\omega^{i_0}) = S^{i_0}(\omega^{i_0})$. The sample path $\mathbf{x}_t(\omega)$ of the hybrid process up to the first jump time is now defined as follows:

$$\begin{aligned}
&\text{if } T_1(\omega) = \infty: \quad \mathbf{x}_t(\omega) = (i_0, x_t^{i_0}(\omega^{i_0})),\ t \geq 0\\
&\text{if } T_1(\omega) < \infty: \quad \mathbf{x}_t(\omega) = (i_0, x_t^{i_0}(\omega^{i_0})),\ 0 \leq t < T_1(\omega)\\
&\qquad\qquad\qquad\quad \mathbf{x}_{T_1}(\omega) \text{ is a r.v. w.r.t. } R((i_0, x_{T_1}^{i_0}(\omega^{i_0})), \cdot).
\end{aligned}$$

The process restarts from $\mathbf{x}_{T_1}(\omega) = (i_1, x_1^{i_1})$ according to the same recipe, using now the process $x_t^{i_1}$. Thus if $T_1(\omega) < \infty$ we define $\omega = (\omega^{i_0}, \omega^{i_1})$ and the next jump time

$$T_2(\omega) = T_2(\omega^{i_0}, \omega^{i_1}) = T_1(\omega^{i_0}) + S^{i_1}(\omega^{i_1})$$

The sample path $\mathbf{x}_t(\omega)$ between the two jump times is now defined as follows:

$$\text{if } T_2(\omega) = \infty: \quad \mathbf{x}_t(\omega) = (i_1, x^{i_1}_{t-T_1}(\omega)),\ t \geq T_1(\omega)$$
$$\text{if } T_2(\omega) < \infty: \quad \mathbf{x}_t(\omega) = (i_1, x^{i_1}_t(\omega)),\ 0 \leq T_1(\omega) \leq t < T_2(\omega)$$
$$\mathbf{x}_{T_2}(\omega) \text{ is a r.v. w.r.t. } R((i_1, x^{i_1}_{T_2}(\omega)), \cdot).$$

and so on.

We denote

$$N_t(\omega) = \sum I_{(t \geq T_k)}$$

Assumption 3 (Non-Zeno executions) *For every starting point* $\mathbf{x} \in X$, $\mathbb{E}N_t < \infty$, *for all* $t \in \mathbb{R}_+$.

3.2.3 Formal Definitions

We can introduce the following definition.

Definition 3 *A General Stochastic Hybrid System (GSHS) is a collection* $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ *where*

- Q *is a countable set of discrete variables;*
- $d : Q \to \mathbb{N}$ *is a map giving the dimensions of the continuous state spaces;*
- $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ *maps each* $q \in Q$ *into an open subset* X^q *of* $\mathbb{R}^{d(q)}$*;*
- $b : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)}$ *is a vector field;*
- $\sigma : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot) \times m}$ *is a* $X^{(\cdot)}$*-valued matrix,* $m \in \mathbb{N}$*;*
- $Init : \mathcal{B}(X) \to [0, 1]$ *is an initial probability measure on* $(X, \mathcal{B}(S))$*;*
- $\lambda : \overline{X}(Q, d, \mathcal{X}) \to \mathbb{R}^+$ *is a transition rate function;*
- $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0, 1]$ *is a transition measure.*

Following [146], we note that if R_c is a transition measure from $(X \times Q, \mathcal{B}(X \times Q))$ to $(X, \mathcal{B}(X))$ and R_d is a transition measure from $(X, \mathcal{B}(X))$ to $(Q, \mathcal{B}(Q))$ (where Q is equipped with the

discrete topology) then one might define a transition measure as follows

$$R(x^i, A) = \sum_{q \in Q} R_d(x^i, q) R_c(x^i, q, A^q)$$

for all $A \in \mathcal{B}(X)$, where $A^q = A \cap (q, X^q)$. Taking in the definition of a GSHS this kind of reset map, the change of the continuous state at a jump depends on the pre jump location (continuous and discrete) as well as on the post jump discrete state.
This construction can be used to prove that the stochastic hybrid processes with jumps, developed in [23], are a particular class of GSHS executions.

A GSHS execution can be defined as follows.

Definition 4 (GSHS Execution) *A stochastic process $x_t = (q(t), x(t))$ is called a GSHS execution if there exists a sequence of stopping times $T_0 = 0 < T_1 < T_2 \leq \ldots$ such that for each $k \in \mathbb{N}$,*

- *$x_0 = (q_0, x_0^{q_0})$ is a $Q \times X$-valued random variable extracted according to the probability measure Init;*
- *For $t \in [T_k, T_{k+1})$, $q_t = q_{T_k}$ is constant and $x(t)$ is a (continuous) solution of the SDE:*

$$dx(t) = b(q_{T_k}, x(t))dt + \sigma(q_{T_k}, x(t))dW_t \tag{3.4}$$

  where W_t is a the m-dimensional standard Wiener;
- *$T_{k+1} = T_k + S^{i_k}$ where S^{i_k} is chosen according with the survivor function (3.2).*
- *The probability distribution of $x(T_{k+1})$ is governed by the law $R\left((q_{T_k}, x(T_{k+1}^-)), \cdot\right)$.*

3.3 Markov strings

In this section we formulate a very general class of Markov processes, which will be called *Markov strings*, loosely based on the so-called "melange" operation of Markov processes [132]. A Markov string is a hybrid state 'jump Markov process'. The 'continuous state' component switches back and forth at random moments of times among a countable collections of Markov

processes defined on some evolution modes. The 'discrete component' keeps track of the index of which Markov process the continuous component is following. This discrete component plays the role of an 'evolution index'. The continuous state is allowed to jump whenever the evolution index changes. For a Markov string the sojourn time in each mode is given as a stopping time with memoryless property for the process which evolves in that mode. Moreover, the continuous state immediately before a switching between modes is allowed to influence that jump.

3.3.1 Informal description

We start with:

1. a countable family of independent Markov processes with some nice properties, for example the strong Markov property, the càdlàg property.

2. a sequence of independent stopping times (for each process is given a stopping time with memoryless property).

3. a renewal kernel is a priori given.

The stopping times play the role of the jump times from one process to another and the renewal kernel gives the distribution of the post-jump state. The probabilistic construction of the Markov string is natural:

1. start with one process, which belongs to the given family;

2. kill the current process at the corresponding stopping time;

3. jump according to the renewal kernel;

4. restart another process (belonging to the given family) from the new state;

5. return to 2. and repeat.

The pieced together process obtained by the above procedure is called Markov string. The main aim of this section is to prove that the Markov string inherits the properties (like the strong Markov property and the càdlàg property) from its component processes.

The Markov string construction is closely related to the mixing operation of Markov processes from [132] and the random evolution process construction from [146].Markov strings differ from the class of processes considered in [132], in that:

1. The jump times are essentially given stopping times, *not necessarily the life times of the component processes*; 2. After a jump, the string is allowed to restart following another process, which might be different from the pre-jump process.

The mixing ("melange") operation in [132] is only sketched and the author claims that it can be obtained using the renewal ("renaissance") operation. We consider that the passing from renewal to mixing is not straightforward. It is necessary to emphasize the construction of all probabilistic elements associated with the resulted string. Lifting the renewal construction to the mixing construction, remarkable changes should be introduced in the Markov string definitions of the state space, probability space, probabilities on the trajectories.

As well, Markov strings can be obtained by specializing the base process and the 'instantaneous' distribution in the structure of the random evolution processes developed by Siegrist in [146], but the proof of the strong Markov property is not given in [146]. There, the author claims this can be derived from the strong Markov property of revival processes introduced by Ikeda, et. al. in [111]. To our knowledge, this property is completely proved by Meyer, in [132], for revival processes.

3.3.2 The Ingredients

Suppose that $M^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i, P^i_{x^i})$, $i \in Q$ is a countable family of Markov processes. We denote the state space of each M^i by $(X^i, \mathcal{B}^i)$ and assume that $\mathcal{B}^i$ is the Borel σ-algebra of X^i if X^i is a topological Hausdorff space. We denote by Δ the cemetery point for all X^i, $i \in Q$. The existence of Δ is assumed for reasons that will be clear below. For each $i \in Q$, the elements $\mathcal{F}^i$, $\mathcal{F}^{i,0}_t$, $\mathcal{F}^i_t$, $\theta^i_t, P^i, P^i_{x^i}$ have the usual meaning as in the Markov process theory.

Let (P^i_t) denote the operator semigroup associated to M^i, which maps $\mathcal{B}^i(X^i)$ into itself, given by

$$P^i_t f^i(x^i) = \mathbb{E}^i_{x^i} f^i(x^i_t),$$

where $\mathbb{E}^i_{x^i}$ is the expectation w.r.t. $P^i_{x^i}$. Then a function f^i is *p-excessive* $(p > 0)$ w.r.t. M^i if $f^i \geq 0$ and $e^{-pt} P^i_t f^i \leq f^i$, for all $t \geq 0$ and $e^{-pt} P^i_t f^i \nearrow f^i$ as $t \searrow 0$.

Assumption 4 *For each $i \in Q$, we suppose that:*

1. M^i is a strong Markov process.

2. *P^i is a complete probability.*

3. *The state space X^i is a Borel space.*

4. *M^i enjoys the càdlàg property, i.e. for each $\omega^i \in \Omega^i$, the sample path $t \mapsto x_t^i(\omega^i)$ is right continuous on $[0,\infty)$ and has left limits on $(0,\infty)$ (inside X_Δ^i).*

5. *The p-excessive functions of M^i are P^i-a.s. right continuous on trajectories.*

Part 3. implies that the underlying probability space Ω^i can be assumed to be $D_{[0,\infty)}(X^i)$, the space of functions mapping $[0,\infty)$ to X^i which are right continuous functions with left limits. Let us consider ω_Δ^i the cemetery point of Ω^i corresponding to the 'dead' trajectory of M^i (when the process is trapped to Δ).

In the terminology of [131], parts 1., 3. and 5. of the Assumption 4 imply that each M^i is a *right process.*

Using this family of Markov processes $\{M^i\}_{i\in Q}$, we define a new Markov process whose realizations consist of concatenations of realizations for different M^i. To achieve this goal, we need to define the transition mechanism from one process to the others. The jumping mechanism will be driven by:

1. A stopping time (which gives the jump temporal parameter) for each process;
2. A renewal kernel, which gives the post jump state.

Formally, in order to define the desired Markov string, M, we need to give:

1. $(S^i)_{i\in Q}$, where, for each $i \in Q$, S^i is a *stopping time* of M^i,
2. The jumping mechanism between the processes M^i is governed by a *renewal kernel,* which is a Markovian kernel

$$\Psi : \{\bigcup_{i\in Q} \Omega^i\} \times \mathcal{B}(X) \to [0,1]$$

Assumption 5 *(i) For each $i \in Q$, S^i is terminal time, i.e. stopping time with the 'memory-less' property:*

$$S^i(\theta_t^i \omega^i) = S^i(\omega^i) - t,\ \forall t < S^i(\omega^i) \tag{3.5}$$

(ii) The renewal kernel Ψ satisfies the following conditions: (a) If $S^i(\omega^i) = +\infty$ then $\Psi(\omega^i, \cdot) = \varepsilon_\Delta$ (here, ε_Δ is the Dirac measure corresponding to Δ); (b) If $t < S^i(\omega^i)$ then $\Psi(\theta_t^i\omega^i, \cdot) = \Psi(\omega^i, \cdot)$.

Note that the component processes have the càdlàg property, therefore they may also have jumps, which are not treated separately in the construction of the Markov strings. The sequence of jump times refers to additional jumps, not to the jumps of the trajectories of component processes.

We consider now, for each $i \in Q$, the killed process $\widetilde{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, \widetilde{x}^i_t, \widetilde{\theta}^i_t, P^i, P^i_{x^i})$ where

$$\widetilde{x}^i_t(\omega^i) = \begin{cases} x^i_t(\omega^i), & \text{if } t < S^i(\omega^i) \\ \Delta, & \text{if } t \geq S^i(\omega^i) \end{cases} \quad \text{and} \quad \widetilde{\theta}^i_t(\omega^i) = \begin{cases} \theta^i_t(\omega^i), & \text{if } t < S^i(\omega^i) \\ \omega^i_\Delta, & \text{if } t \geq S^i(\omega^i) \end{cases}$$

In this case, Ω^i should be thought of as a subspace of $\Omega^i \times [0, \infty)$, the above embedding is made through the map $\omega^i \mapsto (\omega^i, S^i(\omega^i))$. The killed process is equivalent with the subprocess of M^i corresponding to the multiplicative functional $M^i_t = I_{[0,S^i)}(t)$ (see Chapter III, [25]).

3.3.3 The Construction

Using the elements defined in the section 3.3.2 we construct the pieced-together stochastic process $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$, which will be called *Markov string.* We have to point out that M is obtained by the concatenation of the killed processes $\widetilde{M}^i$.

To completely define the Markov string we need to specify the following elements: 1. $(X, \mathcal{B})$ - the state space; 2. $(\Omega, \mathcal{F}, P)$ - the underlying probability space; 3. $\mathcal{F}_t$ - the natural filtration; 4. θ_t - the translation operator; 5. P_x - Wiener probabilities.

State Space $(X, \mathcal{B})$. The state space will be X defined as follows. X is constructed as the direct sum of spaces X^i, with the same cemetery point Δ, i.e.

$$X = \bigcup_{i \in Q} \{(i, x) | x \in X^i\}. \tag{3.6}$$

In the same manner as in the section 3.2, it results that X is a Borel space.

The space X can be endowed with the Borel σ-algebra $\mathcal{B}(X)$ generated by its metric topology. Moreover, we have

$$\mathcal{B}(X) = \sigma\{\bigcup_{i \in Q} \{i\} \times \mathcal{B}^i\}. \tag{3.7}$$

Then $(X, \mathcal{B}(X))$ is a Borel space, whose Borel σ-algebra $\mathcal{B}(X)$ restricted to each component X^i gives the initial σ-algebra $\mathcal{B}^i$ [62].

We can assume, without loss of generality, that $X^i \cap X^j = \emptyset$ if $i \neq j$. Thus the relations (3.6) and (3.7) become

$$X = \bigcup_{i \in Q} X^i; \tag{3.8}$$

$$\mathcal{B}(X) = \sigma(\bigcup_{i \in Q} \mathcal{B}^i). \tag{3.9}$$

Therefore, we can assume, as well, that $\Omega^i \cap \Omega^j = \emptyset$ if $i \neq j$.

Probability Space. The space Ω can be thought as the space generated by the concatenation operation defined on the union of the spaces Ω^i (which are pairwise disjoint), i.e. $\Omega = (\bigcup_{i \in Q} \Omega^i)^*$. Note that, for each $i \in Q$, an arbitrary element ω^i of Ω^i must be thought as a trajectory of the killed process $\widetilde{M^i}$. The cemetery point of Ω is denoted by $\omega_\Delta = (\omega^i_\Delta)_{i \in Q}$. We use to denote by ω (resp. $\widehat{\omega}$ or ω^i) an arbitrary element of Ω (resp. $\bigcup_{i \in Q} \Omega^i$ or Ω^i).

The $\sigma-$algebra $\mathcal{F}$ on Ω will be the smallest $\sigma-$algebra on Ω such that the projection $\pi^i : \Omega \to \Omega^i$ are $\mathcal{F}/\mathcal{F}^i$ measurable, $i \in Q$. The probability P on $\mathcal{F}$ will be defined as a 'product measure'. Let $\widehat{\mathcal{F}}$ be the $\sigma(\bigcup_{i \in Q} \mathcal{F}^i)$ defined on $\bigcup_{i \in Q} \Omega^i$.

Recipe. We give the procedure to construct a sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = x_0^{i_0} \in X^{i_0}$. Let ω^{i_0} be a sample path of the process $(x_t^{i_0})$ starting with x_0. In fact, we give a recipe *to construct a Markov string starting with an initial path* ω^{i_0}. Let $T_1(\omega^{i_0}) = S^{i_0}(\omega^{i_0})$. The event ω and the associated sample path are inductively defined. In the first step

$$\omega = \omega^{i_0}$$

The sample path $x_t(\omega)$ up to the first jump time is now defined as follows:

$$\begin{aligned}
&\text{if } T_1(\omega) = \infty: \quad x_t(\omega) = x_t^{i_0}(\omega^{i_0}),\ t \geq 0 \\
&\text{if } T_1(\omega) < \infty: \quad x_t(\omega) = x_t^{i_0}(\omega^{i_0}),\ 0 \leq t < T_1(\omega) \\
&\qquad\qquad\quad\ x_{T_1} \text{ is a r.v. according to } \Psi(\omega^{i_0}, \cdot).
\end{aligned}$$

The process restarts from $x_{T_1} = x_1^{i_1}$ according to the same recipe, using now the process $(x_t^{i_1})$. Let ω^{i_1} be a sample of the process $(x_t^{i_1})$ starting with $x_1^{i_1}$. Thus, if $T_1(\omega) < \infty$ we define the next jump time

$$T_2(\omega^{i_0}, \omega^{i_1}) = T_1(\omega^{i_0}) + S_{i_2}(\omega^{i_2}).$$

Then, in the second step

$$\omega = \omega^{i_0} * \omega^{i_1}$$

where '$*$' is the concatenation operation of trajectories. The sample path $x_t(\omega)$ between the two jump times is now defined as follows:

$$\begin{aligned}
&\text{if } T_2(\omega) = \infty: \quad x_t(\omega) = x^{i_1}_{t-T_1}(\omega^{i_1}),\ t \geq T_1(\omega) \\
&\text{if } T_2(\omega) < \infty: \quad x_t(\omega) = x^{i_1}_t(\omega^{i_1}),\ 0 \leq T_1(\omega) \leq t < T_2(\omega) \\
&\qquad\qquad\quad\ x_{T_2} \text{ is a r.v. according to } \Psi(\omega^{i_1}, \cdot).
\end{aligned}$$

Generally, if $T_k(\omega) = T_k(\omega^{i_0}, \omega^{i_1}, ..., \omega^{i_{k-1}}) <$ with

$$\omega = \omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}}$$

then the next jump time is

$$T_{k+1}(\omega) = T_{k+1}(\omega^{i_0}, \omega^{i_1}, ..., \omega^{i_k}) = T_k(\omega^{i_0}, \omega^{i_1}, ..., \omega^{i_{k-1}}) + S^{i_k}(\omega^{i_k}) \tag{3.10}$$

The sample path $x_t(\omega)$ between the two jump times T_k and T_{k+1} is defined as:

$$\begin{aligned}
&\text{if } T_{k+1}(\omega) = \infty: \quad x_t(\omega) = x^{i_k}_{t-T_k}(\omega^{i_k}), t \geq T_{k+1}(\omega) \\
&\text{if } T_{k+1}(\omega) < \infty: \quad x_t(\omega) = x^{i_k}_{t-T_k}(\omega^{i_k}), 0 \leq T_k(\omega) \leq t < T_{k+1}(\omega) \\
&\qquad\qquad\quad\ x_{T_{k+1}} \text{ is a r.v. according to } \Psi(\omega^{i_k}, \cdot).
\end{aligned} \tag{3.11}$$

We have constructed a sequence of jump times $0 < T_1 < T_2 < ... < T_n < ...$ Let $T_\infty = \lim_{n\to\infty} T_n$. Then $x_t(\omega) = \Delta$ if $t \geq T_\infty$. A sample path until T_{k_0} (where $k_0 = \min\{k : S^{i_k}(\omega) = \infty\}$) of the process (x_t), starting from a fixed initial point $x_0 = (i_0, x_0^{i_0})$, is obtained as the

concatenation:

$$\omega = \omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k_0-1}}.$$

We denote $N_t(\omega) = \sum I_{(t \geq T_k)}$ the number of jump times in the interval $[0,t]$. To eliminate pathological solutions that take an infinite number of discrete transitions in a finite amount of time (known as Zeno solutions) we impose the following assumption:

Assumption 6 (Non-zeno dynamics) *For every starting point $x \in X$, $\mathbb{E}N_t < \infty$, for all $t \in \mathbb{R}_+$.*

Under Assumption 6, the underlying probability space Ω can be identified with $D_{[0,\infty)}(X)$.

Wiener Probabilities. One might define the expectation $\mathbb{E}^x f$, $x \in X$, where f is a $\mathcal{F}$-measurable function on Ω, which depends only on a finite number of variables, by recursion on the number of variables.

Step1. If $\omega = \omega^{i_0}$ and $f(\omega) = f_1(\omega^{i_0})$ with f_1 a $\mathcal{F}^{i_0}$-measurable function on Ω^{i_0}, then

- if $x = x^{i_0} \in X^{i_0}$ then $\mathbb{E}_x f = \mathbb{E}^{i_0}_{x^{i_0}} f$, where $\mathbb{E}^{i_0}_{x^{i_0}}$ is the expectation corresponding to the probability $P^{i_0}_{x^{i_0}}$;
- if $x = x^j \in X^j$, $j \neq i_0$ then $\mathbb{E}_x f = 0$.

Step2. If $\omega = \omega^{i_0} * \omega^{i_1} * ... * \omega^{i_n}$ and $f(\omega) = f_n(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_n})$ with f_n a $\prod_{k=0}^{n} \mathcal{F}^{i_k}$-measurable function on $\prod_{k=0}^{n} \Omega^{i_k}$ then

$$\begin{aligned} f_{n-1}(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{n-1}}) &= \int_{\Omega^{i_n}} f_n(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{n-1}} * \omega^{i_n}) dP^{i_n}_{\Psi(\omega^{i_{n-1}},\cdot)}(\omega^{i_n}); \\ g(\omega) &= f_{n-1}(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{n-1}}); \\ \mathbb{E}_x f &= \mathbb{E}_x g. \end{aligned} \tag{3.12}$$

Translation Operators. Let us define now the translation operator (θ_t) associated with (x_t). If $t \geq T_\infty(\omega)$, then we take $\theta_t(\omega) = \omega_\Delta$. Otherwise, there exists k such that $T_k(\omega) \leq t <$

$T_{k+1}(\omega)$. In this case we take

$$\theta_t(\omega) = (\theta^{i_k}_{t-T_k(\omega)}(\omega^{i_k}) * \omega^{i_{k+1}} * ...). \tag{3.13}$$

Lemma 5 *(θ_t) is the translation operator associated with (x_t), i.e.*

$$\theta_s \circ \theta_t = \theta_{s+t};\ x_s \circ \theta_t = x_{s+t}.$$

Proof. If $t \geq T_\infty(\omega)$, then $\theta_t(\omega) = \omega_\Delta$ and $x_{s+t}(\omega) = \Delta = x_s(\theta_t(\omega))$.
Suppose that there exist $k, i \geq 0$ such that $T_k(\omega) \leq t < T_{k+1}(\omega)$ and $T_i(\theta_t\omega) \leq s < T_{i+1}(\theta_t\omega)$.
Then

$$x_t(\omega) = x^{i_k}_{t-T_k}(\omega^{i_k});\ (x_s \circ \theta_t)(\omega) = x^{i_l}_{s-T_l}(\theta^{i_l}_{s-T_l}\omega^{i_l}).$$

Since $\theta_t(\omega)$ is given by (3.13) and T_{k+1} is given by (3.10) we obtain

$$\begin{aligned} T_{k+1}(\theta_t\omega) &= S^{i_k}(\theta^{i_k}_{t-T_k(\omega)}(\omega^{i_k})) = S^{i_k}(\omega^{i_k}) - (t - T_k(\omega)) \\ &= T_{k+1}(\omega) - t. \end{aligned}$$

Then

$$T_{i+1}(\theta_t\omega) = T_{k+i+1}(\omega) - t$$

Therefore

$$T_i(\theta_t\omega) \leq s < T_{i+1}(\theta_t\omega) \Leftrightarrow T_{k+i}(\omega) \leq s + t < T_{k+i+1}(\omega).$$

■

Natural Filtrations. Let $(\mathcal{F}_t)$ be the natural filtration with respect to (x_t). The natural filtration $(\mathcal{F}_t)$ on Ω is built such that we have the following definition of $\mathcal{F}_t$-measurability:

Definition 6 *A $\mathcal{F}$-measurable function f on Ω is $\mathcal{F}_t$-measurable if the following property holds: For each k, the function $f \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}$ is equal to $h \circ \eta_k$, where the function $h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})$ is such that for a fixed $(\widehat{\omega}^{i_0} * \widehat{\omega}^{i_2} * ... * \widehat{\omega}^{i_{k-1}})$ with $T_k(\widehat{\omega}^{i_0} * \widehat{\omega}^{i_2} * ... * \widehat{\omega}^{i_{k-1}}) \leq t$, $\omega^{i_k} \mapsto h(\widehat{\omega}^{i_0} * \widehat{\omega}^{i_2} * ... * \widehat{\omega}^{i_{k-1}} * \omega^{i_k})$ is measurable with respect to $\mathcal{F}^{i_k}_{t-T_k}$.*

Because the families of filtrations $(\mathcal{F}_t^i)$ are nondecreasing and right continuous, one can verify that the family $(\mathcal{F}_t)$ has the same properties, as follows.

Proposition 7 *(i) The family $(\mathcal{F}_t)$ is nondecreasing and right continuous.*
(ii) The random variables T_k are stopping times w.r.t. $(\mathcal{F}_t)$.
(iii) Let T a stopping time with respect to $(\mathcal{F}_t)$. For each $k \in \mathbb{N}$, $T \wedge T_k$ is a function on Ω which depends only on $\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}}$. On the other hand, if $\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}}$ is fixed, the function $(T \wedge T_{k+1} - T_k)^+$ with ω^{i_k} as argument is a stopping time with respect $(\mathcal{F}_t^{i_k})$.

Proof. The proof can be obtained with small changes from the similar result proofs given in [132] for the case of rebirth processes. ■

3.3.4 Basic Properties

Mainly, in this section we prove that the Markov string (x_t) constructed in section 3.3.3 is a right Markov process. The proof engine is based on the Markov property of the discrete time Markov chain (p_n), which will be built in the following.

(p_n) is a discrete time Markov chain associated to (x_t) with the state space $(\bigcup_{i \in Q} \Omega_i, \widehat{\mathcal{F}})$ and the underlying probability space $(\Omega, \mathcal{F})$. The chain (p_n) is essentially 'the $n-th$' step of the process (x_t). If its starting point is ω^{i_0} (a trajectory in Ω^{i_0} starting in $x_0^{i_0}$) then $p_n(\omega) = \omega^{i_n}$. The transition kernel associated with (p_n) can be defined as follows:

$$H(\widehat{\omega}, A) = P_\Psi(\widehat{\omega}, A), \ A \in \widehat{\mathcal{F}}.$$

The construction of P_x from subsection 3.3.3 is such that

- H is the transition function of (p_n);
- P_x is the initial probability law of (p_n); i.e. if $\widehat{\omega} \in \bigcup_{i \in Q} \Omega_i$ which starts in $x \in X$

$$P^{\widehat{\omega}}(p_0 \in A) = P_x(A), \ A \in \mathcal{F}.$$

Let η_k be the projection $(p_0, p_1, ..., p_k)$, i.e. $\eta_k(\omega) = (\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})$.

One might construct a jump process (η_t) associated to a Markov string (x_t) following a similar algorithm such that used for Piecewise Deterministic Markov processes, in [62]. We do

not have a one-to-one correspondence between the sample paths of (x_t) and (η_t), as in the case of PDMP. Then the jump process will not serve to study the Markov string. Its role is taken by the Markov chain (p_n).

Remark 1 *For each k on the set $\{T_k(\omega) \leq t < T_{k+1}(\omega)\}$ we have: $x_t = x^{i_k}_{t-T_k} \circ p_k$.*

Proposition 8 (Simple Markov property) *Under Assumptions 4-6, any Markov string $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ is a Markov process.*

Proof. The simple Markov property of (x_t) is equivalent to the following implication [132]:
If f is a positive $\mathcal{F}_t$-measurable function and g is a $\mathcal{F}$-measurable function then

$$\mathbb{E}^x[f \cdot g \circ \theta_t] = \mathbb{E}^x[f \cdot \mathbb{E}^{x_t}[g]]. \tag{3.14}$$

The identity (3.14) can be unfolded into two separated equalities

$$\mathbb{E}^x[f \cdot g \circ \theta_t \cdot I_{\{t \geq T_\infty\}}] = \mathbb{E}^x[f \cdot \mathbb{E}^{x_t}[g] \cdot I_{\{t \geq T_\infty\}}] \tag{3.15}$$

$$\mathbb{E}^x[f \cdot g \circ \theta_t \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}] = \mathbb{E}^x[f \cdot \mathbb{E}^{x_t}[g] \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}] \tag{3.16}$$

The identity (3.15) is clear because on $\{t \geq T_\infty\}$

$$\mathbb{E}^{x_t}[g] = g(\omega_\Delta);\ \theta_t(\omega) = \omega_\Delta;\ x_t(\omega) = \Delta.$$

Let us prove now the identity (3.16). Let $\omega \in \Omega$. By the definition of $\mathcal{F}_t$ we have

$$f(\omega) \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}(\omega) = h(\omega^{i_0} * \omega^{i_1} * \ldots * \omega^{i_k}) \tag{3.17}$$

where h is a measurable function as in the definition 6 and is equal to zero outside of the set $\{T_k(\omega) \leq t < T_{k+1}(\omega)\}$.

In order to prove (3.16) it is enough to treat the case when the function g depends only on a finite number of variables (because the expectation $\mathbb{E}^x$ is defined by the recursion (3.12)).

We start with the case when the function g depends only on a single variable, ω^{i_0}, i.e. $g(\omega) =$

$a(\omega^{i_0})$, where a is $\mathcal{F}^{i_0}$-measurable on Ω^{i_0}. In this case, the left-hand side of (3.16) is equal to

$$\mathbb{E}^x[f \cdot I_{\{T_k(\omega) \le t < T_{k+1}(\omega)\}} \cdot a(\theta^{i_k}_{t-T_k(\omega)}(\omega^{i_k}))]. \tag{3.18}$$

Because the term between [...] depends only on $(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})$, (3.18) becomes

$$\mathbb{E}^x\{\int_{\Omega^{i_k}} h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k}) \cdot a(\theta^{i_k}_{t-T_k(\omega)}(\omega^{i_k}))dP^{i_k}_{\Psi(\omega^{i_{k-1}},\cdot)}(\omega^{i_k})\}. \tag{3.19}$$

Again, the integrand between $\{...\}$ depends only on $(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}})$. Since the function $\omega^{i_k} \to h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})$ is $\mathcal{F}^{i_k}_{t-T_k}$-measurable, we can use the Markov property of the process M^{i_k} and (3.19) becomes

$$\int_{\Omega^{i_k}} h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})\mathbb{E}^{i_k}_{x^{i_k}_{t-T_k}(\omega^{i_k})}[a]dP^{i_k}_{\Psi(\omega^{i_{k-1}},\cdot)}(\omega^{i_k}). \tag{3.20}$$

Since $x_t(\omega) = x^{i_k}_{t-T_k}(\omega^{i_k})$ on $\{T_k(\omega) \le t < T_{k+1}(\omega)\}$ the computation of the right-hand side of (3.16) gives

$$\mathbb{E}^x\{h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k}) \cdot \mathbb{E}^{i_k}_{x^{i_k}_{t-T_k}(\omega^{i_k})}[a]\} \tag{3.21}$$

Using the recursive procedure, as before, (3.21) gives (3.20).

Suppose now that (3.16) is established for all functions g which depend only on $(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}})$. We have to prove that (3.16) is true for

$$g(\omega) = g(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k});\ k > 0.$$

Let

$$c(\omega) = c(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}}) = \int_{\Omega^{i_k}} b(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})dP^{i_k}_{\Psi(\omega^{i_{k-1}},\cdot)}(\omega^{i_k}).$$

Using the recursive procedure, one can check that the functions

$$h(...)g \circ \theta_t \quad \text{and} \quad h(...)c \circ \theta_t$$

have the same expectations.

On the other hand, the functions

$$h(...)\mathbb{E}_{x_t}[g] \quad \text{and} \quad h(...)\mathbb{E}_{x_t}c$$

have the same expectations. Since c depends only on $k-1$ variables, this implies (3.16) for the general case.□ ■

Proposition 9 (Cadlag property) *Under Assumptions 4-6, any Markov string* $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P,$ *has the cadlag property, i.e. for all* $\omega \in \Omega$ *the trajectories* $t \mapsto x_t(\omega)$ *are right continuous on* $[0, \infty)$ *with left limits on* $(0, \infty)$.

Proof. The result is a direct consequence of two facts:

1. the sample paths of (x_t) are obtained by the concatenation of sample paths of component process (i.e. the concatenation is done in such way it preserves the right continuity and the left limits);

2. the component processes enjoy the càdlàg property.

Then the Markov string inherits the càdlàg property. ■

Proposition 10 *Under Assumptions 4-6, any Markov string* $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ *is a strong Markov process.*

Proof. Each T_k is a stopping time for (x_t) (see proposition 7 (ii)). For each $k \geq 1$, T_k can be obtained by the following recursion

$$T_{k+1} = T_k + S^{i_k} \circ \theta_{T_k}$$

Let us prove now that the process (x_t) is a strong Markov process. The filtration $(\mathcal{F}_t)$ is nondecreasing and right continuous (see proposition 7 (i)). Then the process (x_t) satisfies the right hypothesis.

Let (P_t) be the semigroup of the whole Markov process (x_t), $P_t g(x) = \mathbb{E}_x g(x_t)$, where g is bounded $\mathcal{B}$-measurable function. Let $(U_p)_{p>0}$ the resolvent associated to the semigroup, i.e.

$$U_p g = \int_0^\infty e^{-pt} P_t g dt.$$

It is known that the strong Markov property is equivalent with each from the following assertions [133]:

1. If g is a positive bounded continuous function on X_Δ then $f = U_p g$ $(p > 0)$ is nearly Borel and right continuous on the process trajectories.

2. Each p-excessive function $(p > 0)$ is nearly Borel and right continuous on the process trajectories.

Recall that a real function defined on the state space X_Δ is nearly Borel for the process (x_t) if there exist two Borel functions h and h' on X_Δ such that $h' \leq f \leq h$ and

$$P\{\omega | \exists t, h' \circ x_t(\omega) < h \circ x_t(\omega)\} = 0. \tag{3.22}$$

Let g be a positive bounded continuous function on X. We have $g = \sum_{i \in Q} g^i$, where $g^i = g|_{X^i}$ are bounded continuous functions on X^i. Then $P_t g = \sum_{i \in Q} P_t^i g^i$ and

$$U_p g = \int_0^\infty e^{-pt} P_t g dt = \sum_{i \in Q} \int_0^\infty e^{-pt} P_t^i g^i dt = \sum_{i \in Q} U_p^i g^i.$$

It is known that $f = U_p g$ $(p > 0)$ (the restriction to X) is p-excessive function with respect to (P_t) and for each $i \in Q$ and the function $f^i = U_p^i g^i$ is p-excessive function with respect to (P_t^i). Therefore, f^i is nearly Borel and right continuous on the trajectories of the process (x_t^i). It is clear from the construction that the function f is right continuous on the trajectories of the process (x_t).

Let $h^i, h^{i\prime}$ two Borel functions on X_Δ^i such that $h' \leq f^i \leq h^i$ and

$$h^{i\prime} \circ x_t^i(\omega^i) = h^i \circ x_t^i(\omega^i) \; P^i - a.s., \forall t \geq 0. \tag{3.23}$$

Let us consider the function h, h' defined as below:

$$h = \sum_{i \in Q} h^i, \; h' = \sum_{i \in Q} h^{i\prime}. \tag{3.24}$$

It is clear that

$$P\{\omega | \exists t \geq T_\infty, h' \circ x_t(\omega) < h \circ x_t(\omega)\} = 0.$$

Let us compute the probability of the following event:

$$A_k = \{\exists t | T_k \leq t < T_{k+1}, h' \circ x_t(\omega) < h \circ x_t(\omega)\}.$$

We have $A_k \in \mathcal{F}$. Let $a_k = I_{A_k}$ which depends only on $\omega^{i_0} * \omega^{i_2} * ... * \omega^{i_k}$. The recursive method to compute the probability of A_k on $\{T_k \leq t < T_{k+1}\}$ gives

$$\int_{\Omega^{i_k}} a_k(\omega^{i_0} * \omega^{i_2} * ... * \omega^{i_k}) dP^{i_k}_{\Psi(\omega^{i_{k-1}},\cdot)}(\omega^{i_k}). \tag{3.25}$$

Since $a_k(\omega^{i_0} * \omega^{i_2} * ... * \omega^{i_k})$ on Ω^{i_k} is exactly the indicator function of

$$B = \{\omega^{i_k} | \exists u < S^{i_k}(\omega^{i_k}), h^{i_k\prime} \circ x^{i_k}_u(\omega) < h^{i_k} \circ x^{i_k}_u(\omega)\}$$

using (3.23) we obtain that the integral (3.25) is zero. Therefore the functions h, h' defined by (3.24) verify the condition (3.22). Then f will be a nearly Borel function relative to the process (x_t). ■

The Propositions 8, 9, 10 can be summarized in the following theorem:

Theorem 11 *Under Assumptions 4-6, any Markov string $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ has the following properties:*

(i) It is a strong Markov process;

(ii) It has the cadlag property;

(iii) It is a right process.

3.4 Properties of GSHS

Strong Markov property. GSHS, being constructed as particular Markov strings, they inherit the properties of their diffusion component, namely they are *strong Markov processes* with *càdlàg property.*

Proposition 12 (Strong Markov process) *Under the standard assumptions 1-3, any General Stochastic Hybrid Model H is a strong Markov process.*

Proof. To prove that H is a strong Markov process, it is enough to check that a GSHS is, indeed, a Markov string, i.e. it satisfies the Assumptions 4-6 from the Markov string construction. It is easy to see that

- Ass.1 implies Ass.4;
- Ass.3 implies Ass.6.

It remains to prove only that Assumption 2 and the construction of a GSHS implies Assumption 5. We can suppose without loss of generality that $\Omega^i \cap \Omega^j = \emptyset$. Then, the kernel Ψ can be defined as follows

$$\Psi : \{\bigcup_{i\in Q} \Omega^i\} \times \mathcal{B}(X) \to [0,1] \quad \text{such that} \quad \Psi(\omega^i, A) = R(x^i_{S^i(\omega^i)}, A)$$

For any GSHS, we need to check

(a) the memoryless property of kernel, i.e. if $0 < t < S^i(\omega^i)$ then $\Psi(\theta^i_t\omega^i, \cdot) = \Psi(\omega^i, \cdot) \Leftrightarrow R(x^i_{S^i(\theta^i_t\omega^i)}, \cdot) = R(x^i_{S^i(\omega^i)}, \cdot)$.

(b) the memoryless property of the stopping times S^i.

Since the component diffusions are strong Markov processes (b) implies (a). In fact, we have to prove that, if $0 < t < t + s < S^i(\omega^i)$ then stopping times (S^i)

$$P_{x^i}(S^i > t + s | S^i > t) = P_{x^i_t}(S^i > s) \tag{3.26}$$

We have, for each $i \in Q$,

1. the hitting time of the boundary ∂X^i of the diffusion process (x^i_t) has the memoryless property, i.e. $t^*(\theta^i_t\omega^i) = t_*(\omega^i) - t$.
2. the stopping time $S^{i\prime}$ with the survivor function (3.3) has the memoryless property because

$$\begin{aligned} P_{x^i}(S^{i\prime} > t + s | S^{i\prime} > t) &= \frac{P_{x^i}\{\omega^i | m^i(\omega^i) > \Lambda^i_{t+s}(\omega^i)\}}{P_{x^i}\{\omega^i | m^i(\omega^i) > \Lambda^i_t(\omega^i)\}} \\ &= \frac{P_{x^i}\{\omega^i | m^i(\omega^i) > \Lambda^i_t(\omega^i) + \Lambda^i_s(\theta^i_t\omega^i)\}}{P_{x^i}\{\omega^i | m^i(\omega^i) > \Lambda^i_t(\omega^i)\}} \\ &= P_{x^i_t}\{\omega^i | m^i(\omega^i) > \Lambda^i_s(\theta^i_t\omega^i)\} \\ &= P_{x^i_t}(S^{i\prime} > s) \end{aligned}$$

(we have used the fact that m^i has the memoryless property, being an exponentially distributed random variable, and the additivity of Λ_t^i w.r.t. t since this is an additive functional).

Since, for each $i \in Q$, the stopping time S^i is the infimum of t^* and $S^{i'}$, the two above facts easily implies the 'memoryless' property of S^i (it is easy to prove that the infimum of two memoryless stopping times is still a memoryless stopping time).
Thus, H is a Markov string obtained by mixing diffusion processes. Therefore, it inherits the strong Markov property from the component diffusions.□ ■

Corollary 13 *Any General Stochastic Hybrid Model H, under the standard assumptions of section 3.2.2, is a Borel right process.*

Proof. The statement of the corollary is immediate, since the state space is a Lusin space and H is a right process.□ ■

As we discusses in the context of Markov strings, a GSHS might be thought of as a 'restriction' of a random evolution process [146], whose components are diffusion processes defined on different state spaces. We can consider each diffusion component evolving on $\overline{X}$. The first difference is that while a GSHS is defined only on $\bigcup_{i\in Q}\{i\} \times X^i$ a random evolution process should be defined on the entire product space $Q \times \overline{X}$. The second difference is that whilst for a random evolution process the jump times from one process to another are driven only by transition rates, for a GSHS these might be also boundary hitting times of modes.

However, contrary to [146], GSHS are not always standard processes as the random evolution processes.

The Process Generator. We denote by $\mathcal{B}_b(X)$ the set of all bounded measurable functions $f : X \to \mathbb{R}$. This is a Banach space under the norm $\|f\| = \sup_{x\in X} |f(x)|$. Associated with the semigroup (P_t) is its *strong generator* which is the 'derivative' of P_t at $t = 0$. Let $D(L) \subset \mathcal{B}_b(X)$ be the set of functions f for which the following limit exists $\lim_{t\searrow 0} \frac{1}{t}(P_t f - f)$ and denote this limit Lf. This refers to convergence in the norm $\|\cdot\|$, i.e. for $f \in D(L)$ we have $\lim_{t\searrow 0} \|\frac{1}{t}(P_t f - f) - Lf\| = 0$. Specifying the domain $D(L)$ is an essential part of specifying L.

Martingale property (Prop. 1) says that for $f \in D(L)$ we define the real-valued process $(C_t^f)_{t\geq 0}$ by

$$C_t^f = f(x_t) - f(x_0) - \int_0^t Lf(x_s)ds. \tag{3.27}$$

Then for any $x \in X$, the process $(C_t^f)_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$.

There may be other functions f, not in $D(L)$, for which something akin to (3.27) is still true. In this way we get the notion of *extended generator* of the process.

Let $D(\widehat{L})$ be the set of measurable functions $f : X \to \mathbb{R}$ with the following property: there exists a measurable function $h : X \to \mathbb{R}$ such that $t \to h(x_t)$ is integrable $P_x - a.s.$ for each $x \in X$ and the process

$$C_t^f = f(x_t) - f(x_0) - \int_0^t h(x_s)ds$$

is a local martingale. Then we write $h = \widehat{L}f$ and call $(\widehat{L}, D(\widehat{L}))$ the extended generator of the process (x_t).

Following [62], for $A \in \mathcal{B}(\overline{X})$ define p, p^* and $\widetilde{p}$ as follows:

$$p(t, A) = \sum_{k=1}^{\infty} I_{(t\geq T_k)} I_{(x_{T_k} \in A)};$$

$$p^*(t) = \sum_{k=1}^{\infty} I_{(t\geq T_k)} I_{(x_{T_k^-} \in \partial X)};$$

$$\widetilde{p}(t, A) = \int_0^t R(x_s, A)\lambda(x_s)ds + \int_0^t R(A, x_{s-})dp^*(s)$$

$$\widetilde{p}(t, A) = \sum_{T_k \leq t} R(x_{T_k-}, A).$$

Note that p, p^* are counting processes, $p^*(t)$ is counting the number of jumps from the boundary of the process (x_t). $\widetilde{p}(t, A)$ is the compensator of $p(t, A)$ (see [62] for more explanations). The process $q(t, A) = p(t, A) - \widetilde{p}(t, A)$ is a local martingale.

Given a function $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and a vector field $b : \mathbb{R}^n \to \mathbb{R}^n$, we use $\mathcal{L}_b f$ to denote the Lie derivative of f along b given by $\mathcal{L}_b f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) f_i(x)$. Given a function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, we use $\mathbb{H}^f$ to denote the Hamiltonian operator applied to f, i.e. $\mathbb{H}^f(x) = (h_{ij}(x))_{i,j=1\ldots n} \in \mathbb{R}^{n\times n}$,

where $h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. A^T denotes the transpose matrix of a matrix $A = (a_{ij})_{i,j=1...n} \in \mathbb{R}^{n\times m}$ and $Tr(A)$ denotes its trace.

Theorem 14 (GSHS generator) *Let H be an GSHS as in definition 3. Then the domain $D(L)$ of the extended generator L of H, as a Markov process, consists of those measurable functions f on $X \cup \partial X$ satisfying:*

1. $f : \overline{X} \to \mathbb{R}$, $\mathcal{B}-$measurable such that for each $i \in Q$ the restriction $f^i = f|_{X^i}$ is twice differentiable.

2. the boundary condition

$$f(x) = \int_{\mathbb{X}} f(y) R(x, dy),\ x \in \partial X;$$

3. $Bf \in L_1^{loc}(p)$ (see [1]) where

$$Bf(x, s, \omega) := f(x) - f(x_{s-}(\omega)).$$

For $f \in D(L)$, Lf is given by

$$Lf(x) = L_{cont} f(x) + \lambda(x) \int_{\overline{X}} (f(y) - f(x)) R(x, dy) \tag{3.28}$$

where:

$$L_{cont} f(x) = \mathcal{L}_b f(x) + \frac{1}{2} Tr(\sigma(x)\sigma(x)^T \mathbb{H}^f(x)). \tag{3.29}$$

Proof. Let $(\widetilde{L}, D(\widetilde{L}))$ be the extended generator of (x_t). We want to show that $(\widetilde{L}, D(\widetilde{L})) = (L, D(L))$. Suppose first that f satisfies 1-3. Then $Bf \in L_1^{loc}(\widetilde{p})$ and $\int_{[0,t]\times\overline{X}} Bf d\widetilde{p} = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_s)) R(x_s, dy) \lambda(x_s) ds \\ I_2 &= \int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_{s-})) R(x_{s-}, dy) dp^*(s). \end{aligned}$$

[1] *Following [62], f is in $L_1^{loc}(p)$ if for some sequence of stopping times $\sigma_n \uparrow \infty$*

$$E_x \sum_i |f(x_{T_i \wedge \sigma_n}) - f(x_{T_i \wedge \sigma_n -})| < \infty$$

Now the support of p^* is contained in the countable set $\{s : x_{s-} \in \partial X\}$ and because of the boundary condition 2. the second integral I_2 vanishes. Thus

$$\int_{[0,t]\times\overline{X}} Bf dq = \sum_{T_k \le t} (f(x_{T_k}) - f(x_{T_k-})) - \int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_s)) R(x_s, dy)\lambda(x_s) ds.$$

This is a local martingale because of condition 3. Let T_m denote the last jump time prior or equal to t. Then

$$\sum_{T_k \le t} (f(x_{T_k}) - f(x_{T_k-})) = \{f(x_t) - f(x_{Tm})\} + S_m$$

where

$$\begin{aligned} S_m = &\sum_{k=1}^{m} (f(x_{T_k}) - f(x_{T_{k-1}}))\} - \{f(x_t) - f(x_{Tm}) + \\ &+ \sum_{k=1}^{m} (f(x_{T_k-}) - f(x_{T_{k-1}}))\}. \end{aligned}$$

The first bracketed term on the right is equal to $f(x_t) - f(x)$. Note that $x_{T_k-} = x^{i_{k-1}}_{T_k - T_{k-1}}$, if $x_{T_{k-1}} = (i_{k-1}, x^{i_{k-1}}_{k-1})$. Then Itô-formula [73] gives the second term

$$f(x_{T_k-}) - f(x_{T_{k-1}}) = \int_{T_{k-1}}^{T_k} L_{cont} f(x_s) ds + \int_{T_{k-1}}^{T_k} < \sigma(x_s), \nabla f(x_s) > dW(s).$$

The second term is therefore equal to $\int_0^t L_{cont} f(x_s) ds + \int_0^t < \sigma(x_s), \nabla f(x_s) > dW(s)$ and we obtain

$$C_t^f := f(x_t) - f(x) - \int_0^t Lf(x_s) ds = \int_0^t < \sigma(x_s), \nabla f(x_s) > dW(s) + \int_{[0,t]\times\overline{X}} Bf dq$$

is a local martingale (the sum between a continuous martingale and a discrete martingale), where L is given by (3.28). Thus $f \in D(\widehat{L})$ and $\widehat{L}f = Lf$.

Conversely, suppose that $f \in D(\widehat{L})$. Then the process $M_t := f(x_t) - f(x) - \int_0^t h(x_s) ds$ is a local martingale, where $h = \widehat{L}f$. Then M_t must be the sum between a continuous martingale M_t^c and a discrete martingale M_t^d. From Th.(26.12), p.69 [62], we have $M_t^d = M_t^\rho$ for some predictable integrand $\rho \in L_1^{loc}(p)$, where

$M_t^\rho = \int_{\overline{X}\times\mathbb{R}_+} \rho I_{(s\le t)} dq = \sum_{T_k \le t} \rho(x_{T_k}, T_k, \omega) - \int_0^t \int_{\overline{X}} \rho(y, s, \omega)\{R(x_s, dy)\lambda(x_s) ds - R(x_{s-}, dy) dp^*(s)\}$.

Since M_t^d and M_t^ρ agree, their jumps ΔM_t^d and ΔM_t^ρ must agree; these only occur when $t = T_k$ for some k and are given by: $\Delta M_t^d = f(x_t) - f(x_{t-})$; $\Delta M_t^\rho = \rho(x_t, t, \omega) - \int_{\overline{X}} \rho(y, t, \omega) R(x_{t-}, dy) I_{(x_{t-}\in\partial X)}$.

Thus $\rho(x_t, t, \omega) = f(x_t) - f(x_{t-})$ on the set $(x_{t-} \notin \partial X)$, which implies that $\rho(x, t, \omega) =$

$f(x) - f(x_{t-})$ for all (x,t) except perhaps a set to which the process 'never jumps', i.e. $G \subset \mathbb{R}_+ \times X$ such that $\mathbb{E}_z \int_G p(dt,dx) = 0$, $\forall z \in X$.

Suppose that $z = x_{t-} \in \partial X$. Then equating ΔM_t^d and ΔM_t^ρ gives $f(x_t) - f(z) = \rho(x_t,t,\omega) - \int_{\overline{X}} \rho(y,t,\omega) R(z,dy)$ and hence $f(x) - f(z) = \rho(x,t,\omega) - \int_{\overline{X}} \rho(y,t,\omega) R(z,dy)$, except on a set $A \in \mathcal{B}(X)$ such that $R(z,A) = 0$. Integrating both sides of the previous equality with respect to $R(z,dx)$, we obtain $\int_{\overline{X}} f(x) R(z,dx) - f(z) = \int_{\overline{X}} \rho(x,t,\omega) R(z,dx) - \int_{\overline{X}} \rho(y,t,\omega) R(z,dy) = 0$. Thus f satisfies the boundary condition. For fixed z, define $\widetilde{\rho}(x,t,\omega) = \rho(x,t,\omega) - (f(x) - f(z))$. Using the boundary condition we get $\int_{\overline{X}} \widetilde{\rho}(y,t,\omega) R(z,dy) = \int_{\overline{X}} \rho(y,t,\omega) R(z,dy) = \widetilde{\rho}(x,t,\omega)$. Then $\widetilde{\rho}(x,t,\omega) = \int_{\overline{X}} \widetilde{\rho}(y,t,\omega) R(z,dy)$.

However, the right-hand side does not depend on x, and hence $\widetilde{\rho}(x,t,\omega) = u(t,\omega)$ for some predictable process u. The general expression for ρ is thus

$$\rho(x,t,\omega) = f(x) - f(x_{t-}) + u(t,\omega) I_{(x_{t-} \in \partial X)}.$$

Inserting this in the expression of M_t^ρ we find that M_t^ρ does not depend on u, then we can take $u \equiv 0$, obtaining $\rho = Bf$; hence the part 3 of theorem is satisfied.

Finally, consider the sample paths of M_t, $M_t^{Bf} + M_t^c$, for $t < T_1(\omega)$, starting at $x \in X$. We have

$$M_t = f(x_t(\omega^{i_0})) - f(x) + \int_0^t h(x_s(\omega^{i_0}))ds$$

while, because $p = p^* = 0$ on $[0,T_1)$,

$$M_t^{Bf} = -\int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_s(\omega^{i_0}))) R(x_s(\omega^{i_0}),dy) \lambda(x_s(\omega^{i_0}))ds.$$

So, since $M_t = M_t^{Bf} + M_t^c$ for all t a.s., it must be the case that $M_t = M_t^c$ for $t \in [0,T_1)$ and the generator coincides with the generator L_{cont} associated to the stochastic equation, the function $f(x_t(\omega^{i_0}))$ should have second order derivatives on $[0,T_1)$. The general case follows by concatenation. Similar calculations show that

$$M_t^{Bf} + M_t^c = f(x_t) - f(x) - \int_0^t Lf(x_s)ds, \; \forall t \geq 0$$

with L given by (3.28). Hence $f \in D(L)$ and $Lf = \widehat{L}f$. ■

3.5 Examples of Stochastic Hybrid Systems

In this section we prent a collection of SHS examples which can be modeled in the GSHS modelling framework.

3.5.1 Single-server Queues

This example was presented in [63], as a model belonging to the class Piecewise Deterministic Markov Processes, which is a particular class of GSHS.

Customers arrive at a queue at random times $T_1, T_2, ...$, and the customer arriving at time T_i requires $Y_i > 0$ units of time for processing. The total service load (in time units) presented up time t is

$$L_t = L_0 + \sum_i Y_i I_{(t \geq T_i)}$$

where $L_0 \geq 0$ is the service load existing at time 0, and the *virtual time* V_t is the unique solution of the equation

$$V_t = L_t - \int_0^t I_{[V_s > 0]} ds$$

The first term on the right is the total service load presented while the second is the amount of carried out processing; i.e. V_t is the amount of unprocessed load at time t, or equivalently the time a customer arriving at t would have to wait for service to begin. The queue has two possible configurations, namely *busy* and *empty*, and we will denote these by an indicator variable q with $q = 1$ when the queue is busy and $q = 0$ when it is empty.

The queueing systems are characterised via the conventional classification $A/B/n$, where A refers to the arrival process, B to the distribution of Y_i and n to the number of servers ($n = 1$ in this case). Let us consider the $M/G/1$ queue, where M (for 'Markov') means that the interarrival times are independent and exponential (i.e. the arrivals form a Poisson process), and G (for 'general') means that Y_i are independently identically distributed with some arbitrary distribution F. Y_i are assumed to be independent of the arrivals process.

When

- $q = 1$, V_t decreases at unit rate between jumps;

- $q = 0$ there is an exponential waiting time for a new arrival (because of memoryless property of the exponential distribution).

The state space of this model is

$$X = \{0, 0\} \cup \{1\} \times (0, \infty)$$

and take $x_t = (q_t, V_t)$, where q_t is the current configuration of the queue and V_t is the virtual waiting time. Then (x_t) is a Markov process, which evolves as follows: when V_t hits zero, x_t jumps to $(0, 0)$ and waits there until the next arrival, at which point it jumps to $(1, Y)$, where Y is the service requirement of the arriving customer.

In the case of the $GI/G/1$ queue: GI (for 'general independent') means that the interarrival times are independently identically distributed with general distribution. In order to get a Markov process, it is necessary to include the supplementary variable τ, the time since the last arrival. The state space is

$$X = \{0\} \times [0, \infty) \cup \{1\} \times [0, \infty) \times (0, \infty)$$

When $q = 1$, $\zeta = (\zeta^1, \zeta^2)$ is two dimensional with $\zeta_t^1 = \tau$ and $\zeta_t^2 = V_t$, the virtual waiting time. Denote $x_t = (q_t, \zeta_t)$. When $q = 1$, ζ_t^1 and ζ_t^2 increase and decrease respectively at unit rate. If the queue becomes empty (i.e. $\zeta^2 = 0$) then it is necessary to continue accumulating the time since the last jump, so x_t jumps from $(1, \zeta^1, 0)$ to $(0, \zeta^1)$. If the next arrival occurs t time units later, bringing service requirement Y, then x_t jumps from $(0, \zeta^1 + t)$ to $(1, (0, Y))$.

3.5.2 A Hybrid Manufacturing System Model

This model has been studied in [84] and is motivated by the structure of many manufacturing systems. In these systems, discrete entities (referred to as *jobs*) move through a network of workcenters which process the jobs so as to change their physical characteristics according with certain specifications. Associated with each job is a *temporal state* and a *physical state.* The temporal state of a job evolves according to event-driven dynamics and include information such as arrival time, waiting time, service time, or departure time of the job at the various workcen-

ters. The physical state evolves according to time-driven dynamics modeled through differential equations which, depending on the particular problem being studied, describe changes in such quantities as the temperature, size, weight, chemical composition, bacteria level, or some other measures of the "quality" of the job. The interaction of time-driven with event-driven dynamics leads to a natural trade-off between temporal requirements on job completion times and physical requirements on the quality of the completed jobs.

Consider a single-stage manufacturing process modeled as single-server queueing system. The server processes one job at a time on a first-come first-served nonpreemptive basis (i.e., once a job begins service, the server cannot be interrupted, and will continue to work it until the operation is completed). Identical jobs arriving at the system with rate λ wait in an infinite capacity queue until they are processed by the server operating at rate $u \in U$. *Exponential interarrival times are assumed.* The controller is assumed to select processing rates from a finite set

$$U = \{u_1, u_2, ..., u_m\} \text{ where } u_i < u_{i+1},\ i = 1, ..., m-1$$

As job i is being processed, its physical state, denoted by x_i evolves according to time-driven dynamics of the general form

$$\dot{x}_i = g_i(x_i, u_i(t), t, \omega), x_i(\tau_i) = x_i^0 \tag{3.30}$$

where τ_i is the time when processing begins and x_i^0 is the initial state at that time. The control variable $u_i(t)$ is used to attain a final desired physical state corresponding to a target "quality level". On the other hand, the temporal state of the ith job is denoted by T_i^e and represents the time when the job completes processing and departs from the system. Letting T_i be the arrival time of the ith job and S_i be the service time which is a function of $u_i(t)$ during the process, the event-driven dynamics describing the evolution of the temporal state is given by the following "max-plus" recursive equation:

$$T_i^e = \tau_i + S_i = \max(T_i, T_{i-1}^e) + S_i \tag{3.31}$$

where $T_0^e = -\infty$ in which case $\tau_1 = T_1$ and the first job begins service as soon as it arrives.

Equation (3.31) is known in queueing theory as the Lindley equation.

The system is hybrid in the sense that it combines the time-driven dynamics (3.30) with the event-driven dynamics (3.31). If we suppose that there are no delays, i.e. $T_{i-1}^e = T_i$ for $i = 2, ..., m$ then this model is a particular case of GSHS.

3.5.3 A Simplified Model of a Truck with flexible Transmission

This example has been used in [91, 116]. The system is described by

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= -x_2 + x_3 \\ dx_3 &= -x_2 + g_q(x_2)udt + \sigma dw \\ q &= 1, 2, \ -0.1 \leq u \leq 1.1 \ \sigma = 0.01 \end{aligned}$$

where x_1, x_2 and x_3 are the position, velocity, and the rotational displacement of its transmission shaft respectively. The efficiency for gear q is $g_q(x)$, u is the throttle, and dw is a scalar Wiener process. In [116] the model is modified in comparison with that one presented in [91], by assuming that gears switches occur at the speed of equal efficiency between the gears ($x_2 = 0.5$) and therefore, the switching boundary is defined by $A = \{x | x_2 = 0.5\}$.

The objective is to drive the state (x_0, q_0) to the target set

$$E = \{x | \frac{1}{2} x^T x \leq 0.25\}.$$

3.5.4 The Stochastic Thermostat

This is an example of GSHS with two discrete states, which models the temperature in a house with n rooms, $n \geq 1$, regulated by a single thermostat. This is the generalization of the one-dimensional process that was studied in [130].

Let $z = (z_1, z_2, ..., z_3) \in \mathbb{R}^n$ describe the temperature in the n rooms of the house, and $q \in Q = \{0, 1\}$ the binary state of the thermostat. The global state of the system is then described by the hybrid state $x = (q, z) \in Q \times \mathbb{R}^n$, which has both a discrete and a continuous component. For a given discrete state $q \in Q$ of the thermostat, the temperature z_t evolves in

$\mathbb{R}^n$ according to an SDE of the form

$$dz_t = b(q, z_t) + \sigma dW_t \tag{3.32}$$

where $\sigma \in \mathbb{R}^{n\times n}$ and $b_q = b(q, \cdot)$ describe the action of the thermostat, the effect of the exterior environment, and the coupling between the temperatures of adjacent rooms. The switching of the thermostat is controlled by a linear criterion $\Psi(z) = \sum_{i=1}^{n} \alpha_i z_i$. The thermostat switches on when $\Psi(z_t)$ crosses some threshold $\Psi_{\min}$ downwards, and switches off when it crosses another threshold $\Psi_{\max} > \Psi_{\min}$ upwards. This can be described in the GSHS framework as follows: we define

$$\begin{aligned} X_0 &= \{z \in \mathbb{R}^n | \Psi(z) > \Psi_{\min}\} \\ X_1 &= \{z \in \mathbb{R}^n | \Psi(z) < \Psi_{\max}\} \end{aligned}$$

and then the hybrid state space will be

$$\overline{X} = \{0\} \times \overline{X}_0 \cup \{1\} \times \overline{X}_1 \subset Q \times \mathbb{R}^n.$$

The process z_t evolves continuously in X_q according to the SDE (3.32) as long as the thermostat is in state $q_t = q$, and Q_t switches when z_t reaches ∂X_q. Therefore, the hybrid process $x_t = (q_t, z_t)$ is a GSHS with only forced discrete transitions.

3.6 Some Remarks

In this chapter we set up the notion of Markov string, which is roughly speaking, a concatenation of Markov processes. This notion has raised as a result of our research on stochastic hybrid system modelling [106, 40, 41, 137] and it aims to be a very general formalization of all existing models of stochastic hybrid systems. The Markov string concept has been proved to be a very powerful tool in the studying of the general models of stochastic hybrid processes GSHS introduced at the beginning of the chapter.

One of the main contributions of this work is the proof of the strong Markov property. Since

GSHS are a particular class of Markov strings, this property holds also for them.

In the end of this chapter, based on the strong Markov property of GSHS we have developed the extended generator of this model.

Further developments of our model will include two main tracks.

- First it is necessary a study of the reachability problem for GSHS. One possible approach in this direction is the introduction of a bisimulation concept for GSHS. Reachability analysis and model checking are much easier when a concept of bisimulation is available. The state space can be drastically abstracted in some cases.

- Second it is natural to generalize the results on dynamic programming, relaxed controls, control via discrete-time dynamic programming, non-smooth analysis, from PDMP to GSHS.

Chapter 4

Further Developments of GSHS

4.1 Overview

This chapter is a first step in the direction of stochastic hybrid system composition. The DSHS model, which will be defined in the following, extends the CPDMP model (communicating of Piecewise Deterministic Markov Processes) defined in [154] towards GSHS. In this way, the DSHS can be viewed as an extension of the composition formalism for Markov models à la [96], since CPDMP extends that formalism without considering equational theories (process algebra).

4.2 Distributed Stochastic Hybrid Systems

In real life, distribution is present in many various ways. Finding its adequate definition in a specific context is often a difficult task. In ATM systems one can easily discover that centralised control coexists with many (semi) autonomic behaviours. This situation is not very common for discrete systems. Moreover, what can be logically seen as centralised control is *distributed* from a physical point of view. In example analysis, where space aspects are important, one might need a distributed model. Simultaneous executions in ATM are, obviously everywhere. From a continuous mathematics viewpoint parallelism (simultaneous executions) is very easy to be modelled essentially because of the lack of interaction. Basically, the Cartesian product of two (continuous) processes does the job.

But also, it is known that the interaction is an essential aspect in all distributed systems we are talking about. These systems are large scale, possibly involving the collaboration of multiple agents (both human and non-human). Sensing and control are often distributed, and information exchange and co-ordination has to be achieved through communication. Pilots talk to controllers and sometimes one with each other. Embedded controllers usually control components that interact and are aware of each other existence. In the case of ubiquitous systems controllers are even self-aware having the possibility to modify their own behaviour.

Due to the distributed nature of air traffic management, even advanced engineering design approaches have proven to fall short. It is quite relevant to know that in the Europe and the USA this has already led to the involvement of distributed control theorists and stochastic analysis in air traffic management studies.

Taking in account the above considerations about the features of the ATM systems, a further development which we make in this chapter is to enrich the GSHS model with two capabilities: compositionality and communication. The result is what we have called *distributed stochastic hybrid systems* (DSHS). In ATM , these models can illustrate architectures for handling conflicts between multiple aircraft, while maintaining the situational awareness of human operators.

4.2.1 Model Description

In this section we introduce the DSHS formalism. First we formally define its structure and after we give an algorithm which describes its executions.

Definition 15 *A DSHS, denoted by $\mathcal{DH}$, is a collection $((Q,d,m,\mathcal{X}),(f,\sigma),L,A,P)$ where*
(i) $(Q,d,m,\mathcal{X})$ describes the state space, which is countable union of open sets from an euclidean space (modes), each one corresponding to a discrete location. Note that the dimension of embedding euclidean space might be different for different locations.
(ii) (f,σ) gives the continuous dynamics between jumps of the continuous state within the locations.
(iii) L is the set of labels.
(iv) A are the set of active transitions. *These are the union of the forced (boundary-hit) transitions B and the spontaneous transitions S. The forced transitions B depend on the*

transition-choice function C. P *is the set of* passive transitions. *Passive transitions are* discrete *transitions defined in a similar manner as the forced transitions which take place in the communication process with the environment.*

Formally, the elements of (i) are defined as follows:

- Q is a countable set of locations.
- $d : Q \to \mathbb{N}$ is a map giving for each location the dimension of the continuous state space in that location.
- $m : Q \to \mathbb{N}$ is a map giving the dimension of the Weiner processes that govern the evolution of the continuous state.
- $\mathcal{X} : Q \to \mathbb{R}^{d(.)}$ maps each $q \in Q$ into an open subset $\mathcal{X}(q) = X^q$ of $\mathbb{R}^{d(q)}$; this means that for each $q \in Q$, X^q is the *mode* (the *invariant* set) associated to the location q. Let us denote by X the whole space, $X = \cup\{(q, X^q) | q \in Q\}$. We also define the boundary set $\partial X^q := \overline{X^q} \backslash X^q$ of X^q and the boundary of the whole space $\partial X = \cup\{(q, \partial X^q) | q \in Q\}$.

The continuous motion parameters from (ii) are given as follows:

- $f : X \to \mathbb{R}^{d(.)}$ is a vector field and $\sigma : X \to \mathbb{R}^{d(.) \times m(.)}$ is a $X^{(.)}$-valued matrix. For all $q \in Q$, the functions $f^q : X^q \to \mathbb{R}^{d(q)}$ and $\sigma^q : X^q \to \mathbb{R}^{d(q) \times m(q)}$ are bounded and Lipschitz continuous and the continuous motions is governed by the following stochastic differential equation (SDE):

$$dx(t) = f^q(x(t))dt + \sigma^q(x(t))dW_t \qquad \text{(SDE)}$$

  where $(W_t, t \geq 0)$ is an $m(q)$-dimensional standard Wiener process in a complete probability space.

The active transitions of (iii) are given as follows:

- B is the set of forced transitions. Each element $b \in B$ is a quadruple (q, l, q', R_b) where q is the origin mode (discrete state), l is the label of the jump, q' is the target mode, and R_b is the reset map of the jump, i.e. for each $x \in \partial X^q$ with $C(b, q, x) > 0$ (see next item)

and for all Borel sets E of $X^{q'}$ the quantity $R_b(x, E)$ is the probability to jump in the set E when the transition b is taken from the boundary state x.

- The function $C : B \times Q \times \partial X \to [0, 1]$ is defined such that for all $q \in Q$, all $x \in \partial X^q$, and all $b \in B$, which are outgoing transitions of q, the quantity $C(b, q, x)$ is the probability of executing a forced transition b. Moreover, $\sum_{b \in B_{q\rightarrow}} C(b, l, x) = 1$ where $B_{q\rightarrow}$ is the set of all elements of B that are outgoing transitions of q.

- S is the set of spontaneous transitions. Each element $s \in S$ is a pentuple (q, l, q', R_s, λ), where q is the origin mode, l is the label of the jump, q' is the target mode, R_s is the reset map of the jump, and λ_s is the jump rate (it determines the rate of process jumping).

The passive transitions of (iv) are given as follows:

- P is the set of passive transitions. Each element $p \in P$ is a quadruple (q, l, q', R_p), where q is the origin mode, l is the label of the jump, q' is the target mode, and R_p is the reset map of the jump.

A GSHS can be defined in a similar way as DSHS. The only difference is that the discrete transitions do not have labels and there is no choice function defined on the boundary.

Remark 2 *In the case of GSHS the parameters λ, R are globally defined. For DSHS these functions can be globally defined as following:*

1. $\boldsymbol{\lambda} : S \times X \to \mathbb{R}^+$ *is a transition rate function (a bounded Borel measurable function), i.e for all spontaneous transitions $s \in S$, with $s = q \rightsquigarrow q'$, the function $\boldsymbol{\lambda}(s, x)$ is nonzero iff $x \in X^q$ and is the transition rate associate to the transition s.*

2. $R : (B \cup S \cup P) \times \overline{X} \times \mathcal{B}(X) \to [0, 1]$ *is a transition measure such that, for all transitions $t \in B \cup S \cup P$ with $t = q \rightsquigarrow q'$ (where $q, q' \in Q$), we have that for each $x \in X^q$ and all Borel sets A of $X^{q'}$ the quantity $R(t, x, A)$ is the probability to jump in A when the transition t is taken from state x to A.*

Since a DSHS inherits the stochastic features of its associated GSHS, then it allows:

1. Diffusion processes in the continuous evolution (i.e. the continuous evolution is governed by some stochastic differential equations);

2. Spontaneous discrete transitions (according to a transition rate);

3. Forced transitions (driven by a boundary hitting time);

4. Probabilistic reset of the discrete and continuous state as a result of discrete transitions.

Moreover, for DSHS there exist passive transitions whose stochastic parameter is the reset map of the post jump location.

A realization of a DSHS given by the definition 15, generates a stochastic process. The above remark and the structure of a DSHS assure that this process is a realization of a GSHS. We will refer to this process as the associated GSHS to the given DSHS. In this context, a DSHS execution can be defined as a sample path of this stochastic process. For the generation of the DSHS executions we assume that no communication takes place, i.e. we consider that the passive transitions do not play any role in the generation of executions.

To eliminate pathological solutions that take an infinite number of discrete transitions in a finite amount of time (known as Zeno solutions) we impose the following assumption.

Assumption 7 *For each location $q \in Q$ the number of outgoing transitions is finite.*

Execution of a DSHS

We assume that an initial hybrid state (q_0, x_0) is given. The continuous dynamics in the mode X^{q_0} is determined by the stochastic differential equation (SDE) with q replaced by q_0. Let ω_{q_0} be an arbitrary diffusion sample path starting in x_0. Suppose that ω_{q_0} reaches the boundary ∂X^{q_0} at time τ_b and suppose that location q_0 has n_0 outgoing spontaneous transitions. During the continuous motion, for every outgoing spontaneous transition, can generate a hybrid jump. The probability density functions of these processes is equal to $\lambda_i(x_t(\omega_{q_0}))\exp(-\int_0^t \lambda_i(x_t(\omega_{q_0})))$, where λ_i is the jump rate of the i-th outgoing spontaneous transition s_i. Define

$$\tau_{q_0}(\omega_{q_0}) := \min_{i=1..n_0} \tau_i(\omega_{q_0})$$

where τ_i is the jump time corresponding to the i-th outgoing spontaneous transition s_i.

There are two possibilities:

1. If $\tau_b < \tau_{q_0}$, the boundary is hit before any spontaneous transition is about to be executed, which means that a forced transition is executed at time τ_b.

2. If $\tau_{q_0} < \tau_b$, the spontaneous transition corresponding to τ_{q_0} causes a jump at time τ_{q_0} before hitting the boundary.

Remark 3 *The above discussions show that the first jump time (corresponding to the diffusion path ω_{q_0}) is a minimum of $n_0 + 1$ stopping times (the first boundary hitting time and the stopping times given by the Poisson probability distributions corresponding to the n_0 outgoing spontaneous transitions from X^{q_0}).*

A *forced transition* at time τ_b from the boundary state $x_{\tau_b}(\omega_{q_0}) \in \partial X^{q_0}$ is executed as follows. It could be the case that multiple forced transitions are active in state $x_{\tau_b}(\omega_{q_0})$, therefore we use the choice function C to choose one of the active transitions. A transition $b \in B_{q_0\rightarrow}$ is taken according to the probability measure determined by $C(\cdot, q_0, x_{\tau_b})$. Then $b = (q_0, l_b, q_b', R_b)$ is the transition which takes place. The post-jump location is q_b' and the continuous state after the jump is $x' \in X^{q_b'}$ which is drawn according to the reset measure $R_b(\cdot, x_{\tau_b})$. From the new hybrid state (q_b', x') at time τ_b, the above recipe can be repeated to continue the execution.

A *spontaneous transitions* $s = (q_0, l_s, q_s', R_s)$ at time τ_{q_0} from the continuous state $x_{\tau_{q_0}}(\omega_{q_0}) \in X^{q_0}$ is taken as follows. The target location is q_s' is given according to the probability measure $R_s(x_{\tau_{q_0}}(\omega_{q_0}), \cdot)$. Starting with the new location (q_s', x') we can repeat the recipe given above to continue the execution.

The executions of the DSHS can be thought of as being generated by the following algorithm.

Algorithm. Generation of DSHS Executions.

set $T = 0$

select X-valued random variable $\widehat{x}$

repeat

set $q = \mathcal{X}^{-1}(\widehat{x})$, n_q the number of spontaneous transitions from X^q

set x_t as solution of (SDE) with initial condition equal to $\widehat{x}$

select ω_q a sample path for the process (x_t) with the start point $\widehat{x}$

select $\mathbb{R}^+$-valued random variable $\widehat{S}$ such that

$$\widehat{S}(\omega_q) = \min(\tau_1(\omega_q), ...\tau_{n_q}(\omega_q), \tau_b(\omega_q))$$

where τ_i is the jump time of the i-th outgoing spontaneous and

τ_b is the first hitting time of the boundary ∂X^q

select the transition $t = q \rightsquigarrow q'$ associated to $\widehat{S}$

select $X^{q'}$-valued random variable $\widehat{x}$ according to $R_t(., x_{\widehat{S}})$

set $T = T + \widehat{S}$

until true

In the structure of DSHS we can take some particular cases for the continuous evolution or for the discrete transitions. The results will be constituted in some particular distributed models for stochastic hybrid system. As a conjecture, we examplify:

1. If the continuous motion within the modes is deterministic then its associated stochastic process is a piecewise deterministic Markov process. The result is the notion of distributed piecewise deterministic Markov process;

2. If there is no forced transition then its associated stochastic process is a switching-jump diffusion. The result is the notion of distributed switching-jump diffusion.

3. If there is no spontaneous transition and the choice function is deterministic then its associated stochastic process is exactly the model proposed in [41]. Evidently, the result is a distributed variant of that model.

4.3 Parallelism and Communication of DSHS

In this section we introduce a composition operator || on the set of DSHS. In the DSHS framework, communication takes places by means of the passive transitions. The execution of an active transition is always independent on the environment. In a context with two composed

DSHS, one of DSHS can execute a passive transition with label l if and only if at the same time the other DSHS executes an active event with label l.

We have to point out that we introduce the communication and parallelism for GSHS in the same style it has been done for Piecewise Determisnistic Markov processes in the paper [154]. The main difference is that our model allows diffusions between jumps. The main problem which results from here is related by the parallel composition of diffusions. Mathematically, this leads to the concept of product of stochastic processes [49]. The infinitesimal generator of the product of two processes is not the product of component generators [28]!

Notations. We write $q \overset{l,R}{\mapsto} q'$, $q \overset{l,R}{\rightarrow} q'$, $q \overset{l,R,\lambda}{\rightsquigarrow} q'$ to denote the existence of respectively forced, passive and spontaneous transitions from q to q' with label l, reset map R and, in the case of spontaneous transition, jump rate λ.

Parallelism

The parallel composition of two DSHS is defined as follows:

Definition 16 *Let* $\mathcal{DH}_i = ((Q_1, d_i, m_i, \mathcal{X}_i, f_i, \sigma_i), L, B_i, C_i, S_i, P_i)$ *be two given DSHS for* $i = 1, 2$. *The parallel composition* $\mathcal{DH} = \mathcal{DH}_1 || \mathcal{DH}_2$ *is the collection* $((Q, d, m, \mathcal{X}, f, \sigma), L, B, C, S, P)$ *whose components are defined as following:*

1. $Q = Q_1 \times Q_2$;
2. $d : Q \rightarrow \mathbb{N}$ *such that* $d(q_1, q_2) = d_1(q_1) + d_2(q_2)$;
3. $m = m_1 + m_2$;
4. $\mathcal{X} : Q \rightarrow \mathbb{R}^{d(.)}$ *such that* $\mathcal{X}(q_1, q_2) = \mathcal{X}_1(q_1) \times \mathcal{X}_2(q_2)$;
5. $f^{(q_1,q_2)} = \begin{pmatrix} f^{q_1} \\ f^{q_2} \end{pmatrix}$ *and* $\sigma^{(q_1,q_2)} = \begin{pmatrix} \sigma^{q_1} \\ \sigma^{q_2} \end{pmatrix}$;
6. $b \in B$, $s \in S$ *and* $p \in P$ *if* b, s *and* p *can be derived from the rules r1 till r6 defined below, and r1' till r6' which are the mirrored versions of r1 till r6. These rules are derived in a classical process algebra style (see, for e.g. [96], for similar rules defined for Markov chains). The nominator represents the transitions of the components and the denominator gives the transition of the composed agent.*

$$r1.\ \frac{q_1 \overset{l,R_1}{\mapsto} q_1', q_2 \overset{l}{\nrightarrow}}{(q_1,q_2) \overset{l,R}{\mapsto} (q_1',q_2)} \qquad r2.\ \frac{q_1 \overset{l,R_1}{\mapsto} q_1', q_2 \overset{l,R_2}{\to} q_2'}{(q_1,q_2) \overset{l,R}{\mapsto} (q_1',q_2')}$$

$$r3.\ \frac{q_1 \overset{l,R_1,\lambda_1}{\rightsquigarrow} q_1', q_2 \overset{l}{\nrightarrow}}{(q_1,q_2) \overset{l,R,\lambda}{\rightsquigarrow} (q_1',q_2)} \qquad r4.\ \frac{q_1 \overset{l,R_1,\lambda_1}{\rightsquigarrow} q_1', q_2 \overset{l,R_2}{\to} q_2'}{(q_1,q_2) \overset{l,R,\lambda}{\rightsquigarrow} (q_1',q_2')}$$

$$r5.\ \frac{q_1 \overset{l,R_1}{\to} q_1', q_2 \overset{l}{\nrightarrow}}{(q_1,q_2) \overset{l,R}{\to} (q_1',q_2)} \qquad r6.\ \frac{q_1 \overset{l,R_1}{\to} q_1', q_2 \overset{l,R_2}{\to} q_2'}{(q_1,q_2) \overset{l,R}{\to} (q_1',q_2')}$$

7. *R, the reset map is given as the product measure, namely in case r1, r3, r5*

$$R((x_1,x_2),\cdot) = R_1(x_1,\cdot) \otimes 1_{x_2}$$

and in the case r2, r4, r6

$$R((x_1,x_2),\cdot) = R_1(x_1,\cdot) \otimes R_2(x_2,\cdot)$$

where $x_1 \in \partial X^{q_1}$ in case r1, r2, and $x_1 \in X^{q_1}$ (for the cases left) and $x_2 \in X^{q_1'}$;

8. *the transition map λ for cases r3 and r4 is given by*

$$\lambda(x_1,x_2) = \lambda_1(x_1)$$

for all $x_1 \in X^{q_1}$ and $x_2 \in X^{q_1'}$;

9. *the choice function C should be specified for any $q = (q_1,q_2) \in Q$ and any $b : (q_1,q_2) \mapsto (q_1',q_2') \in B$. It is clear that b has been derived from one of the following cases:*

$$\begin{aligned}
c1 &: \quad q_1 \overset{b_{11}}{\mapsto} q_1', q_2 \nrightarrow, q_2 = q_2', \\
c2 &: \quad q_1 \overset{b_{12}}{\mapsto} q_1', q_2 \to q_2', \\
c3 &: \quad q_2 \overset{b_{23}}{\mapsto} q_2', q_1 \nrightarrow, q_1 = q_1', \\
c4 &: \quad q_2 \overset{b_{24}}{\mapsto} q_2', q_1 \to q_1'
\end{aligned}$$

Then for all $(x_1,x_2) \in \partial\mathcal{X}(q)$ the function is defined as

$$C(b,a,(x_1,x_2)) = \begin{cases} C_1(b_{1i},q_1,x_1); & ci,\ i=1,2,\ x_1 \in \partial X^{q_1},\ x_2 \in X^{q_2} \\ C_2(b_{2i},q_2,x_2) & ci,\ i=3,4,\ x_1 \in X^{q_1},\ x_2 \in \partial X^{q_2} \\ \textit{undefined;} & x_1 \in \partial X^{q_1},\ x_2 \in \partial X^{q_2} \end{cases}$$

and C takes the zero value in rest.

Remark 4 *The idea to obtain the coefficients of the diffusion processes for the composed agent $\mathcal{DH}$ by superposition operation of coefficients of components is natural. It can be shown that under mild assumptions the product of two diffusion processes is again a diffusion process. For this kind of results the reader is referred to [28].*

Communication

The parallel composition of two DSHS can be defined by the following four possible cases:

- If one agent is able to execute an active transition and the other agent does not have a matching passive transition, then no communication takes place (the active agent executes the transition whilst the second agent stays in the same location, according to the rules r1, r1', r3, r3').

- If contrary, the first agent is able to execute an active transition and the second agent has a matching passive transition, then the communication takes place (both agents execute respectively the active and passive transition at the same time, according to the rules r2, r2', r4, r4').

- If the first agent has a passive transition with label l and the second agent has no passive transition with label l, then the composed system has a passive transition with label l outgoing from the joint location, which gives the possibility to interact with other systems, in another composition context (rules r5, r5').

- If both agents have a passive transition with the same label, then the composed system inherits the passive transition with this label. Then both agents can execute the passive transitions at the same time in another composition context where a third agent executes an active transition with the same label (rules r6, r6').

A state (x_1,x_2) is a *double boundary state* if both x_1 and x_2 are boundary points.

Remark 5 *[154] The choice function C is undefined for the double boundary states. In the composed system execution, these points are considered only when the two components reach their boundary at exactly the same time. This kind of situation is nondeterministic. Two forced transitions have to take place simultaneously in time. If the two transitions have different labels then a simultaneous execution of these transitions gives difficulties in the composition operation: If a third agent has both labels available in passive transitions, which passive transition should be chosen? As a theoretical assumption one might suppose that the probability that two separate agents reach their boundary at exactly the same time is almost zero. Then the case of double boundary states remains open and we say that the composed DSHS is undefined on the double boundary states.*

Remark 6 *Because the reset maps are not defined for the double boundary states, the forced transitions are not defined for these states.*

Remark 7 *The choice function C defined by the point (9) of the definition 16, being defined by using the choice functions C_1, C_2 of the components, meets the conditions imposed to the choice function from the definition 15 of DSHS.*

From the definition 16 and the remarks 6, 7, 4 it results that the parallel composition $\mathcal{DH}$ of the two DSHS $\mathcal{DH}_1$ and $\mathcal{DH}_2$ is a DSHS which is undefined on double boundary states. The proof can be derived from the similar result presented in [154].

4.4 Implementation in Charon

General models of stochastic systems are specified with enough accuracy in the classical language of continuous mathematics (topology, functional analysis, and differential equations). After adding concurrency the system dynamics become very complex, thus there is the danger of confusion. In this context system specification is becoming increasingly important. The tools of traditional control engineering specifications like block diagrams are not sufficient to deal with distributed systems. Modern control engineering specification is based on various extensions of the UML. A specific implementation of UML, considering extensions to support control and hybrid systems is the Charon system [7, 8].

In the following, we sketch how it is possible to implement our model in an extension of the modelling language Charon obtained in a similar manner as in [20].

First, we provide a very brief summary of Charon. Charon is a modelling language for hierarchical hybrid automata. In the Charon formalism the specifications are constructed as networks of communicating agents, internally modeled by hybrid automata with a hierarchical structure. Charon provides formal semantics which constitute an important aspect necessary for reasoning about the model. The language permits specification of architectural as well as behavioral hierarchy. Charon only allows communications through the shared variables.

The behavioral hierarchy of Charon is encoded as the hierarchy of modes. Each mode describes a continuous behavior and a single thread of discrete control. Each mode has its own constraints in terms of differential equations, algebraic equations, and invariants, just as a location in a simple hybrid automaton. As well, a mode in Charon model may also contain a set of submodes. At any given moment, at most one submode is active in an active mode. Transitions link a mode with its sibling modes, parent mode, and child modes. The constraints imposed by an active mode is a collection of constraints of the modes and all its active descendant modes.

The architectural hierarchy of Charon is implemented by agents. Each agent stands for a hybrid automaton. An agent may be compositional, in which case it contains several subagents, or atomic. Atomic agents are building blocks of architectural hierarchy, and a compositional agent functions as the composition of all its descendant atomic agents.

The language supports the operations of composition of agents to model concurrency, hiding of variables to restrict sharing of information, and instantiation of agents to support reuse.

The Charon capability of resource hiding is implemented by defining the scope of variables. At any level of hierarchy, a mode or an agent may specify the attributes of variables, it can asses as "read" meaning that a variable defined in a higher level can be read in the current agent or mode, "write", meaning that a variable defined in a higher level may be read and written to, and "private", meaning that a new variable is introduced. Resource hiding will help define the interface between the tester and the system by specifying what variables may or not be seen by the tester.

The definition of semantics of an agent is based on the definition of the semantics of modes.

Basically, for hybrid systems there are ways of acting: either some transition is taken, where variables are changed instantaneously, or some time passes, where the system resides in the same control location (modeled by a mode) and the continuous variables change according to their specified constraints (differential equations, algebraic equations, etc).

In order to make Charon suitable for implementation of DSHS, it is possible to extend the current version with syntax for specifying initial probabilities, jumps, and stochastic differential equations. In the terminology of Charon any DSHS is an agent.

The syntax for specifying an invariant is

```
inv <condition>
```

where condition can depend on the variables of the agent.

Specification of stochastic systems in Charon is facilitated by a number of predefined (Java) functions, that can generate, for example, the transition probabilities. These functions call the predefined distributions such as:

1. `randUniform(begin,end)` specifies uniform distribution on the interval `[begin,end]`;
2. `randPareto(parameter_a,parameter_b)` specifies Pareto distributions;
3. `randExp(parameter)` specifies exponential distribution;
4. `randNorm(mean,variance)` specifies normal distribution.

Therefore the syntax for specifying a jump can be defined as follows

```
jump from <source_mode> when <guard>
  (to <destination_mode> do {<update_cv>}
  weight <weight>)+
```

where the guard depends on the variables of the agent and defines a part of the complement of the invariant assigned to the source mode. The union of the guards of all jumps from a mode must be equivalent to the complement of the invariant of the mode. A jump has multiple transition branches. Each branch is specified by its destination mode, post jump location, and the weight assigned to it. The weight can depend on the variables of the agent. The post jump location `<update_cv>` is an assignment of the following form:

```
variable_name=f(...)
```

where f is a function specifying the distribution of the transition measure. Function f can depend on the variables of the agent. The post jump location specifies the probability measure

on the set of valuations of the destination mode. The function f is built using the predefined distributions.

The syntax for specifying an SDE is

SDE {d(<variable name>)==f(...)*dt+σ(...)*dW(t)}

where f(...) and σ(...) are functions which depend on the variables of the agent.

For each DSHS (agent) we can introduce some extra variables which are associated to the passive/active transition and capture the communication. A passive transition (with a label a) of an agent $\mathcal{DH}_1$ and an active transition with the same label of another agent $\mathcal{DH}_2$ must have associated the same communication variable. For the first agent this variable is an external variable (input) and for the second agent is an observable variable (output). In this way the communication defined in section 4.3 can be described in the Charon formalism.

4.5 Some Remarks

The next steps towards an extended DSHS model might be:

- Introduce active synchronization and guards for the active transitions.
- Investigate possible ways to define continuous dynamics interaction.

These extensions would make the DSHS model more expressive, however:

- Non-determinism is introduced by the guards and this means diverging from the GSHS model.
- The continuous interaction could cause non-trivial problems concerning the stochastic reset-maps.

The problems listed might find elegant solutions using the mappings between GSHS and Stochastically and Dynamically Coloured Petri Nets [74]. This approach does not make the subject of this thesis. For a detail presentation of this approach the reader is referred to [74] and the references therein.

Chapter 5

Hybrid Systems Verification

5.1 Overview

The verification problem consists in deciding whether some system satisfies some given invariance property describing the set of correct (safe) behaviors. In fact, safety requirements are the most important part of the specification of a hybrid system. Invariance properties are expressed as constraints on the values of the variables of the system.

It is well known that invariance properties are duals of the *reachability properties* asserting that some (dangerous) configuration is reachable. Hence, the verification problem of invariance properties reduces to reachability problem in the considered models of hybrid systems [31].

The safety criticality of many applications requires the use of formal methods to guarantee that an unsafe region of the state space is not reachable from a set of initial conditions. A reachability problem is to answer the following question: given a set of initial condition, will a target set be reached by some trajectories of a hybrid system?

There are two different approaches in hybrid system verification:

- *engineering methods*: simulations.
- *formal methods*: theorem proving and model checking.

Simulation is a procedure of generating partial traces by executing the model, and then checking the set of partial traces against its specification. This standard approach to verifying certain

properties of a hybrid system consists of finding an equivalent transition system called a *bisimulation* with a finite number of states. Bisimulations are reachability preserving quotient systems, in the sense that they preserve the closed properties of the original systems and can be used to reduce the complexity of verifying properties of very large systems. If a hybrid automaton has a finite state bisimulation, then checking properties for the hybrid automata can be equivalently performed on the finite, discrete, quotient graph. Since the quotient graph is finite, the algorithm will terminate. However, a finite state bisimulation exists only for certain classes of hybrid automata for which the reachability problem is decidable [121].

In this chapter we present a quick tour of the (classical) formal verification methods for (deterministic) hybrid systems.

5.2 Formal Verification Methods

Formal verification is a hot topic nowadays in the field of system engineering, specially for the development of critical dependable systems. The use of formal methods for specification and verification of properties of systems is one methodological improvement of the system production process, which together with other techniques, can make it possible to reach high quality standards: the use of formal methods is increasingly required by the international standards and guidelines for the development of safety critical systems.

Formal methods are mathematically-based techniques that can offer a rigorous and effective way to model, design and analyse computer systems, and they have been a topic of research for many years and the question now is whether these methods can be effectively used in industry. To achieve this aim, there is a clear need for tool support and for improved integration of formal method techniques with other software engineering practices.

Formal verifications methods and tools can be roughly classified into two categories, the so-called model-theoretical approaches and proof-theoretical ones.

In the proof theoretical approaches, the system state is modelled in terms of set-theoretical structures on which invariants are defined, while operations on the state are modelled by specifying their pre- and post-conditions in terms of the system state. Properties are described by invariants, which must be proved to hold through the system execution, by means of theorem

proving.

Model theoretical approaches give, on the other hand work on a finite state representation of the system behaviour. Verification is usually carried out by checking the satisfiability of some desired properties over the system model by checking algorithms or equivalence relations. In particular, safety requirements may be expressed as temporal logic formulae and may be checked on the model of the system.

Model theoretical approaches give a direct automatic verification method of system properties. Unfortunately, this approach suffers of the so called "State Space Explosion" problem: systems composed of several subsystems can be associated to a finite state model with a number of states which is exponential in the number of the component subsystems. Moreover, systems which are highly dependent on data values, share the same problem producing a number of states exponential in the number of data variables. Hence, traditional model checking techniques have shown themselves not powerful enough to cope with many real systems.

On the controverse, proof-theoretic approaches, which can exploit their generalization capability in order not to be affected by the state explosion problem, require in general more skill in the use of theorem proving tools, and, therefore, more investment, in terms of knowhow and training. This is because proofs usually need to be guided by humans and so the theorem proving process is not entirely automatic.

5.3 Formal Verification Methods for Hybrid Systems

Complex behaviors that can be exhibited by hybrid systems make the verification of such systems both important and challenging. Netherless, the ever-increasing presence of hybrid systems in safety critical applications makes it evident that verification is an issue that has to be addressed. Scalable automated methods for verification of hybrid systems are definitely in demand. From computer science, there exist comprehensive bodies of techniques for verifying temporal logic for discrete /hybrid/probabilistic systems; they fall into two mainstream approaches: model checking and deductive verification.

5.3.1 Model-checking and theorem proving

The most effective methods to verify systems from control engineering are model-checking and theorem proving.

Deductive verification verifies system properties through formal deduction based on a set of inference rules. Deductive verification is applicable to infinite state systems, but has a drawback in the sense that guidance from a user is almost always needed in the process. Theorem proving for control engineering has been introduced and developed by U. Martin and collaborators (see [30] for an overview). Essentially, specific control system properties are formalised in a kind of model logic, and the proving support is based on huge continuous mathematics formal theory libraries. This approach, although perfectly legitimated, encountered resistance from control engineers. The method requires a very specialised background, complex interdisciplinary collaborations and time-consuming activities. The ongoing project of Queen Mary University of London [45] might produce stronger arguments in favour of this method.

Model-checking is by far more effective and it has achieved a relatively wider acceptance in control engineering. In model-checking, a property is given as a formula of a propositional temporal logic and automatically compared with a state-transition graph representing the actual behavior of the system. One of the advantage of this method is its efficiency: model-checking is linear in the product of the size of the structure and the size of the formula, when the logic is the branching-time temporal logic CTL (computation tree logic) [57]. The model checkers can be entirely automated and now many of them can support system descriptions that contain plenty of continuous mathematics. Model checking is applicable usually to finite state systems, and basically performs an exhaustive exploration of all possible system behaviors in a fully automated way. The drawback of model checking is the state explosion problem, i.e., the number of system trajectories that need to be explored grows very quickly as the number of states increases, although the use of an efficient data structure called ordered binary decision diagrams [35] has allowed model checking of systems with an astronomical number of states. Still, when the number of possible states is infinite, such as when the state space is continuous, model checking is no longer applicable. The disadvantage of this approach is that only specialised situations can be checked and in many cases infinite state systems can not get finite abstractions. Indeed, the main difficulty of applying model checking to hybrid systems is caused by the

continuous part of their state space.

Symbolic model-checking procedure and its implementation HYTECH for linear hybrid automata have been developed using manipulating and simplifying $(\mathbb{R}, \leq, +)$-formulae [92, 6]. The underlying system model is a hybrid automata, an extension of finite automata with continuous variables that are governed by differential equations. The requirement specification language is the integrator computation tree logic ICTL, a branching-time logic with clocks and stop-watches for specifying timing constraints. Safety, liveliness, real-time, and duration requirements of hybrid systems can be specified in ICTL. Given a hybrid automaton describing a system and an ICTL describing a requirement, HYTECH computes the state predicate that characterizes the set of system states that satisfy the requirement.

The core of HYTECH is a symbolic model-checking procedure, whose primitive are pre, post, and boolean operations on regions. The original implementation of HYTECH represented regions as state predicates and manipulated regions by syntactic operations on formulas. The performance of HYTECH was improved by representing and manipulating regions geometrically: each data region is representing as a union of convex polyhedra. The current implementation of HYTECH consists of a MATHEMATICA main program and a collection of C++ subroutines that make use of a polyhedron-manipulation library by Halbwachs.

UPPAAL [90] is a tool suite for automatic verification of safety and boundness liveliness properties of real-time systems modeled as networks of timed automata. It includes: a graphical interface that supports graphical and textual representations of networks of timed automata, and automatic transformation from graphical representations to textual format, a compiler that transforms a certain class of linear hybrid systems to networks of timed automata, and a model-checker which is implemented based on constraint-solving techniques. UPPAAL also supports diagnostic model-checking providing diagnostic information in case verification of a particular real-time systems fails. The current version of UPPAAL is implemented in C++. UPPAAL allows linear hybrid automata where the speed of clocks is given by an interval. Hybrid automata of this form may be transformed into ordinary timed automata using a specific translator.

In the current version UPPAAL is able to check for reachability properties, in particular whether certain combinations of control-nodes and constraints on clocks and integer variables are reachable from an initial configuration.

5.3.2 Testing for Hybrid Systems

An alternative method to tackle the hybrid system validation is software testing. We must point out, from the beginning, that software testing for hybrid systems necessarily implies the use of a formal language to specify the system considered.

Hybrid systems can be considered a class of real-time reactive systems. The term *reactive* was introduced by Harel and Pnueli in [88] to designate systems that continuously interact with their environment. When the behavior of the reactive system depends on time, the system is called a *real-time reactive system.*

According to [135], test cases for reactive systems can not be simply described by means of a pre-state, test inputs and the expected state after the execution of program. Because of the continuous interaction between system and its environment, test cases must verify the correct causal relationships between inputs and outputs, the promptness to engage into specific actions, real-time requirements. In the reactive system context a *Test Execution* denotes a *Trace* observable at the interface between the test driver and the component under consideration.
A *test case for hybrid systems (or reactive systems, too)* consists of:

- *input data and causal execution conditions*: This describes initial inputs and consecutive inputs depending on the target system reactions.
- *timed execution conditions*: This describes time-dependent conditions necessary to stimulate a certain input.
- *(timed) expected results:* This describes the traces which are regarded as correct execution of the test case.

Some characteristics of the hybrid system testing [135] might be summarized as follows. Testing hybrid systems requires a combination of very different techniques for test generation, execution and evaluation because such systems have a specific construction. Usually, hybrid systems are built from components of the following types:

- sequential components;
- untimed concurrent components;

- discrete-time concurrent components;
- dense-time/ discrete value concurrent components;
- dense-time/ continuous value concurrent components.

In [135] test cases for hybrid systems are generated using Hybrid CSP specifications. This methods reduce to the theory for testing against timed or untimed CSP specifications.

It seems that testing reactive systems in a black-box fashion requires a model based specification language (which can specify the system state changes) as Z, Object-Z, VDM, Charon [7], Masaccio [95], Shift. Rigorous methods exist to generate test cases from Z, Object-Z, VDM [127], Larch. In the literature, in general, we can not find much work for rigorously testing timed systems. A recent example is [149]. This approach is based on timed input/output automata.

5.4 Other Verification Methods

From control theory, there exist also comprehensive bodies of techniques for verifying properties of continuous systems such as stability, performance, robust stability, robust performance, and so on [161]. These techniques are deductive in nature, since the systems considered have an infinite number of states. If the systems have a special structure (e.g., linear), then the verification can be automated. Unfortunately, the techniques are geared to verify properties that are expresses in terms of Lyapunov stability or signal/system norms, and as such are not directly applicable to verification of properties such as safety, reachability, let alone more general temporal logic formulas.

Chapter 6

Verification Methods for General Stochastic Hybrid Systems

6.1 Overview

Stochastic hybrid systems can exhibit very complex bahaviors, which makes their analysis both critical and challenging. Verification of SHS is an area where deductive formal methods, relying on mathematical inferences and proofs to produce exact statements about the system, are indispensable. Formal methods are also needed in system synthesis, particularly when correctness, robustness, and optimality are of paramount importance, which renders design by informal reasoning combined with trial and error ineffective.

Besides the more traditional properties such as stability and performance, properties of interest in (probabilistic or stochastic) hybrid systems also include safety and reachability. In principle, safety verification aims to show that starting from any initial condition in some prescribed set, a system can not evolve to some unsafe region in the state space. On the other hand, reachability verification aims to show that for some initial conditions in some prescribed set, the system will evolve to some target region in the sate space. The above properties are the most relevant when the system specifications are given in temporal logic formulas [108, 129]. These verification questions are no mean easy to answer, as for very simple classes of hybrid systems they are known to be undecidable already [94].

One of the most important goals of this thesis is to develop methods for verification of

stochastic hybrid systems. Since these systems are specified using complex tools of stochastic analysis, the first main step is to define the mathematical formulation of the verification problem in this context. Then we have to investigate mathematical properties, analytical solutions of this problem. The ultimate goal is to investigate if the formal verification methods can be further developed for SHS. This can be a long research track because SHS being multidimensional systems (continuous time/ state space, stochastic dimension, hybrid dimension) do not always admit finite state abstractions.

In the next section we briefly encounter different formal verification methods for different probabilistic systems developed in the literature. Then we define the reachability problem for SHS and develop some possible mathematical approaches based on stochastic analysis.

6.2 Formal Verification Methods for Probabilistic Systems

There is an increasing awareness that unexpected behavior from interacting processes cause serious problems.. This observation applies not only to programs and digital systems, but also to physical processes, such as robots, automobiles, manufacturing processes, and so on.

Formal verification of these systems seeks mathematical methods for reasoning about their behavior. Automatic formal verification is particularly promising, because it requires far less labor than the manual techniques.

Physical systems work in real time and behave probabilistically, so we need formal methods for analyzing real-time stochastic systems.

In the last fifteen-twenty years, model-checking has been extended to models of real-time and probabilistic systems. Model-checking algorithms have been devised for several different models of real time [4, 2], for discrete-time Markov chains (DTMC) [136, 59], for semi-Markov Processes [3]. In [87], it is introduced a branching-time logic PCTL for DTMC and several model checking algorithms are presented for this logic.

More recently, model-checking techniques have been extended to stochastic processes as continuous-time Markov chains (CTMC). In particular, efficient verification algorithms have been developed for CSL (Continuous Stochastic Logic [13, 14, 15]), a stochastic variant of CTL. CSL supports the specification of sophisticated steady-state and time dependent properties.

The initial proposal for a temporal logic for specifying performance properties over CTMC is given in [14]. In [14], it is shown that the model checking problem for this logic is decidable for rationale time bounds. The first full proposal for CSL, its formal semantics, and Volterra fixpoint characterization for the probabilistic time-bounded until operator are presented in [15]. The paper [16] describes how model checking of probabilistic timed-bounded until-formulas can be done using transformations of the Markov chains, yielding a standard transient analysis problem. Besides it is shown that lumping preserves the validity of all CSL formulas. CTMC are widely used in practice, mainly because they combine a reasonable modelling flexibility with well-established efficient analysis techniques for transient and steady-state probabilities. The stochastic processes described by CTMC are characterised by the fact that the state holding times, indicating the amount of time the system has spent in a state, are restricted to negative exponential distributions. As a result of their memoryless property, the probability of moving from one state to another is independent of the amount of time the system has spent in the current state.

6.3 Stochastic Reachability

Stochastic analysis tools have been proposed [40, 41] as an alternative verification method for situations when uncertain/probabilistic systems do not admit finite state abstractions. Although computer science communities do prefer understandably, probabilistic models, many more complex stochastic models record a wide spread use in different branches of engineering. The most important verification method for these models is reachability analysis. The stochastic features of these models have to be captured in the formulation of the reachability problem. This leads to the concept of *stochastic reachability.*

6.3.1 Motivation

Understanding the reachability problem is one of the most important objectives in the context of a stochastic hybrid modelling.

In general terms, a reachability problem consists of determining if a given system trajectory will eventually enter a prespecified set starting from some initial state. For deterministic hy-

brid systems, the reachability analysis refers to the problem of computing bounds on the set of states that can be reached by system trajectories. Reachability analysis is relevant to a variety of control applications. The available reachability tools consider various uncertainty. In this context 'uncertainty' is synonym with nondeterminism. descriptions such as differential inclusions [140, 158], polygonal approximations [56], ellipsoidal approximations [119], and general nonlinear systems with set disturbances [124].

In many control applications, the dynamics of the system under study is subjected to the perturbation of random noises that are either inherent or present in the environment. Typically, a certain part of the state space is "unsafe" and the control input to the system has to be chosen so as to keep the state away from it, despite the presence of the random noises. This can happen in many safety-critical situations. Therefore, in these applications it is very important to have some measure of criticality for evaluating whether the selected control input is appropriate or a corrective action should be taken to timely steer the system out of the unsafe set. A natural choice for the measure of criticality is the probability of intrusion into the unsafe set within a finite/infinite time horizon: the higher the intrusion probability, the more critical is the situation. Within the ATM context, safety critical situations arise during the flight when an aircraft comes closer than a minimum allowed distance to another aircraft or enters a forbidden region of the airspace. In the current ATM system, air traffic controllers are in charge of guaranteeing safety by issuing to pilots correcting actions on their flight plans when a safety critical situation is predicted. The limit to the air traffic system capacity due to its human-operated structure can be pushed forward by introducing automatic tools for supporting air traffic controllers in detecting safety critical situations.

One of the most important goals of our work [40, 41, 137] was to develop formal mathematical models for the *safety critical air traffic management situations.* A central problem in air traffic control is determining the *collision probability* (rare events [18]), i.e. the probability two aircraft come closer than a minimum allowed distance. If this probability can be computed, an alert can be issued when it exceeds a certain threshold.

In the context of stochastic hybrid systems, the computation of the conflict probability reduces to a *reachability problem*: *computing the probability that the stochastic hybrid process*

modelling the aircraft motion reaches an unsafe part of the state space (where two aircraft come closer than the minimum allowed distance). This is the key approach in new air traffic control philosophies like the *free flight concept* [103]. Free flight - sometimes referred to as *self separation assurance* [103] - is a concept where pilots are allowed to select their trajectory freely at real time, at the cost of acquiring responsibility for conflict prevention. It changes air traffic management in such a fundamental way, that one could speak of a paradigm shift: the centralised control becomes a distributed one, responsibilities transfer from ground to air, air traffic sectorization and routes are removed and new technologies are brought in. In the RTCA Free Flight Task Force, Free Flight is presented as a range of concepts, allowing self optimization of the routes by the airlines. The document also describes a mechanism for airborne separation as a part of the Free Flight concept. The basic assumption is that the collision probability is getting smaller as the sky is getting less crowded. Safety analysis of the Free Flight is essentially stochastic. The Achilles heel is then that reachability analysis needs to be stochastic.

GSHS could be used as the aircraft dynamics model, which incorporates the information on the aircraft flight plan, and takes into account the presence of wind as the main source of uncertainty on the aircraft actual motion. Therefore, the problem of estimation of the probability that the aircraft enters in an "unsafe" set can be formally specified as the reachability problem for GSHS.

Developing a methodology for the reachability analysis of SHS will involve dealing with two aspects:

1. the theoretical aspect of the measurability of the reachability sets (the set of trajectories which enter a prespecified set of state space);

2. the computational aspect regarding how to estimate the probability of the reachable events and how to quantify the level of approximation introduced.

6.3.2 Mathematical Formulation

The Markov property holds for almost all hybrid system models discussed in this thesis, in particular for GSHS (see section 3.2). Because of this, we formulate the reachability problem in the general setting of Markov processes. Assume that we have a given strong Markov process,

(x_t). We further assume that (x_t) has the state space $(X, \mathcal{B})$ (where X is a Borel and $\mathcal{B}$ is the Borel σ-algebra of X) and it has the càdlàg property.

Let E a Borel set of the state space X, i.e. $E \in \mathcal{B}(X)$. In this context the trajectories of the system can be identified with the elementary events of the underlying probability space Ω. Thus the reachable "event" associated to E (i.e. the set of trajectories that reach E within a finite/infinite time horizon) can be defined as follows:

$$Reach_T(E) = \{\omega \in \Omega \mid \exists t \in [0,T] : x_t(\omega) \in E\} \tag{6.1}$$

$$Reach_\infty(E) = \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in E\} \tag{6.2}$$

(keep in mind that an element $\omega \in \Omega$ is, in fact, a trajectory of the system).

Problem 1: We would like to have some information about the measure $P[R_T(E)]$ and $P[R_\infty(E)]$ of these sets in the underlying probability space. But, for this, we should know if we can apply the measure P to the sets defined by (6.1) and (6.2). Thus the first problem can be formulated as follows: Are $R_T(E)$ and $R_\infty(E)$ really events?

Remark 8 *Note that if $R_T(E)$ is an event then $R_\infty(E)$ is also an event, since $R_\infty(E) = \bigcup_{n=0}^{\infty} R_n$.*

Problem 2: If it turns out that we can assign a probability to $R_T(E)$ and $R_\infty(E)$. Can we compute this probabilities?

The measurability of the reachable events can be proved employing some standard techniques for Markov processes using the properties of the so-called analytic sets [46]. The method is very general and it works for all stochastic hybrid system models provided that they are Markov processes with càdlàg property.

6.3.3 Estimation of reach set probabilities. Possible Approaches

We encounter, in the following, some different approaches used to estimate the probabilities to reach a Borel set belonging to the state space.

Via Markov process theory

A very first approach used to estimate the reach set probabilities is based on *hitting times and hitting time moment approximations.* This approach has been investigated in [40]. The main advantage of this approach is the fact that many analytical results for exit (hitting) times are known. The method, proposed in [40], allows to estimate the number of steps after a certain set is reached and is based on a very old result (of Dynkin) about hitting times. The difficulty of this method is that, in many cases, the computation of the hitting time expectations is quite complicated.

Via Potential theory

This method employs potential theory notions such as*: excessive functions, Choquet capacities [54] and potentials.*

The reach set probabilities can be related to the so-called *hitting probabilities* associated to a Borel set E (which are, roughly speaking, probabilities to hit E). Then, it is known that the hitting probabilities can be treated as excessive functions for the corresponding Markov process [25]. Therefore, we can invoke approximation results for the excessive functions in order to obtain approximations of the hitting probabilities.

Another direction is to write the reach set probability associated to E as a *Choquet capacity* (in the way described in[126]), which is, roughly speaking, a nonlinear extension of a measure. Then we can employ approximation results for capacities. The main advantage is that the analytical and numerical theories of capacities are very well-developed, richer than the similar theories for hitting times, so this approach could be better than the first one. But again, because of the overlapping between the hitting time theory and capacity theory; we expect the same big computational effort in the approximations of these capacities.

Another very interesting method is based on the so-called *Dirichlet forms* (that are the quadratic forms, defined in section 2.2.4, which satisfy some supplementary axioms [126]) and it was investigated by the author in [41]. This approach shows how bounds on the measure of the reach event can be computed using the infinitesimal generator of the process and the corresponding Dirichlet forms. It has already been proved in the literature that Dirichlet forms

constitute a powerful tool for studying Markov processes (see, for example, [1, 126] and the references therein). Dirichlet form techniques have found striking applications in the study of stochastic partial differential equations [1, 29]. This is mainly due to the fact that they allow to develop a highly nontrivial stochastic analysis under some minimal regularity hypothesis, for instance, on very irregular spaces without differentiable structure like fractals, or on infinite dimensional spaces like path spaces or spaces of measures.

For Dirichlet forms, a lot of work was carried out on axiomatizations and representation results. This provides a mathematical vehicle for zooming in and out at different levels of abstraction in a consistent way. For example, in the most abstract view, the Dirichlet forms can be seen as mixing a linear space structure with a partial order structure, by providing simple compatibility axioms. In more concrete applications, a Dirichlet form defines a logical type of functions with an inner product given explicitly by a logical expression. The advantage of Dirichlet forms which derives from this is that they might be easily implemented. There are two main streams: one is symbolic (like using a model-checker or a theorem prover or their combination like PVS) and another one is numerical [52].

For the reachability problem the symbolic approach has been intensively applied (see e.g. the papers in [128]), especially because the accessible states can be generated. In the case of PVS, we can link these techniques with the huge mathematical libraries made available by the theorem provers.

Via Optimal Control

This method consists of drawing parallels between the computations based on the extended generator of the stochastic hybrid system model (as Markov process) and the Hamilton-Jacobi equations [60]. It involves the following steps:

- *Start point*: for any Borel set E belonging to the state space, take

$$\begin{aligned} P[Reach_T(E)] &= \mathbb{E}[\max_{t\in[0,T]} I_E(x_t)] \\ P[Reach_\infty(E)] &= \mathbb{E}[\max_{t\geq 0} I_E(x_t)]. \end{aligned}$$

- $Reach_T(E)$, $Reach_\infty(E)$ measurable$\rightarrow \mathbb{E}[\max_{t\in[0,T]} I_E(x_t)], \mathbb{E}[\max_{t\geq 0} I_E(x_t)]$ well-defined.

- The possibility to characterize the previous probabilities as viscosity solutions [60] to a partial differential equation.

6.4 Reachability Estimation via Hitting Times

In this section, reachability questions are characterized as hitting time problems. We extend our approach for Piecewise Deterministic Markov Processes from [40] to GSHS. This is possible since in Chapter 3 we have studied the main properties of this model and we have obtained the expression of the generator.

Let H be a GSHS. Let us consider a fixed starting point $x_0 \in X$ and the sequence $T_1 < T_2 < ...T_k < ...$ of jump times associated with x_0. Let E a Borel subset of X. Let L be the generator of the process H, given by formula (3.28), and let $\mathcal{D}(L)$ denote the domain of the generator. It is known that the process $(C_t^u)_{t\in\mathbb{R}^+}$ defined by

$$C_t^u = u(x_t) - u(x_0) - \int_0^t Lu(x_s)ds$$

is a martingale for each u in the domain $\mathcal{D}(L)$. This fact implies that for each $t > 0$ we have

$$\mathbf{E}[u(x_{t\wedge T_E}) - \mathbf{E}[u(x_0)] - \mathbf{E}[\int_0^{t\wedge T_E} Lu(x_s)ds] = 0$$

Since, we have supposed that $T_E < \infty$ almost sure with respect to P, letting $t \rightarrow \infty$ gives

$$\mathbf{E}[u(x_{T_E}) - \mathbf{E}[u(x_0)] - \mathbf{E}[\int_0^{T_E} Lu(x_s)ds] = 0 \tag{6.3}$$

We define the *occupation measure* μ_0 and *hitting distribution* μ_1 by

$$\begin{aligned} \mu_0(B) &= \mathbf{E}[\int_0^{T_E} I_B(x_s)ds] \\ \mu_1(B) &= P(x_{T_E} \in B) \end{aligned}$$

for all $B \in \mathcal{B}(X)$, where I_B is the indicator function of B.

It is clear that if $B \subset E$ then $\mu_0(B)$ is zero, and if $B \subset E^c = X \backslash E$ then $\mu_1(B)$ is zero. This means that μ_0 is concentrated in E^c and μ_1 is concentrated in E. With an integrability argument one can obtain from formula (6.3) the so-called *adjoint equation*, as follows:

$$\int_{E^c} Lu(x)\mu_0(dx) + u(x_0) - \int_E u(x)\mu_1(dx) = 0. \tag{6.4}$$

It is clear that

$$\mu_0(E^c) = \mathbf{E}(T_E) \text{ and } \mu_1(E^c) = 0. \tag{6.5}$$

We consider (x_t^*) the process (x_t) 'killed' when it enters to E. Then the adjoint equation (6.4) becomes

$$\int_{E^c} L^* u(x)\mu_0(dx) + u(x_0) = 0$$

if we suppose that $x_0 \notin E$. Note that L^* has a similar expression with L, i.e.

$$L^* f(x) = L_{cont} f(x) + \lambda(x) \int_{\overline{E^c}} (f(y) - f(x)) R(x, dy)$$

where $L_{cont} f$ is given by (3.29). If we take $u = I_{E^c} \in \mathcal{D}(L^*)$ then the adjoint equation becomes

$$\int_{E^c} L^* I_{E^c}(x)\mu_0(dx) + 1 = 0$$

or

$$\int_{E^c} \lambda(x) \int_{\overline{E^c}} (I_{E^c}(y) - I_{E^c}(x)) R(x, dy)\mu_0(dx) + u(x_0) = 0 \tag{6.6}$$

If $y \in \overline{E^c} - E^c$ then $I_{E^c}(y) = 0$ then (6.6) becomes

$$\int_{E^c} \lambda(x) R(x, \overline{E^c} - E^c)\mu_0(dx) + u(x_0) = 0$$

Thus, once the measure μ_0 has been determined (using for e.g. linear programming methods), the hitting time mean can be obtained from (6.5).

The following lemma is extremely useful to estimate the expectations of the moments of hitting times.

Lemma 17 *[68] Let E be a Borel subset of X, $T = T_E$ and let $m(t) = \sup\{P_x[T > t], x \in X\}$. Then, for all $t \geq 0, x \in X$ we have $\mathbb{E}_x T \leq \frac{t}{1-m(t)}$. Moreover, if $m < 1$, for $0 < p < -\frac{1}{t}\ln m$ then $\mathbb{E}_x e^{pT}$ is an analytic function of p in a neighborhood of the origin, i.e. we have $\mathbb{E}_x e^{pT} = \sum_{k=1}^{\infty} \frac{p^k}{k!}\mathbb{E}_x T^k$.*

If we suppose that T_E is finite then there exists T_k such that $T_E \in [T_k, T_{k+1})$. Using lemma 17 and the survivor function execution of a GSHS leads to the following estimation for all $k \geq 1$

$$\mathbb{E}_{x_0} T_E \leq \frac{T_{k+1}}{1 - P_{x_{T_k}}[T_E > T_{k+1}]} \leq \frac{T_{k+1}}{\exp(-\int_{T_k}^{T_E} \lambda(\phi(x_{T_k}, \tau))d\tau)}$$

Then we compute the quantities $\tau_k = \frac{T_k}{\mathbb{E}_{x_0} T_E}$, $k = 0, 1, 2, 3, ...$ (with the convention $T_0 = 0$) and let τ_{k_0} be the biggest one. Then $T_E \in [T_{k_0}, T_{k_0+1})$, i.e. the number of steps after the set E is reached is k_0.

6.5 Reachability Estimation via Quadratic Forms

This section is developed using our paper [41].

The basic idea of the reachability method proposed here is to employ the characterization of the strong Markov processes based on the associated quadratic forms, called *Dirichlet forms*, defined using the process generator (see Section 2.2.4). A Dirichlet form makes possible the use of the operator theory for a given Markov process. Also, it comes with a nonlinear extension of a measure, called *capacity.* The capacity associated with a Dirichlet form can be expressed in terms of the hitting times of the corresponding Markov process [126]. We investigate the possible benefits of applying a Dirichlet form based method to study the reachability problem of GSHS.

Let us briefly explain the methodology to obtain upper bounds for reach set probabilities. First we suppose that the target sets in the state space of H are given as level sets. Then we start with a Borel set E given by a nice function F. Markovian properties of the GSHS model H allow us to define the corresponding quadratic form $\mathcal{E}$. This form is a special kind of Dirichlet form (i.e. it satisfies the axioms of the Dirichlet form definition since H is a Borel right process).

We do not give in this thesis the background for Dirichlet forms. For this, the reader is referred to [126]. We consider that for the reachability problem studied in this section is not relevant to present the axioms which define a Dirichlet form. Intuitively, a Dirichlet form is a quadratic form with some Markovian properties. What is very important for us is that we can obtain results on the estimation of reachability probabilities using properties of the capacity associated to a Dirichlet form.

Secondly, using the function F we define the *induced process* $F(x_t)$, which is a stochastic process with values in $\mathbb{R}$, and the induced Dirichlet form $\mathcal{E}^*$ (see formula (6.9) below). The form $\mathcal{E}^*$ is associated with $F(x_t)$ if the induced process is Markov.

The mechanism used to obtain the upper bounds for the reach set probabilities is based on the inequality which exists between the capacity of the initial Dirichlet form $\mathcal{E}$ and the capacity corresponding to the induced Dirichlet form $\mathcal{E}^*$ (the inequality (6.13) below). The relation existing between hitting times and capacities allows us to get the result we are interested in, namely, the formula (6.14).

6.5.1 Target Sets

Usually a target set E in the state space is a *level set* for a given function $F : X \to \mathbb{R}$, i.e.

$$E = \{x \in X | F(x) > l\};$$

(F can be chosen as the Euclidean norm or as the distance to the boundary of E). The probability of the set of trajectories which hit E until time horizon $T > 0$ can be expressed as

$$P\{\sup_{t\in[0,T]} F(x_t) > l\}. \tag{6.7}$$

6.5.2 Dirichlet Forms

The strong Markov property of GSHS allows us to define an associated quadratic form, as follows.

Let H be a GSHS. Let $D(L)$ be the domain (not the extended domain) of its infinitesimal generator L. Using the Lebesgue measure λ^i on $\mathbb{R}^{d(i)}$, we define a new measure m on the state

such that for each $i \in Q$ the projection of m to each mode X^i is exactly the restriction of the Lebesgue measure λ^i to that mode, $\lambda^i|_{X^i}$, i.e.

$$m(\{i\} \times A) := \lambda^i(A),\ A \in \mathcal{B}(\mathbb{R}^{d(i)}).$$

Let m^* the image of m through the map F.

Remark 9 *Since any GSHS is a Borel right process, the process semigroup (P_t), defined by (2.3), may be viewed as a strongly continuous semigroup of operators on $L^2(X,m)$ [126]. Its generator is defined by the same limit*

$$\lim_{t \searrow 0} \frac{1}{t}(P_t f - f) \tag{6.8}$$

with respect to the norm of $L^2(X,m)$. The domain of the generator consists of those $f \in L^2(X,m)$ for which the limit (6.8) exists in the norm of $L^2(X,m)$.

Remark 10 *[126] Under the standard assumptions of section 3.2.2, there exists a* quasi-regular Dirichlet form[1] *$(\mathcal{E}, D[\mathcal{E}])$ on $L^2(X,m)$ associated with the process (x_t), given by*

$$\begin{cases} D(L) \subset D[\mathcal{E}] \\ \mathcal{E}(u,v) = (-Lu, v),\ u \in D(L),\ v \in D[\mathcal{E}]. \end{cases}$$

We can think of a Dirichlet form $\mathcal{E}$ as a recipe for a Markov process $(x_t)_{t \geq 0}$, in the sense that $\mathcal{E}$ describes the behavior of the composed process $u(x_t)$ for every u in the domain of $\mathcal{E}$. There is no guarantee that the 'coordinates' $(u(x_t))_u$ can be put together in a consistent way to form a process with reasonable sample paths.

6.5.3 Induced Dirichlet Forms

Let us denote the sub-σ-algebra of $\mathcal{B}$, generated by $\sigma(F)$, and the projection operator from $L^2(X,\mathcal{B},m)$ to $L^2(X,\sigma(F),m)$ by $\mathcal{F}.\mathcal{F} \equiv \mathbb{E}_m[\cdot|F]$ in case m is a probability measure.
More precisely, the function F induces a form $\mathcal{E}^*$ on $L^2(\mathbb{R},m^*)$ by

$$\mathcal{E}^*(u^*,v^*) = \mathcal{E}(u^* \circ F, v^* \circ F);\ u^*, v^* \in D[\mathcal{E}^*] \tag{6.9}$$

[1] *See the definition 3.1 from [126]*

where

$$D[\mathcal{E}^*] = \{u^* \in L^2(\mathbb{R}, m^*) | u^* \circ F \in D[\mathcal{E}]\}.$$

Proposition 18 *[112] If $\mathcal{F}(\mathcal{D}) \subset D[\mathcal{E}]$ where $\mathcal{D}$ is some L^2-dense subset of $D[\mathcal{E}]$, then $\mathcal{E}^*$ is a Dirichlet form on $L^2(\mathbb{R}, m^*)$.*

Assumption 8 *Suppose that the Dirichlet form $\mathcal{E}^*$ is quasi-regular*[2].

In [112], it is shown that, under a mild condition on the function F, the assumption 8 can be accomplished. This assumption ensures that there exists a right Markov process, (x_t^*), with the state space $\mathbb{R}$, associated with the Dirichlet form $\mathcal{E}^*$ [1]. If $F(x_t)$ happens to be Markovian then $\mathcal{E}^*$ is its associated Dirichlet form (see [141], for conditions on F which imply the Markov property of $F(x_t)$).

Assumption 9 *Suppose that the Dirichlet forms $\mathcal{E}$, $\mathcal{E}^*$ are symmetric*[3]

$$\mathcal{E}(u,v) = \mathcal{E}(v,u),\ u,v \in D[\mathcal{E}];$$

$$\mathcal{E}^*(u^*,v^*) = \mathcal{E}^*(v^*,u^*),\ u^*,v^* \in D[\mathcal{E}^*].$$

Assumption 9 is not restrictive (any result valid for regular Dirichlet forms and invariant under quasi-homeomorphisms is applicable to quasi-regular Dirichlet forms [53]).

Each (quasi-regular) symmetric Dirichlet form can be expressed as the sum of its parts: continuous, jumping and killing corresponding to the same parts of the Markov process considered. Precisely, a regular Dirichlet form $\mathcal{E}$ can be decomposed using *the Beurling-Deny representation* [76]:

$$\begin{aligned} \mathcal{E}(u,v) &= \mathcal{E}_c(u,v) + \int_{X\times X\setminus d} [u(x)-u(y)][v(x)-v(y)]J(dx,dy) \quad (6.10) \\ + \int_X u(x)v(x)k(dx),\ u,v &\in D[\mathcal{E}] \cap C_0(X). \end{aligned}$$

Here $\mathcal{E}_c$ is a symmetric form with domain $D[\mathcal{E}_c] = D[\mathcal{E}]$ which satisfies the property

[2] *See the definition 3.1 from [126]*

[3] *See [76, 126] for the theory of symmetric and non-symmetric Dirichlet forms.*

$\mathcal{E}_c(u,v) = 0$ if $u, v \in D[\mathcal{E}]$ have support compact and v is constant on a neighbourhood of $supp[u]$

J is a symmetric positive measure on $X \times X \backslash d$, d being the diagonal; and k is a positive measure on X. The form $\mathcal{E}_c$ and measures J and k are uniquely determined by $\mathcal{E}$; $\mathcal{E}_c$ is called the diffusion part of $\mathcal{E}$, and J and k are called the *jump* measure and the *killing* measure, respectively, associated with $\mathcal{E}$.

Note if E^* is open in $\mathbb{R}$ and $E = F^{-1}(E^*)$ then we can define for $p > 0$, the *p-capacity* of E

$$Cap_p(E) = \inf\{\mathcal{E}(u,u) + p(u,u)_m | u \in D[\mathcal{E}],\ u \geq 1\ m-a.e.\ \text{on}\ E\} \tag{6.11}$$

where $(\cdot,\cdot)_m$ is the inner product of $L^2(X,m)$ and the *p-capacity* of E^*

$$Cap_p^*(E^*) = \inf\{\mathcal{E}^*(u^*,u^*) + p(u^*,u^*)_{m^*} | u^* \in D[\mathcal{E}^*],\ u^* \geq 1\ m^*-a.e.\ \text{on}\ E^*\} \tag{6.12}$$

where $(\cdot,\cdot)_{m^*}$ is the inner product of $L^2(\mathbb{R}, m^*)$.

Proposition 19 *[112] Under the assumptions 8 and 9, if E^* is open and $E = F^{-1}(E^*)$ then*

$$Cap_p(E) \leq Cap_p^*(E^*). \tag{6.13}$$

We can consider the two first hitting times T_E (with respect to (x_t)) and T_{E^*} (with respect to (x_t^*)). Intuitively, the capacity (7.4.1) (resp. (6.12)) is the Laplace transform of the hitting time T_E (resp. T_{E^*}) of the target set (resp. of the 'induced' target set).

6.5.4 Upper Bounds for Reach Set Probabilities

An upper estimation for the probability (6.7) will be given in terms of the Dirichlet form induced by F on $\mathbb{R}$. This form corresponds to the process $F(x_t)$.

Assumption 10 *Assume that $m(X) < \infty$, $1 \in D[\mathcal{E}]$ and $k(X) < \infty$, where the killing measure k is described in the Beurling-Deny representation (6.10).*

The translation of the capacitary inequality (6.13) into probabilistic terms for the right Markov processes (x_t) and (x_t^*) associated with $\mathcal{E}$ and $\mathcal{E}^*$ gives rise to the following result:

Proposition 20 *[112] Under the assumption 10, if $E^* \subset \mathbb{R}$ is an open set of finite Cap^*-capacity and $E = F^{-1}(E)$ then for all $p > 0$,*

$$P_m(T_E \leq T) \leq e^p \{E_{m^*} e^{-pT_{E^*}/T} + Tp^{-1} \int_{\mathbb{R}} E_r e^{-pT_{E^*}/T} k^*(dr)\}.$$

where k^* is the killing measure associated with the killing part of $\mathcal{E}^*$. Also

$$P_m(T_E \leq T) \leq p^{-1} e^p \min\{T\mathcal{E}^*(u^*, u^*) + p(u^*, u^*)_{m^*} | u^* \in D[\mathcal{E}^*],\ u^* \geq 1,\ m^* - a.e.\ on\ E^*\} \tag{6.14}$$

One might, for instance, use the small induced processes rather then the huge original process to deal with the reachability problem. The induced Dirichlet form capacity (of $E^* = (l, \infty)$) plays an essential role in obtaining the reach event probability estimation. If the model H is discretized then the induced process is a one-dimensional jump process and therefore the computation of Laplace transform and the mean level-crossing time is feasible. It is interesting to note that the capacity of the target set is subadditive. So even if the target set were very complex, then the capacity of target set is at most the sum of capacities of its parts.

6.6 Some Remarks

In this chapter we have presented some different analytical approaches used to obtain estimations reach set probabilities. These approaches we have investigated in our papers [40, 41]. By far, it is clear that the theoretical results we have obtained are very nice from the mathematical point of view, but, to make them practically useful we have to deal only with quite simple models of stochastic hybrid systems. Then, naturally, our next research investigation was directed towards methods for abstractions of SHS models. To achieve this goal the next step is to define the concept of bisimulation for SHS (see the following chapter).

On the other hand, the method based on Dirichlet forms to obtain estimations for reach set probabilities gives us the idea to define a coarser version of bisimulation for SHS which preserves these probabilities with respect to some certain sets of their state spaces. Intuitively, two SHS will be bisimilar with respect to some target sets if the induced Dirichlet forms are 'equivalent'. We have not investigated yet this idea, but we believe this might be worthy since

in practice we want to be ensured that, in the abstraction process, we preserve the reach set probabilities for the unsafe sets. Then this bisimulation concept will not imply anymore the equivalence of the initial processes, but the equivalence of the induced process (which are much more simpler since they take real values).

Chapter 7

Bisimulation for Stochastic Hybrid Systems

7.1 Overview

Significant progress in verification of probabilistic systems has been done mostly for discrete distributions or Markov chains. Continuous stochastic processes are notably more difficult to verify. It is notorious that theorem proving of stochastic properties (with the probability one) can be carried out on the unit circle only. Model checking and reachability analysis are strongly conditioned by abstraction techniques. When the state space is not only infinite but also continuous, abstraction techniques must be very strong. Hybrid systems add an extra level of complexity because of the hybrid nature of the state space (discrete and continuous states coexist) and stochastic hybrid systems push further this complexity by adding non-determinism and uncertainty. Therefore, it is imperious necessary to have an abstraction theory for stochastic processes that can be used for verification and analysis of stochastic hybrid systems.

Reachability analysis and model checking are much easier when a concept of bisimulation is available. The state space can be drastically abstracted in some cases. In this chapter, we focus on defining bisimulation relations for stochastic hybrid systems, as a first step towards creating a framework for verification.

Besides of different bisimulation concepts in the concurrency theory, the notion of bisimu-

lation is present

• in the 'deterministic world': continuous and dynamical systems [142] or hybrid systems [85];

• or in the 'probabilistic world': probabilistic discrete systems [120], labelled Markov processes [26], piecewise deterministic Markov processes [153].

In this chapter we define different concepts of bisimulation for a very large class of Markov processes. This work is motivated by the fact that different models for stochastic hybrid systems are indeed Markov processes. Our interest is related with the *general stochastic hybrid systems*, introduced in Chapter 3. We mainly present two approaches of defining stochastic bisimulation:

The first one is based on category theory tools. The definition of bisimulation builds on the ideas of Edalat [26, 70] and of Larsen and Skou [120] and of Joyal, Nielsen and Winskel [114]. We extend the Edalat's definition of bisimulation for labelled Markov processes to continuous time strong Markov processes defined on analytic spaces. The main result is that this bisimulation is indeed an equivalence relation. This turns out to be a rather hard mathematical result, which employs the whole stochastic analysis apparatus associated to a strong Markov process defined on analytic space.

Being defined in a category theory context, this stochastic bisimulation, as a notion of system equivalence, enjoys some fundamental mathematical properties. Moreover, we give a characterization of this bisimulation through a measurable relation between the state spaces which induces equivalent quotient processes. For the case of GSHS, we prove that, this is a natural notion of bisimulation because the bisimilarity of two GSHS implies the bisimilarity of their diffusion components and respectively of their jumping parts.

The second approach to define bisimulation between Markov processes is more robust. Two processes are bisimilar if there exist a relation between their state spaces which induces two quotient processes with the same probabilities to reach the bisimilar measurable sets. The reach set probabilities are described by the statistical notion of Choquet capacity associated to a Markov processes. A capacity is non-additive set-function used to represent uncertainty. The mathematical theory of non-additive set-functions got its first contribution with Gustave Choquet's "Theory of Capacities" [54] in 1953. Choquet's interest was applications to statistical mechanics and potential theory. Later this theory found applications in decision theory [66, 143],

robust Bayesian inference [104], Artificial Intelligence and automated reasoning [67], finance and asset pricing [69], etc.

The rest of the chapter is organized as follows. Next section presents some specific aspects of stochastic hybrid systems. In section 3, we present two alternative ways to define the concept of stochastic bisimulation for the category of strong Markov processes defined on analytic spaces. The difference lies in the methodology to define the arrows in the respective category. A characterization of this concept of bisimulation is also given. Specific properties for the case of GSHS are also derived. In section 4 a bisimulation concept "free of category theory" is proposed. This can be considered as a weak version of bisimulation because its purpose is to preserve only some statistical parameters of the initial processes. The chapter ends with some conclusions and further work.

7.2 Stochastic Hybrid Systems

7.2.1 Models

In the Chapter 3 we have defined a general model for SHS which encompasses most of the models for SHS studied in the literature. SHS can be described as stochastic hybrid automata, which can be thought as the syntax, and then their executions represent the semantics.

7.2.2 SHS Realizations

The realization of a GSHS (which might be considered as the most general model of SHS) is built as a *Markov string* H obtained by the concatenation of some diffusion processes together with a jumping mechanism given by a family of stopping times. We have proved that the realization of any GSHS, H, under standard assumptions (about the diffusion coefficients, non-Zeno executions, transition measure, etc see Chapter 3) is a strong Markov process.

In this chapter, different concepts of stochastic bisimulation for SHS will be defined between their realizations. From this perspective, the most important aspect of SHS is that their realizations are stochastic processes. Then we can combine stochastic analysis tools with classical concepts of bisimulations, in order to get new definitions for stochastic bisimulation.

Stochastic processes, we consider here, are non-deterministic systems with a continuous

state space, where "non-determinism" can be measured using transition probability measures. Markov processes form a subclass of stochastic systems for which, at any stage, future evolutions are conditioned only by the present state (in other words, they do not depend on the past).

A probability space $(\Omega, \mathcal{F}, P)$ is fixed and all $X-$valued random variables are defined on this probability space. The trajectories in the state space are modelled by a family of random variables (x_t), where t denotes the time. The reasoning about state change is carried out by a family of probabilities P_x one for each state $x \in X$. The construction is similar to the coalgebraic reasoning in the semantics of specification languages: the system behavior is described by given for each state the possible evolutions. For Markov processes, for each state x, the probability $P_x(x_t \in A)$ to reach a given set of state $A \subset X$ (provided that A is measurable) starting from x describes the system evolution. We remark two ingredients that make the difference from the deterministic case: the evolutions are described from an initial state to a set of final set (nondeterminism) and all we know is a probability to have such trajectories (uncertainty).

Formally, let $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ be a strong Markov process. In the following sections, usually M will represent the realization of a stochastic hybrid system model, H.

Moreover, we suppose that M is a transient Markov process (i.e. there exists a strict positive Borel measurable function q such that Vq is a bounded function). The transience means that for any Borel set E in X for almost all trajectories there exists a finite stopping time t^*such that $x_t \notin E$ for all $t > t^*$ (for more explanations about the transience property see [55]). On the state space X we define a preorder relation $\prec_M$ given $x \prec_M y \Longleftrightarrow Vf(y) \leq Vf(x), \forall f \in \mathcal{B}^b(X), f \geq 0$. Intuitively, $\prec_M$ is the order on the trajectories of M. In particular, if M degenerates in a semi-dynamical system, $\prec_M$ is exactly the order relation on the trajectories.

One can define on X the *fine topology* [133], which consists of the sets $G \subseteq X$ with the following property: for each $x \in G$ there exists a measurable set $A \supset X \backslash G$ and $P_x(T_A > 0) = 1$, where

$$T_A(\omega) = \inf\{t > 0 | x_t(\omega) \in A\}$$

is the first *hitting time* of A. Intuitively, this means that each trajectory starting from x remains for a while in G. The fine topology is separated and is finer than the initial topology.

7.3 Categorical Approaches for Defining Bisimulation of Markov Processes

Having defined the structure of stochastic hybrid systems and their abstractions, we now consider relationships between abstract realizations that preserve their structure and therefore can be seen as homomorphisms of (abstractions of) Markov processes. We shall introduce simulation morphism which transfer the study of properties of an initial system to the study of properties of a smaller system. In this context we will define a notion of bisimulation as a special simulation. This concept will induce a new stochastic bisimulation for those models of stochastic hybrid systems, whose realizations are strong Markov processes (for example: Piecewise deterministic Markov processes [40], stochastic hybrid systems [41], switching diffusion processes [81], general stochastic hybrid systems [43]). The state space of these processes is usually a Borel space. In most of the cases, the state space can be identified with the Euclidean space or an open subset of it. In the following we consider a slightly more general case, namely we consider the case of strong Markov processes defined on analytic spaces. An analytic space is the image of a Polish space under a continuous function from one Polish space to another. A Polish space is a topological space homeomorphic with a complete separable space. Any Borel space is an analytic space.

In the first subsection, we discuss a general view of the methodology for defining bisimulation for Markov processes. In the next two subsection we give two possible methods to define the category of strong Markov processes with analytic state spaces. The difference consists in the way to define the arrows in such category. In each case we define the concept of bisimulation and we show that the respective category has semi-pullback. The later result implies that the bisimulation is an equivalence relation. The resulting concept of bisimulation will be compared with a concept of bisimulation via open maps (as introduced by Winskel et.al. and applied to continuous dynamical system by Tabuada et.al.) for semi-dynamical. After that we present some specific properties for bisimulation of GSHS.

7.3.1 Defining the methodology

The definition of bisimulation builds on the ideas of Edalat [70] and of Larsen and Skou [120] and of Joyal, Nielsen and Winskel [114].

The classical paper of Joyal, Nielsen and Winskel [114] presents a general categorical view of what bisimulation is for deterministic systems. The bisimulation concept is given in terms of open maps and simulation morphisms.

For the continuous case of the Markov processes, this definition can not be adapted straightforward. The main problem is how to define the simulation morphisms and the open maps. In this case, we say that a Markov process M^1 *simulates* another Markov M^2 if there exist a surjective continuous morphism $\psi : X^2 \to X^1$ between their state spaces such that each transition probability on X^2 'is matched' by a transition probability on X^1. The meaning of this 'matching' is that for each measurable set $A \subset X^1$ and for each $u \in X^2$ we have

$$p_t^2(u, \psi^{-1}(A)) \leq p_t^1(\psi(u), A) \qquad \forall t \geq 0 \qquad (*)$$

where (p_t^2) and (p_t^1) are the transition functions corresponding to M^2, respectively to M^1. A such morphism ψ is called *simulation morphism.*

The open maps are replaced by the so-called *zigzag morphisms*, which are simulation morphism for which the above condition holds with equality.

Practically, a simulation condition as before is hard to be checked because the time t runs in a 'continuous' set. Then, it is necessary to require supplementary assumptions about the transition probabilities of the processes we are talking about. This kind of simulation morphisms and zigzag morphisms have been defined for some particular Markov processes: for labelled Markov processes and for stationary Markov processes with discrete time defined on Polish or analytic spaces (see [70] and the references therein). The categories considered there have the above Markov processes as objects and the zigzag morphisms as morphisms. Then the bisimulation notion for these processes is given in a 'classical' way. Two labelled Markov processes, for example, are probabilistically bisimilar if there exists a *span of zigzag morphisms* between them. In this context, we can point out another reason why only some special kind of Markov processes are considered, as follows. This bisimulation relation is always reflexive and symmetric. But, the transitivity of a such relation (the bisimulation must be an equivalence relation) is usually implied by the existence of *semi-pullbacks* in the Markov process category

considered [114, 70]. That means, in the respective category, for any pair of morphisms $\varphi^1 : M^1 \to M$ and $\varphi^2 : M^2 \to M$ (M^1, M^2, M are objects in that category) there exist an object M^0 and morphisms $\pi^i : M^0 \to M^i$ $(i = 1, 2)$ such that $\varphi^1 \circ \pi^1 = \varphi^2 \circ \pi^2$.
The construction of the semi-pullback in the above categories of Markov processes is strongly based on the stationarity property of the Markov processes considered [70]. In this case the transition probabilities do not depend on time! Then the construction mechanism of the semi-pullback in a such categories of Markov processes is reduced to the construction of the semi-pullback in the category of transition probability functions and surjective transition probability preserving Borel maps (as morphisms in the respective category) (see [70] for the detailed construction).

We develop a novel concept of *stochastic bisimulation* for strong Markov processes defined on analytic spaces. The novelty consists of the way to define the simulation morphisms and the zigzag morphisms. Specifically, we replace the condition $(*)$ by a 'global condition' given in terms of kernel operators. A zigzag morphism between two Markov processes should 'commute' with the kernel operators of the processes considered. Then the bisimulation relation is naturally given via zigzag morphism spans between Markov processes. Moreover, the category of strong Markov processes defined on analytic spaces with these zigzag morphisms as arrows has semi-pullback. Therefore, the bisimulation relation is an equivalence relation.

The zigzag morphisms for Markov processes can be defined in an alternative way not between the state spaces, but between their cone of excessive functions. These kind of functions can be thought of as general solutions associated to the processes generator. In this case the zigzag morphisms change the directions of arrows. The simulator process has a larger cone of excessive functions. Then the zigzag morphism spans between Markov processes used to define the bisimulation relation become co-spans of morphims between the excessive function cones. Also, in this case the category of strong Markov processes defined on analytic spaces with these zigzag morphisms as arrows has semi-pullback.

7.3.2 First Approach

The Category of Markov Processes

We consider the category **GMP** of the strong Markov processes, defined on analytic spaces, with continuous time, as objects. In this category, the arrows will be the zigzag morphisms, which will be defined in the following. The aim of this subsection is to give an appropriate definition of these *zigzag morphisms (and of simulation morphisms)* between such processes, which will allow us to give a 'natural' concept of stochastic bisimulation in this category.

Comparing with the similar notions from [70], the main difference is that we impose to these morphisms to satisfy some *global conditions* written in terms of kernel operators associated to the Markov processes considered. Our choice is motivated by the fact that, in the non-stationary case, the transition probabilities of the two processes depend on time and their computation, for each moment of time $t > 0$, is not practically possible.

Let M^1 and M^2 be two objects of **GMP**. The state space of M^1 (resp. M^2) is $X^{(1)}$ (resp. $X^{(2)}$).

Definition 21 *A simulation morphism between the processes M^2 and M^1 (the process M^1 simulates the process M^2) is a measurable, monotone, finely continuous application $\psi : X^{(2)} \to X^{(1)}$ such that*

$$V^2(f \circ \psi) \leq V^1 f \circ \psi, \forall f \in \mathcal{B}^b(X^{(1)}),\ f \geq 0,$$

where V^1 (resp. V^2) is the kernel operator associated to M^1 (resp. M^2).

In some papers [138], an application ψ as in the Def. 21 is called *H-map*. The Def. 21 illustrates, in terms of kernel operators, that the simulating process can make all the transitions of the simulated process with greater probability than in the process being simulated.
A surjective H-map $\psi : X^{(2)} \to X^{(1)}$ induces an equivalence relation $\sim_\psi$ on $X^{(2)}$

$$u \sim_\psi v \Leftrightarrow \psi(u) = \psi(v). \tag{7.1}$$

In this way, to each $x \in X^{(1)}$ we can associate an equivalence class $[u]_\psi$ w.r.t. $\sim_\psi$ such that $[u]_\psi = \psi^{-1}(x)$. We call $\sim_\psi$ the *simulation relation* induced by ψ.

Definition 22 *A surjective simulation morphism ψ between the processes M^2 and M^1 is called zigzag morphism if the condition from the Def. 21 holds with equality, i.e.*

$$V^2(f \circ \psi) = V^1 f \circ \psi, \forall f \in \mathcal{B}^b(X^{(1)}),\ f \geq 0. \tag{7.2}$$

Proposition 23 *A surjective simulation morphism ψ between the processes M^2 and M^1 is a zigzag morphism iff for almost all $t \geq 0$ (i.e. except with a zero Lebesgue measure set of times) the following equality holds*

$$P_t^2(f \circ \psi)(u) = (P_t^1 f)(\psi(u)), \forall f \in \mathcal{B}^b(X^{(1)}),\ f \geq 0, u \in X^{(2)}, \tag{7.3}$$

where (P_t^1) (resp. (P_t^2)) is the semigroup of operators associated to M^1 (resp. M^2).

Remark 11 *The monotonicity of a zigzag morphism ψ can be derived from condition satisfied by a zigzag morphism. Roughly speaking, this means that whilst the process M^2 evolves from u to $\psi^{-1}(A)$ ($A \in \mathcal{B}(X^{(1)})$) on a trajectory with a given probability, the process M^1 evolves from $\psi(u)$ to A with the same probability.*

Remark 12 *A zigzag morphism $\psi : X^{(2)} \to X^{(1)}$ induces a morphism between the lattices of measurable functions associated with the two processes: $\Psi : \mathcal{B}^b(X^{(1)}) \to \mathcal{B}^b(X^{(2)})$ such that*

$$\Psi(f) = f \circ \psi \tag{7.4}$$

for all $f \in \mathcal{B}^b(X^{(1)})$. Then the condition (7.3) can be written as follows

$$\Psi(P_t^1 f) = P_t^2(\Psi(f)) \tag{7.5}$$

for all $f \in \mathcal{B}^b(X^{(1)})$, or equivalently the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}^b(X^{(2)}) & \stackrel{P_t^2}{\to} & \mathcal{B}^b(X^{(2)}) \\ \Psi \uparrow & & \uparrow \Psi \\ \mathcal{B}^b(X^{(1)}) & \stackrel{P_t^1}{\to} & \mathcal{B}^b(X^{(1)}) \end{array}$$

The Remark 12 shows that a zigzag morphism from two Markov processes can be thought of

as a generalization for the stochastic case of the simulation concept for abstract control systems defined in [156].

More explicitly, the operator semigroups (P_t^1) and (P_t^2) define two dynamical systems (the *abstractions* of M^1, resp. M^2) on the Banach spaces $\mathcal{B}^b(X^{(1)})$ and $\mathcal{B}^b(X^{(2)})$, respectively

$$\begin{aligned} \phi^1 &: \mathbb{R}_+ \times \mathcal{B}^b(X^{(1)}) \to \mathcal{B}^b(X^{(1)});\ \phi^1(t,f) = P_t^1 f \\ \phi^2 &: \mathbb{R}_+ \times \mathcal{B}^b(X^{(2)}) \to \mathcal{B}^b(X^{(2)});\ \phi^2(t,f) = P_t^2 f. \end{aligned}$$

A zigzag morphim between M^2 and M^1 implies that ϕ^1 is simulated by ϕ^2. The condition (7.5) is, in fact, the characterization of an open map between these dynamical systems [86]: when ϕ^1 evolves from f to P_t^1 then ϕ^2 evolves from Ψf to $\Psi(P_t^1 f) = P_t^2 \Psi f$.

Stochastic Bisimulation

We consider the category **GMP** with the strong Markov processes, defined on analytic spaces, with continuous time as objects and with zigzag morphisms as arrows.

Then, we define the *stochastic bisimulation* between two processes in this category as the existence of a span of zigzag morphisms between them.

Definition 24 *Let M^1 and M^2 be two objects in* **GMP**. *M^1 is stochastic bisimilar to M^2 (written $M^1 \sim M^2$) if there exists a span of zigzag morphisms between them, i.e. there exists a Markov process M^{12} (object in* **GMP***) and the zigzag morphisms ψ^1 (where $\psi^1 : X^{12} \to X^{(1)}$) and ψ^2 (where $\psi^2 : X^{12} \to X^{(2)}$) such that*

$$\begin{array}{ccc} & M^{12} & \\ {\scriptstyle \psi^1}\swarrow & & \searrow{\scriptstyle \psi^2} \\ M^1 & & M^2 \end{array}$$

Proposition 25 *M^1 and M^2 are stochastic bisimilar iff there exists a co-span between their lattices of measurable functions, i.e. there exists a Markov process M^{12} and the zigzag morphisms ψ^1 (where $\psi^1 : X^{12} \to X^{(1)}$) and ψ^2 (where $\psi^2 : X^{12} \to X^{(2)}$) such that*

$$\begin{array}{ccc} & \mathcal{B}^b(X^{12}) & \\ {\scriptstyle \Psi^1}\nearrow & & \nwarrow{\scriptstyle \Psi^2} \\ \mathcal{B}^b(X^1) & & \mathcal{B}^b(X^2) \end{array}$$

where Ψ^1 and Ψ^2 are induced from ψ^1 and ψ^2 by the formula (7.4).

Theorem 26 *The category* **GMP** *has semi-pullbacks.*

Proof. Let M^1, M^2, M be three strong Markov processes defined on the analytic spaces $X^{(1)}, X^{(2)}, X$, respectively. Suppose that there exist two zigzag morphisms $\psi^1 : X^{(1)} \to X$, $\psi^2 : X^{(2)} \to X$. We have to prove that there exist another object M^0 (a strong Markov process defined on a analytic space $X^{(0)}$) and two zigzag morphisms $\pi^1 : X^{(0)} \to X^{(1)}$ and $\pi^2 : X^{(0)} \to X^{(2)}$ such that the following diagram commutes

$$\begin{array}{ccccc} & & X^{(0)} & & \\ & {\scriptstyle \pi^1}\swarrow & & \searrow{\scriptstyle \pi^2} & \\ X^{(1)} & & & & X^{(2)} \\ & \searrow{\scriptstyle \psi^1} & & {\scriptstyle \psi^2}\swarrow & \\ & & X & & \end{array}$$

Let $X^{(0)} = \{(x^1, x^2) | \psi^1(x^1) = \psi^2(x^2)\}$ equipped with the subspace topology of the product topology on $X^{(1)} \times X^{(2)}$. Note that $X^{(0)}$ is nonempty since ψ^1 and ψ^2 are supposed surjective. Clearly, $X^{(0)}$ is a analytic space and it is a closed subset of the analytic space $X^{(1)} \times X^{(2)}$. We take M^0 as the part of the product of the Markov processes M^1, M^2 restricted to $X^{(0)}$, i.e. M^0 is the product process $M^1 \otimes M^2$ "killed" outside of $X^{(0)}$. More explicitly, M^0 is the subprocess of $M^1 \otimes M^2$ with respect to the multiplicative functional $N_t = I_{[0,T)}(t)$, where T is the first exit time of $X^{(0)}$ and $I_{[0,T)}$ is the indicator function of $[0, T)$ (see [25], Ch.3 for background on multiplicative functionals and subprocesses).

Let $(P_t^1), (P_t^2)$ be the operator semigroups associated with M^1 and M^2. Let $\widehat{P_t^1}$ and $\widehat{P_t^2}$ the semigroups defined on $\mathcal{B}^b(X^{(1)} \times X^{(2)})$ by

$$\widehat{P_t^1} f(x^1, x^2) = P_t^1(f(\cdot, x^2))(x^1), \widehat{P_t^2} f(x^1, x^2) = P_t^2(f(x^1, \cdot))(x^2).$$

The semigroup associated with $M^1 \otimes M^2$ is $P_t = \widehat{P_t^1 P_t^2} = \widehat{P_t^2 P_t^1}$ [49]. Then according with the Th. 3.3 [25], the process M^0 is a Markov process and the semigroup associated with it is $Q_t f(x^1, x^2) = \mathbb{E}_x[f(x_t^1 \otimes x_t^2) N_t]$ for any $f \in \mathcal{B}^b(X^{(1)} \times X^{(2)})$ or, equivalently, $Q_t f(x^1, x^2) = P_t f(x^1, x^2)$ for any $f \in \mathcal{B}^b(X^{(0)})$.

Moreover, M^0 is a strong Markov process since N_t is a strong functional multiplicative (see Prop. 3.12 [25]).

Then π^1 and π^2 can be taken as the projection maps. Using product semigroup and the Prop.23, it follows that these projection maps are indeed zigzag morphisms. For example, for π^1, we have $f \circ \pi^1$ (for $f \in \mathcal{B}^b(X^{(1)})$) depends only on x^1 and P_t^2 does not change it. Then $Q_t(f \circ \pi^1)(x^1, x^2) = (P_t^1 f)(\pi^1(x^1, x^2))$ for all $(x^1, x^2) \in X^{(0)}$. The surjectivity of π^1 or π^2 can be easily derived using the sujectivity of ψ^1 and ψ^2 and the definition of $X^{(0)}$. The equality $\psi^1 \circ \pi^1 = \psi^2 \circ \pi^2$ trivially holds.□

An immediate consequence of the existence of semi-pullbacks in the category **GMP**, proven in Th.26 is the following result:

Proposition 27 *The stochastic bisimulation in the category* **GMP** *is an equivalence relation.*

7.3.3 Second Approach

The Category of Markov Processes

We consider the category $\widetilde{\mathbf{GMP}}$ of the strong Markov processes defined on analytic spaces as the objects and the $\mathcal{E}$-zigzag morphisms (which will be defined in the following) as the arrows.

The zigzag morphisms between Markov processes can be defined also as morphisms between their cones of excessive functions. Let M^1, M^2 be two strong Markov processes defined on analytic spaces $X^{(1)}$, respectively $X^{(2)}$. Let $\mathcal{E}_{M^1}, \mathcal{E}_{M^2}$ the associated cones of excessive functions.

Definition 28 *An* $\mathcal{E}$-morphism *(between these two cones) can be defined as an application*

$$\Psi : \mathcal{E}_{M^1} \to \mathcal{E}_{M^2} \tag{7.6}$$

such that the following properties hold: (i) $\Psi(f+g) = \Psi(f) + \Psi(g), \forall f, g \in \mathcal{E}_{M^1}$; *(ii)* $f \leq g \Rightarrow$

$\Psi(f) \leq \Psi(g)$; $f_k \nearrow f \Rightarrow \Psi(f_k) \nearrow \Psi(f)$; *(iv)* $\Psi(f \cdot g) = \Psi(f) \cdot \Psi(g), \forall f, g \in \mathcal{E}_{M^1}$; *(v)* $\Psi(1) = 1$. *An $\mathcal{E}$-morphism Ψ is called* finite *if* $f < +\infty \Rightarrow \Psi(f) < +\infty$.

Proposition 29 *If $\psi : X^{(2)} \to X^{(1)}$ is a H-*map *then $\Psi : \mathcal{E}_{M^1} \to \mathcal{E}_{M^2}$ given by*

$$\Psi(f) = f \circ \psi \tag{7.7}$$

for all $f \in \mathcal{E}_{M^1}$, is a finite $\mathcal{E}$-morphism.

Intuitively, in the formula (7.7) the H-map ψ can be thought of as a *variable change*, i.e. for all $f \in \mathcal{E}_{M^1}$

$$\Psi(f)(u) = f(\psi(u)), \forall u \in X^{(2)}. \tag{7.8}$$

Remark 13 *(i) The map Ψ defined by (7.7) can be extended as a map between the two cones of measurable positive functions defined on $X^{(1)}$, respectively $X^{(2)}$, loosing the property of finely continuity. Prop.29 shows how a function between the state spaces of M^1, M^2 can provide an $\mathcal{E}$-morphism.*

(ii) Conversely, if Ψ is an $\mathcal{E}$-morphism as in (7.6) then there exists a unique measurable monotone and finely continuous application $\overline{\psi}$ from $X^{(2)}$ to an extension of $X^{(1)}$ such that: $\Psi(f) = f \circ \overline{\psi}, \forall f \in \mathcal{E}_{M^1}$. To obtain this result one can use results from [138].

Using (7.8), each function g belonging to the range of Ψ can be extended to $X^{(2)}/_{\sim\psi}$, i.e. $g([u]_\psi) = f(x)$ provided that $[u]_\psi = \psi^{-1}(x)$ and $g = \Psi(f)$.

Proposition 30 *If $\psi : X^{(2)} \to X^{(1)}$ is a surjective and finely open H-map such that each excessive function $g \in \mathcal{E}_{M^2}$ has the property*

$$u \sim_\psi v \Rightarrow g(u) = g(v) \tag{7.9}$$

then the $\mathcal{E}$-morphism $\Psi : \mathcal{E}_{M^1} \to \mathcal{E}_{M^2}$ given by formula (7.7) is surjective.

Proof. For each $g \in \mathcal{E}_{M^2}$ we have to define $f \in \mathcal{E}_{M^1}$ such that $\Psi(f) = g$. Let $f : X^{(1)} \to [0, \infty)$ defined by $f(x) = g(u)$ for each $x \in X^{(1)}$, where $u \in X^{(2)}$ is such that $\psi(u) = x$ (there exists a such u since ψ is surjective). The function f is well defined because of (7.9). Then f can be

written as $f = g \circ \psi^{-1}$ and for any open set $D \subset [0, \infty)$ we have $f^{-1}(D) = \psi(g^{-1}(D))$. Since ψ is a finely open map we obtain that $f^{-1}(D)$ is finely open in $X^{(1)}$. Then $f \in \mathcal{E}_{M^1}$.□

Remark 14 *It is easy to check that if in the Prop. 29 both ψ and Ψ are surjective then Ψ must be bijective. Therefore the two excessive function cones can be identified and the two processes are equivalent.*

Definition 31 *A simulation $\mathcal{E}$-morphism between M^1, M^2 is an $\mathcal{E}$-morphism such that*

$$V^2 \circ \Psi \leq \Psi \circ V^1. \tag{7.10}$$

A surjective $\mathcal{E}$-morphism Ψ is called zigzag $\mathcal{E}$-morphism if

$$V^2 \circ \Psi = \Psi \circ V^1 \tag{7.11}$$

i.e. the following diagram commutes

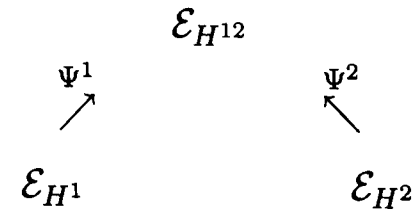
$$\begin{array}{ccc} \mathcal{E}_{M^1} & \stackrel{\Psi}{\rightarrow} & \mathcal{E}_{M^2} \\ V^1 \uparrow & & \uparrow V^2 \\ \mathcal{E}_{M^1} & \stackrel{\Psi}{\rightarrow} & \mathcal{E}_{M^2} \end{array}$$

Remark 15 *It is clear that if ψ is a H-map which is a zigzag morphism in the sense of the first approach, i.e. it satisfies the condition (7.2) then the $\mathcal{E}$-morphism generated by (7.7) is a zigzag $\mathcal{E}$-morphism.*

Stochastic Bisimulation

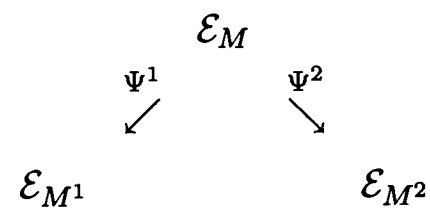
We can define a weak version of the stochastic bisimulation via $\mathcal{E}$-morphisms:

Definition 32 *Let M^1 and M^2 be two objects in $\widetilde{\mathbf{GMP}}$. M^1 is stochastic bisimilar to M^2 (written $M^1 \sim M^2$) if there exists a cospan of $\mathcal{E}$-zigzag morphisms between them, i.e. there exists a Markov process M^{12} (object in $\widetilde{\mathbf{GMP}}$) and the $\mathcal{E}$-morphisms Ψ^1 and Ψ^2 between their excessive function cones*

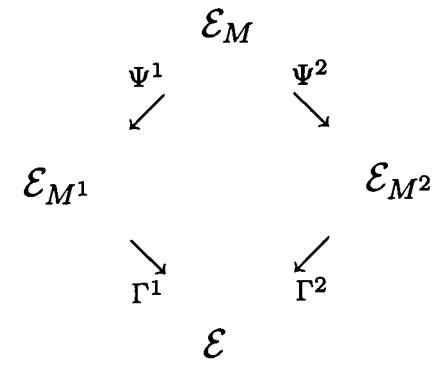


Proposition 33 *The category* $\widetilde{\mathbf{GMP}}$ *has semi-pullbacks.*

Proof. If we define the stochastic bisimulation defined via zigzag $\mathcal{E}$-morphisms, then the semi-pullback existence for the category of Markov processes (with morphisms given by zigzag $\mathcal{E}$-morphisms) is equivalent with the *pushout existence* in the category of their excessive function cones (with the morphisms given by zigzag $\mathcal{E}$-morphisms). Let us take the following span of morphims between the excessive function cones



Naturally, we consider $\mathcal{E}$ as the product $\mathcal{E}_{M^1} \otimes \mathcal{E}_{M^1}$ of the cones $\mathcal{E}_{M^1}, \mathcal{E}_{M^1}$ (which correspond to the product of operator semigroups or to Markov process product defined on $X^{(1)} \times X^{(2)}$). Then the 'inclusions' $\mathcal{E}_{M^1} \overset{\Gamma^1}{\hookrightarrow} \mathcal{E}$, $\Gamma^1(f^1) = \Psi^1(f) \otimes \Psi^2(f)$ if $f^1 = \Psi^1(f)$ and $\mathcal{E}_{M^2} \overset{\Gamma^2}{\hookrightarrow} \mathcal{E}$, $\Gamma^2(f^2) = \Psi^1(f) \otimes \Psi^2(f)$ if $f^2 = \Psi^2(f)$ (essentially, Ψ^1 and Ψ^2 are surjective) gives the desired pushout construction, i.e. the following diagram commutes



Proposition 34 *The stochastic bisimulation defined by Def. 32 in* $\widetilde{\mathbf{GMP}}$ *is an equivalence relation.*

7.3.4 Characterization of stochastic bisimulation

Let us consider two bisimilar processes M^1 and M^2 and ψ^1 and ψ^2 are zigzag morphisms as in the Def.24. Then we can define a relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$, called *bisimulation relation*, given by

$$x^1 \mathcal{R} x^2 \Leftrightarrow (\psi^1)^{-1}(x^1) \cap (\psi^2)^{-1}(x^2) \neq \emptyset \tag{7.12}$$

Proposition 35 *$x^1 \mathcal{R} x^2$ if and only if*

$$x^1 \in \psi^1[(\psi^2)^{-1}(x^2)].$$

For $A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$, we define

$$\begin{aligned} \mathcal{R}^{-1}(A^1) &= \{(x^1, x^2) | x^1 \in A^1, x^1 \mathcal{R} x^2\} \\ \mathcal{R}^{-1}(A^2) &= \{(x^1, x^2) | x^2 \in A^2, x^1 \mathcal{R} x^2\} \end{aligned}$$

$\mathcal{R}$ is called *measurable* if for all $A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$ the sets $\mathcal{R}^{-1}(A^1)$, $\mathcal{R}^{-1}(A^2)$ are measurable w.r.t. σ-algebra product $\mathcal{B}(X^{(1)}) \otimes \mathcal{B}(X^{(2)})$.

Then we can extend the bisimulation relation (7.12) to the measurable sets

$$A^1 \mathcal{R} A^2 \Leftrightarrow (\mathcal{R}^1)^{-1}(A^1) = (\mathcal{R}^2)^{-1}(A^2).$$

or, equivalently,

$$A^1 \mathcal{R} A^2 \text{ iff } \forall x^1 \in A^1 \exists x^2 \in A^2 \text{ s.t. } x^1 \mathcal{R} x^2 \text{ and viceversa.}$$

$\mathcal{R}$ is called *weak measurable* if for all $A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$ with $A^1 \mathcal{R} A^2$ then $\mathcal{R}^{-1}(A^1)$, $\mathcal{R}^{-1}(A^2)$ are measurable w.r.t. σ-algebra product $\mathcal{B}(X^{(1)}) \otimes \mathcal{B}(X^{(2)})$.

Let $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ be the equivalence relation given by (7.12). This relation will induce other two relations $\mathcal{R}^1$ and $\mathcal{R}^2$ define on $X^{(1)}$ and $X^{(2)}$, respectively, as follows.

$$x^1 \mathcal{R}^1 y^1 \Leftrightarrow \exists x^2 \in X^{(2)} \text{ s.t. } x^1 \mathcal{R} x^2 \text{ and } y^1 \mathcal{R} x^2 \tag{7.13}$$

and a similar definition for $\mathcal{R}^2$.

Lemma 36 *$x^1\mathcal{R}^1y^1$ iff there exist $u, v \in X^{12}$ such that $x^1 = \psi^1 u$ and $y^1 = \psi^1 v$ provided that $u \sim_{\psi^2} v$. Similarly, for $\mathcal{R}^2$.*

Proposition 37 *$\mathcal{R}^1$ and $\mathcal{R}^2$ are equivalence relations.*

Proof. The previous lemma ensures the transitivity property and the surjectivity of ψ^1, ψ^2 gives the reflexivity property. The symmetry is clear.□

Remark 16 *In a similar way, for a relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ with $\Pi^1(\mathcal{R}) = X^{(1)}$ and $\Pi^2(\mathcal{R}) = X^{(2)}$ we can define the induced equivalence relations $\mathcal{R}^1, \mathcal{R}^2$ (If it is necessary we have to take the transitive closure of these relations).*

Let

$$\mathcal{B}^*(X^{(1)}) = \mathcal{B}(X^{(1)}) \cap \{A^1 \subset X^{(1)} | \text{ if } x^1 \in A^1 \text{ and } [x^1] = [y^1] \text{ then } y^1 \in A^1\}$$

be the collection of all Borel sets in which any equivalence class of $X^{(1)}$ is either totally contained or totally not contained. Here, for $x^1 \in X^{(1)}$ (resp. $x^2 \in X^{(2)}$) we denote its class of equivalence by $[x^1]$ (resp. $[x^2]$) w.r.t. $\mathcal{R}^1$ (resp. $\mathcal{R}^2$). It can be checked that $\mathcal{B}^*(X^{(1)})$ is a σ-algebra. Let $X^{(1)}/_{\mathcal{R}^1}$ be the set of equivalence classes of $X^{(1)}$, let $\pi_{X^{(1)}} : X^{(1)} \to X^{(1)}/_{\mathcal{R}^1}$ be the mapping that maps each $x^1 \in X^{(1)}$ to its equivalence class and let

$$\mathcal{B}(X^{(1)}/_{\mathcal{R}^1}) = \{A^1 \subset X^{(1)}/_{\mathcal{R}^1} | \pi^{-1}_{X^{(1)}}(A^1) \in \mathcal{B}^*(X^{(1)})\}.$$

Then $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}(X^{(1)}/_{\mathcal{R}^1}))$, which is a measurable space, is called the quotient space of $X^{(1)}$ with respect to $\mathcal{R}^1$. The quotient space of $X^{(2)}$ with respect to $\mathcal{R}^2$ is defined in a similar way. Clear, $\mathcal{B}(X^{(i)}/_{\mathcal{R}^i})$ can be identified with $\mathcal{B}^*(X^{(i)})$. Then, for $i = 1, 2$, the space $X^{(i)}/_{\mathcal{R}^i}$ can be endowed with the σ-algebra $\mathcal{B}^*(X^{(i)})$, which is the "saturation" of the Borel σ-algebra of $X^{(i)}$ w.r.t. $\mathcal{R}^i$.

The following proposition shows that only the saturated sets can be bisimilar.

Proposition 38 *If $A^1 \in \mathcal{B}(X^{(1)})$ is such that there exists $A^2 \in \mathcal{B}(X^{(2)})$ with $A^1\mathcal{R}A^2$ then A^1 is saturated, i.e. $A^1 \in \mathcal{B}^*(X^{(1)})$.*

Proof. If $y^1\mathcal{R}^1x^1 \in A^1$ then there exist $u, v \in X^{12}$ such that $y^1 = \psi^1 u$ and $x^1 = \psi^1 v$ with $\psi^2 u = \psi^2 v$. Since $\psi^2 v \in \psi^2(\psi^1)^{-1}(A^1) = A^2$ there exists $x^2 \in A^2$ such that $\psi^2 v = x^2$. Therefore, $\psi^2 u = x^2 \in A^2$ and $\psi^1 u \in \psi^1(\psi^2)^{-1}(A^2) = A^1$, i.e. $y^1 \in A^1$. That means A^1 is saturated.□

Lemma 39 *If ψ^1, ψ^2 are finely open H-maps then*
(i) $A^1\mathcal{R}A^2$ iff $(\psi^1)^{-1}(A^1) = (\psi^2)^{-1}(A^2)$.
(ii) $\mathcal{R}$ is a weak measurable relation.

Proof. (i) It is clear that $A^1\mathcal{R}A^2$ iff

$$\begin{aligned} A^1 &= \psi^1[(\psi^2)^{-1}A^2] \\ A^2 &= \psi^2[(\psi^1)^{-1}A^1]. \end{aligned}$$

Then $u \in (\psi^1)^{-1}A^1$ implies $\psi^1 u \in A^1 = \psi^1[(\psi^2)^{-1}A^2]$, i.e. there exists $v \in (\psi^2)^{-1}A^2$ such that $\psi^1 u = \psi^1 v$ and $\psi^2 v \in A^2$. Since A^2 is saturated we get that $\psi^2 u \in A^2$, i.e. $(\psi^2)^{-1}A^2$. Then $(\psi^1)^{-1}(A^1) \subset (\psi^2)^{-1}(A^2)$. The inverse inclusion is similar.

(ii) The conclusion is clear since for $A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$ we have

$$\begin{aligned} \mathcal{R}^{-1}(A^1) &= \{(x^1, x^2) | x^1 \in A^1 \text{ and } x^2 \in \psi^2[(\psi^1)^{-1}x^1]\} \\ \mathcal{R}^{-1}(A^2) &= \{(x^1, x^2) | x^2 \in A^2 \text{ and } x^1 \in \psi^1[(\psi^2)^{-1}x^2]\}.\square \end{aligned}$$

Proposition 40 *If ψ^1, ψ^2 are finely open H-maps then the quotient spaces $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)}))$, $(X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$ are homeomorphic..*

Proof. We can define an application $\varphi : (X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)})) \to (X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$ such that, for all $[x^1] \in X^{(1)}/_{\mathcal{R}^1}$ we have

$$\varphi([x^1]) = [x^2] \tag{7.14}$$

provided that $x^1\mathcal{R}x^2$. Definition of $\mathcal{R}^1$ and $\mathcal{R}^2$ ensure that φ is well-defined and bijective. For measurability, let us consider an arbitrary $A^2 \in \mathcal{B}^*(X^{(2)})$ then

$$\varphi^{-1}(A^2) = \psi^1[(\psi^2)^{-1}A^2]$$

is a measurable set in $X^{(1)}$, where A^2 is considered as a measurable set in $X^{(2)}$. The Prop.35 and the fact that A^2 is saturated w.r.t. $\mathcal{R}^2$ ensure that $\varphi^{-1}(A^2)$ is saturated w.r.t. $\mathcal{R}^1$. Then φ is measurable. Similarly, φ^{-1} is measurable. □

Remark 17 *The map (7.14) from the previous proposition shows that an equivalence class* $[x^2] \in X^{(2)}/_{\mathcal{R}^2}$ *is identified with an equivalence class* $[x^1] \in X^{(1)}/_{\mathcal{R}^1}$ *where* x^1 *corresponds to* $[u]_{\psi^1}$ *given by* $(\psi^2)^{-1}([u]_{\psi^1}) = [x^2]$.

Proposition 41 (reachability equivalence) *If* M^1 *and* M^2 *are stochastic bisimilar via finely open zigzag morphisms then for all pairs* $(x^1, x^2) \in X^{(1)} \times X^{(2)}$ *and* $(A^1, A^2) \in \mathcal{B}(X^{(1)}) \times \mathcal{B}(X^{(2)})$ *such that* $x^1 \mathcal{R} x^2$ *and* $A^1 \mathcal{R} A^2$ *the equality between the transition probabilities*

$$p_t^1(x^1, A^1) = p_t^2(x^2, A^2) \tag{7.15}$$

is fulfilled for almost all $t > 0$.

Proof. Since $A^1 \mathcal{R} A^2$ then, from the Prop.38, we get that $(A^1, A^2) \in \mathcal{B}^*(X^{(1)}) \times \mathcal{B}^*(X^{(2)})$. Formula (7.3) can be written for the sets A^1 and A^2 as follows

$$\begin{aligned} p_t^1(x^1, A^1) &= p_t^{12}[u, (\psi^1)^{-1}(A^1)],\ u \in (\psi^1)^{-1}(x^1) \\ p_t^2(x^2, A^2) &= p_t^{12}[v, (\psi^2)^{-1}(A^2)],\ v \in (\psi^2)^{-1}(x^2) \end{aligned}$$

But, from Lemma 39

$$\begin{aligned} A^1 &= \psi^1[(\psi^2)^{-1} A^2] \\ A^2 &= \psi^2[(\psi^1)^{-1} A^1] \end{aligned}$$

which implies that $(\psi^1)^{-1}(A^1) = (\psi^2)^{-1}(A^2)$ since A^1, A^2 are saturated. In fact, the equality (7.15) results from the definition of zigzag morphism and the existence of the homeomorphism φ between the quotient spaces (see Prop.40).□

Now with these two results (Prop.40 and Prop.38) in hand we can introduce the quotient stochastic processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ with

- the quotient spaces $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)}))$, $(X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$, respectively, as state spaces;

- transition probabilities given by

$$\widetilde{p}_t^1([x^1], A^1) = p_t^1(x^1, A^1), \text{ for all } A^1 \in \mathcal{B}^*(X^{(1)}); x^1 \in X^1$$
$$\widetilde{p}_t^2([x^2], A^2) = p_t^1(x^2, A^2), \text{ for all } A^2 \in \mathcal{B}^*(X^{(2)}); x^2 \in X^2$$

  defined for all $t > 0$. The way to define the induced equivalence relations $\mathcal{R}^1$, $\mathcal{R}^2$ ensures that these transition probabilities are well-defined, i.e. they do not depend on the representants of equivalence classes $[x^1]$ or $[x^2]$.

Proposition 42 *The quotient stochastic processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are Markov processes.*

From Prop.40, we are able now to make the connection between stochastic bisimulation and equivalence of stochastic processes as follows.

Proposition 43 *If ψ^1, ψ^2 are finely open H-maps then the quotient stochastic processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are equivalent.*

Proof. According to the Prop.40 the quotient spaces $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)}))$, $(X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$ are homeomorphic. Then the equality (7.15) becomes

$$\widetilde{p}_t^1([x^1], A^1) = \widetilde{p}_t^2([x^2], A^2)$$

for all $A^1 \in \mathcal{B}^*(X^{(1)})$; $x^1 \in X^1$; $A^2 \in \mathcal{B}^*(X^{(2)})$; $x^2 \in X^2$ and for almost all $t > 0$ provided that $\varphi([x^1]) = [x^2]$ and $\varphi(A^1) = A^2$ with φ defined as in the Prop.40. This means that $\mathcal{R}$ preserves the transition probabilities, i.e. $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are equivalent.□

The properties of the bisimulation relation $\mathcal{R}$ induced by the existence of a span of zigzag morphisms between M^1 and M^2 give the idea to introduce a general concept of bisimulation relation, which will not depend on a given span.

Definition 44 *A relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ is called a bisimulation relation between M^1 and M^2 if the following conditions are satisfied:*

1. $\Pi^1(\mathcal{R}) = X^{(1)}$ *and* $\Pi^2(\mathcal{R}) = X^{(2)}$;

2. $\mathcal{R}$ is measurable;

3. the quotient stochastic processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are equivalent.

For a bisimulation relation $\mathcal{R}$, let us define

$$X^{12} = \{(x^1, x^2) \in X^{(1)} \times X^{(2)} | x^1 \mathcal{R} x^2\} \tag{7.16}$$

The σ-algebra of X^{12} is defined as the product σ-algebra

$$\mathcal{B}(X^{12}) = \sigma\{\mathcal{R}^{-1}(A^1) \text{ and } \mathcal{R}^{-1}(A^2) | A^1 \times A^2 \in \mathcal{B}(X^{(1)}) \otimes \mathcal{B}(X^{(2)})\}. \tag{7.17}$$

Assumption 11 (Analiticity of $\mathcal{R}$) *We suppose that if $X^{(1)}$ and $X^{(2)}$ are analytic spaces then X^{12} is analytic.*

Theorem 45 (Characterization of Stochastic Bisimulation) *Under the Ass.11 the following assertions hold:*
(A) M^1 is stochastic bisimilar with M^2 via finely open zigzag morphisms then there exists a weak measurable bisimulation relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ between them.
(B) If there exists a measurable bisimulation relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ between M^1 and M^2 then they are stochastic bisimilar.

Proof. Given two bisimilar processes M^1 and M^2 via finely open zigzag morphisms, the construction of the bisimulation relation $\mathcal{R}$ is given by (7.12) and the assertion (A) follows from Prop.40, Prop.37, and Prop.43.
Suppose now there exists a bisimulation relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ which satisfies the conditions of Def.44. The direct sum of the quotient spaces $(X^{(1)}/_{\mathcal{R}^1}, \mathcal{B}^*(X^{(1)}))$, $(X^{(2)}/_{\mathcal{R}^2}, \mathcal{B}^*(X^{(2)}))$ is embedded in $(X^{12}, \mathcal{B}(X^{12}))$.
We construct the Markov process M^{12} with the following transition probabilities

$$\begin{aligned} p_t^{12}[(x^1, x^2), \mathcal{R}^{-1}(A^1)] &= p_t^1(x^1, A^1) \\ p_t^{12}[(x^1, x^2), \mathcal{R}^{-1}(A^2)] &= p_t^2(x^2, A^2). \end{aligned}$$

Then we define for $i = 1, 2$ two maps ψ^i from X^{12} to $X^{(i)}$ as the canonical projections

$$\begin{aligned} \psi^i &: X^{12} \to X^{(i)} \\ \psi^i(x^1, x^2) &= x^i \end{aligned}$$

Clear, ψ^i are surjective since $\Pi^i(\mathcal{R}) = X^{(i)}$, $i = 1, 2$. If $A^{(i)}$ is a measurable set of $X^{(i)}$ then

$$(\psi^i)^{-1}(A^{(i)}) = \mathcal{R}^{-1}(A^i) \in \mathcal{B}(X^{12}),$$

i.e. ψ^i is measurable. On the other hand we have

$$\psi^i[\mathcal{R}^{-1}(A^i)] = A^i$$

but for $i \neq j$ we do not have the measurability of $\psi^i[\mathcal{R}^{-1}(A^j)]$. If $A^i \mathcal{R} A^j$ then $\mathcal{R}^{-1}(A^i) = \mathcal{R}^{-1}(A^j)$ and

$$\psi^i[\mathcal{R}^{-1}(A^j)] = \psi^i[\mathcal{R}^{-1}(A^i)] = A^i.$$

Then ψ^i are open maps only w.r.t. the σ-algebras generated by bisimilar sets.□

7.3.5 Specific Features of Bisimulation for GSHS

Let H^1 and H^2 be two GSHS with the realizations M^1 and M^2, respectively.

Definition 46 *H^1 and H^2 are stochastic bisimilar if their realizations M^1 and M^2 are stochastic bisimilar.*

Properties of zigzag morphims

A zigzag morphism $\psi : X^{(2)} \to X^{(1)}$ between M^1 and M^2, induces a relation $\mathcal{R}_\psi \subset X^{(2)} \times X^{(1)}$ as follows: $u \mathcal{R}_\psi x \Leftrightarrow \psi(u) = x$. Then the equivalence relation $\sim_\psi$on $X^{(2)}$ can be thought of as the equivalence relation induced by $\mathcal{R}_\psi$ in sense of [153], i.e. $u \sim_\psi v$ iff there exists $x \in X^{(1)}$ such that $u \mathcal{R}_\psi x$ and $v \mathcal{R}_\psi x$ (which is exact the meaning of (7.1)). The equivalence relation induced by $\mathcal{R}_\psi$ on $X^{(2)}$ is the trivial one (x can be equivalent only with itself).

The space $X^{(2)}/_{\sim_\psi}$ can be endowed with the σ-algebra $\mathcal{B}^*(X^{(2)})$, which is the "saturation" of the Borel σ-algebra of $X^{(2)}$ w.r.t. $\sim_\psi$ (i.e. the collection of all Borel sets of $X^{(2)}$ in which any equivalence class of $X^{(2)}$ is either totally contained or totally not contained). A function on $g : X^{(2)} \to \mathbb{R}$, which is measurable w.r.t. $\mathcal{B}^*(X^{(2)})$ will be called *saturated measurable function.* It is clear that a function measurable g is saturated measurable iff (7.9) holds. Each function $f : X^{(1)} \to \mathbb{R}$ measurable w.r.t. $\mathcal{B}(X^{(1)})$ can be identified with a saturated measurable function g such that $g = f \circ \psi$.

The morphism ψ can be viewed as a bijective mapping $\psi : X^{(2)}/_{\sim_\psi} \to X^{(1)}$. It is clear that ψ is a measurable application. To identify the two above measurable spaces ψ^{-1} must be measurable. The main idea, which results from this reasoning, is that the measurable space $(X^{(1)}, \mathcal{B}(X^{(1)}))$ can be embedded in the measurable space $(X^{(2)}, \mathcal{B}(X^{(2)}))$ and the measurable function on $X^{(1)}$ can be identified with the saturated measurable functions on $X^{(2)}$.

Based on the theory of semigroups of Markov processes, one can obtain from the zigzag condition (7.2): for almost all $t \geq 0$ (i.e. except with a zero Lebesgue measure set of times) the following equalities hold

$$\begin{aligned} p_t^2(u, \psi^{-1}(A)) &= p_t^1(x, A), \forall x \in X^{(1)}, \forall u \in [u]_\psi = \psi^{-1}(x), \forall A \in \mathcal{B}(X^{(1)}) \\ P_t^2(f \circ \psi)(u) &= P_t^1 f(x), \forall x \in X^{(1)}, \forall u \in [u]_\psi = \psi^{-1}(x), \forall f \in \mathcal{B}^b(X^{(1)}) \end{aligned} \tag{7.18}$$

Note that $\psi^{-1}(A) \in \mathcal{B}^*(X^{(2)})$. Therefore the transition probabilities of M^1 simulates 'equivalence classes' of transition probabilities of M^2.

Remark 18 *The connection between the kernel operator and the infinitesimal generator of the strong process Markov process allows us transform the conditions (7.11) and (7.2) as follows*

$$\begin{aligned} L^{(2)} \circ \Psi &= \Psi \circ L^{(1)} \\ L^{(2)}(f \circ \psi) &= L^{(1)} f \circ \psi, \forall f \in \mathcal{D}(L^{(1)}) \end{aligned} \tag{7.19}$$

where $L^{(1)}$ (resp. $L^{(2)}$) is the infinitesimal generator of M^1 (resp. M^2). The equality (7.19) holds provided that for each $f \in \mathcal{D}(L^{(1)})$ (the domain of L^1) the function $f \circ \psi$ belongs to $\mathcal{D}(L^{(2)})$ (the domain of $L^{(2)}$).

Since for a GSHS realization the expression of the infinitesimal generator is known, to check if the formula (7.19) is true for two given GSHS is only a computation exercise..
Recall that the realization of an GSHS has been constructed as a Markov string, i.e. a sequence of diffusion processes with a jumping structure. Then the cone of excessive functions associated to a GSHS can be characterized as a 'sum' of the excessive function cones associated to the diffusion components. This characterization 'explains' the following result.

Proposition 47 *A zigzag morphism ψ between the realizations of two GSHS H^1 and H^2 defined as in Def. 22 preserves the continuous parts of the two models.*

Proof. Suppose that the two GSHS state spaces are $X^{(1)} = \underset{i\in Q^1}{\cup} \{i\} \times X^{i(1)}$ and $X^{(2)} = \underset{q\in Q^2}{\cup} \{q\} \times X^{q(2)}$. We can suppose without loosing the generality that each two modes have empty intersection and therefore $X^{(1)} = \underset{i\in Q^1}{\cup} X^{i(1)}$ and $X^{(2)} = \underset{q\in Q^2}{\cup} X^{q(2)}$. The function ψ maps $X^{(2)}$ into $X^{(1)}$. From the construction of H^1, as Markov string, we have $V^1 f = \sum_{i\in Q^1} V^{i1} f^i, \forall f \in \mathcal{B}^b(X^{(1)})$, where, for each $i \in Q^1$, V^{i1} is the kernel operators of the component diffusion of H^1 which operates on $X^{i(1)}$ and $f^i = f|_{X^{i(1)}} \in \mathcal{B}^b(X^{i(1)})$. A similar expression can be written for V^2 (i.e. $V^2 g = \sum_{q\in Q^2} V^{q2} g^q$, $g \in \mathcal{B}^b(X^{(2)})$).
Let f be an arbitrary positive bounded measurable function on $X^{(1)}$. Then for each $i \in Q^1$ consider f^i as before. Let $Y^{i(2)} = \psi^{-1}(X^{i(1)})$ (note that $Y^{i(2)}$ is an open set) and ψ^i be the restriction of ψ, which maps $Y^{i(2)}$ into $X^{i(1)}$. Denote $g^i = f^i \circ \psi^i \in \mathcal{B}^b(Y^{i(2)})$ and $g^{iq} = g^i|_{Y^{i(2)}\cap X^{q(2)}}$. The zigzag condition (7.2) becomes $W^{i2}(f^i \circ \psi^i) = V^{i1} f^i \circ \psi^i$, where W^{i2} is the 'restriction' of V^2 to $Y^{i(2)}$, i.e. $W^{i2} g^i = \sum_{q\in Q^2} V^{q2} g^{iq}$ (more intuitively, W^{i2} is the sum of kernels associated to the component diffusions of H^2, which operate on $Y^{i(2)}$). Then, for all $x \in X^{i(1)}$ we have

$$W^{i2} g^i(u) = V^{i1} f^i(x), \tag{7.20}$$

provided that $\psi^i(u) = x$. Because V^{i1} corresponds to a diffusion process, it must be the case that in the left hand side of (7.20) the 'jumping part' to not longer exist (at least for the saturated measurable functions). Then the kernel W^{i2} corresponds to a continuous process (which might be a diffusion or a switching diffusion process).□

Any zigzag morphism ψ can be extended by (finely) continuity to the boundary of the

state spaces. Or, we can suppose from the beginning that the zigzag morphims operate on the closures of the state spaces. We have to assume that the zigzag morphims 'keep' the boundary points, or, in other words, $\psi : \partial X^{(2)} \to \partial X^{(1)}$ is surjective.

Remark 19 *The finely continuity of a zigzag morphism between the realizations of two GSHS is important only when we use the connection with the associated excessive function cones. Otherwise, we can replace this continuity with the continuity w.r.t. to the initial topologies of the state spaces.*

Proposition 48 *A zigzag morphism ψ between the realizations of two GSHS H^1 and H^2 defined as in Def. 22 preserves the jumping structure of the two models.*

Proof. For each $x \in X^{(1)}$ there exist, by surjectivity of ψ, some elements $u \in X^{(2)}$ such that $\psi(u) = x$. Then, for each $f \in \mathcal{D}(L^{(1)})$, a simple computation of the right hand side of (7.19) gives

$$L^{(1)}f(x) = L^{(1)}_{cont}f(x) + \lambda^1(x)\int_{\overline{X}^{(1)}}(f(y) - f(x))R^1(x, dy) \tag{7.21}$$

and after, the left hand side of (7.19) is

$$L^{(2)}(f \circ \psi)(u) = L^{(2)}_{cont}(f \circ \psi)(u) + \lambda^2(u)\int_{\overline{X}^{(2)}}[(f \circ \psi)(v) - (f \circ \psi)(u)]R^2(u, dv). \tag{7.22}$$

From the Prop. 47 we have the equality of the continuous parts of (7.21) and (7.22). Then the jumping parts (7.21) and (7.22) must coincide. Then

$$\lambda^1(x)\int_{\overline{X}^{(1)}}(f(y) - f(x))R^1(x, dy) = \lambda^2(u)\int_{\overline{X}^{(2)}}[(f \circ \psi)(v) - (f \circ \psi)(u)]R^2(u, dv).$$

The construction of GSHS H^1 and H^2, as Markov strings, shows that the transition measures R^1 and R^2 play the role of the transition probabilities when the processes jump from one diffusion to another. Then they satisfy (7.18), i.e.

$$R^2(u, \psi^{-1}(A)) = R^1(x, A), \forall A \in \mathcal{B}(X^{(1)}).$$

It easily follows that $\lambda^1(x) = \lambda^2(u), \forall u \in [u]_\psi = \psi^{-1}(x)$.□

Properties of bisimulation

Consider now two bisimilar GSHS, H^1 and H^2, with the realizations M^1 and M^2, respectively. Let M^{12} and ψ^1, ψ^2 as in the Def.24. Define the bisimulation relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ by formula (7.12).

If ψ^1, ψ^2 are finely open maps then the quotient processes $M^1/_{\mathcal{R}}$ and $M^2/_{\mathcal{R}}$ are equivalent (Th.45, (A)). This means that families of trajectories starting in bisimilar points can be identified in the quotient processes. The trajectories of the quotient processes will be classes of equivalent trajectories of the initial processes.

The stochastic bisimulation between two GSHS reduces to the bisimulations between their continuous components and between their jump structures. In this way our concept of bisimulation can be related with the bisimulation for piecewise deterministic Markov processes (which are particular class of GSHS) defined in terms of an equivalence relation between the deterministic flows and the probabilistic jumps [153].

7.4 A Non-Categorical Approach for Defining Bisimulation of Markov Processes

In this section we define a new concept for stochastic bisimulation which better suits to stochastic hybrid system models whose realizations are Markov processes. This section is based on our paper [48]. This new concept of bisimulation is motivated by the idea to construct bisimulations which preserve the reach set probabilities. It is well known that reachability analysis can provide useful information for diagnosis purposes and corrective action design like controller design based on reachability analysis. In this context it is worthy to remind that model checking consists of automatic methods for safety verification through reach sets computation, i.e. it requires to be able to "compute" with sets (to represent and to manipulate). This bisimulation concept is very robust because it is not based on the equality of transition probabilities of the quotient processes as in the previous section. In practice probabilities are approximated by various statistical methods and therefore equality of transition probabilities is difficult to be checked. In this context, the preservation of reach set probabilities is major breakthrough result towards applying model checking to reachability analysis.

The probability of the reachable defines a statical concept of *capacity* used to model imprecise probabilities.

The information input into different real-world models may be imprecise for several reasons. Prior information is sometime recorded in the literature as intervals without any information about probability distributions[66].

The extension of probabilistic analysis to include imprecise information is now well established in the theory of imprecise probabilities [165], robust Bayesian analysis [105] and fuzzy statistics [160].

The imprecise probabilities are modelled by sets of probability measures which might generate upper/lower probabilities [66, 77], Choquet capacities [54, 104] and others.

In the following, first, we shortly present the concept of Choquet capacity and then we give the construction of the capacity associated to a Borel right Markov process. This later concept will be connected with the stochastic reachability and then used to give a new definition for stochastic bisimulation.

7.4.1 Capacity

Intuitively, a capacity is a set function which extend the concept of measure. The additivity property is not longer true for a capacity. For every space X and algebra $\mathcal{A}$ of subsets of X a set-function $c : \mathcal{A} \to [0,1]$ is called a *normalized capacity* if it satisfies the following:

(i) $c(\emptyset) = 0$, $c(X) = 1$,

(ii) $\forall A, B \in \mathcal{A}$: $A \subset B \Rightarrow c(A) \leq c(B)$.

A capacity is called *convex (or supermodular)* if in addition to (i)-(ii) it satisfies the additional property

(iii) $\forall A, B \in \mathcal{A}$: $c(A \cup B) \geq c(A) + c(B) - c(A \cap B)$.

A special type of convex capacities are the *belief functions* presented and discussed by Dempster [66] and Shafer [144]. A capacity is called a probability if (iii) holds everywhere with equality, i.e. it is additive. If a capacity satisfies the inverse inequality in (iii) then it is called *submodular or strongly subadditive.*

Since we allow the possibility that c is not additive, we can not use the integral in the Lebesgue sense to integrate with respect to c. The notion of integral we will use is due originally

to Choquet [54] and it was independently rediscovered and extended by Schmeidler [143]. If $f : X \to \mathbb{R}$ is bounded $\mathcal{A}$-measurable function and c is any capacity on X we define the Choquet integral of f with respect to c to be the number

$$\int_X f(x)dc(x) = \int_0^\infty c(\{x \in X | f(x) \geq \alpha\})d\alpha + \int_{-\infty}^0 [c(\{x \in X | f(x) \geq \alpha\}) - 1]d\alpha$$

where the integrals are taken in the sense of Riemann.

7.4.2 Capacity associated to a Markov process

Suppose that $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x), \in Q$ is a Markov process. We denote the state space of M by $(X, \mathcal{B})$ and assume that $\mathcal{B}$ is the Borel σ-algebra of X if X is a topological Hausdorff space. Let Δ be the cemetery point for X, which is an adjoined point to X, $X_\Delta = X \cup \{\Delta\}$. The existence of Δ is assumed in order to have a probabilistic interpretation of $P_x(x_t \in X) < 1$, i.e. at some 'termination time' $\zeta(\omega)$ when the process M escapes to and is trapped at Δ. The elements $\mathcal{F}$, $\mathcal{F}_t^0$, $\mathcal{F}_t$, P_x have the usual meaning as in Section 2.2.5.

In the following of this chapter, $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ will be a Borel right Markov process (see. for example, [62] and the references therein) on $(X, \mathcal{B})$. This means that:

- Its state space $(X, \mathcal{B})$ will be in a Lusin state space (i.e. X is a separable metric space homeomorphic to a Borel subset of some compact metric space, with Borel σ-algebra $\mathcal{B}(X)$ or shortly $\mathcal{B}$) and it will be equipped with a σ-finite measure m.

- M is a strong Markov process with cadlag property.

We assume also that M is *transient*. More, we suppose that $\sup_{x \in X} V1(x) < \infty$ (where $Vf = \int_0^\infty P_t f dt$ is the *kernel operator*). For each $x \in X$, the kernel V will provide a measure V_x defined by

$$V_x(A) = VI_A(x), \forall A \in \mathcal{B}$$

and for any measurable positive function f on X we have $Vf(x) = \int f dV_x$.

More, we have

$$V_x(A) = (m \otimes P_x)(\{(t, \omega) | x_t(\omega) \in A\}).$$

Therefore, $V_x(A)$ 'measures' two aspect: (i) the length of time spent by the process in A and (ii) the probability of the trajectories which start in x and reach A at some times $t \in [0, \infty)$.

One can take the sample space Ω for M to be the set of all paths $(0, \infty) \ni t \mapsto \omega(t) \in X_\Delta$ such that (i) $t \mapsto \omega(t)$ is X-valued and cadlag on $(0, \zeta(\omega))$ where $\zeta(\omega) := \inf\{s > 0 | \omega(s) = \Delta\}$, (ii) $\omega(t) = \Delta$ for all $t \geq \zeta(\omega)$, and (iii) $\zeta(\omega) < \infty$. In this way, M is realized as the coordinate process on Ω: $x_t(\omega) = \omega(t)$, $t > 0$. We complete the definition of M by declaring $x_0(\omega) = \lim_{t \searrow 0} \omega(t)$, $t > 0$.

Because of transience condition, the measure m is purely excessive [79]:

$$\lim_{t \to \infty} (m < P_t >)(A) = 0, \forall A \in \mathcal{B} \text{ with } m(A) < \infty,$$

where $(m < P_t >)(A) = \int p_t(x, A) m(dx)$ and $p_t(x, A) = P_t(I_A)(x) = P_x(x_t \in A)$.
Consequently there is a unique entrance law $(\mu_t)_{t>0}$ (a family of σ-finite measures on $(X, \mathcal{B})$ with $\mu_t < P_s >= \mu_{t+s}$ for all $t, s > 0$) such that

$$m(A) = \int_0^\infty \mu_t(A) dt, \forall A \in \mathcal{B}.$$

See, for example, [79] for more details. Then there is a σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_t^0, \mathcal{F}^0)$ (see [78]) under which the coordinate process $(x_t)_{t>0}$ is Markovian with transition semigroup $(P_t)_{t \geq 0}$ and one-dimensional distributions

$$\mathbb{P}(x_t \in A) = \mu_t(A), \forall A \in \mathcal{B}, t > 0.$$

The *capacity* associated to M is defined as follows (see [78] and the references therein): for all $B \in \mathcal{B}$

$$Cap_M(B) = \mathbb{P}(T_B < \infty) = \mathbb{P}(T_B < \zeta),$$

where T_B is the first hitting time of B, i.e. $T_B = \inf\{t > 0 | x_t \in B\}$.
The initial definition of this notion gives the capacity Cap_M as an upper envelope of e non-empty class of probability measures on $\mathcal{B}$. It can be shown that this capacity is monotone increasing, submodular, countably subadditive [78]. Then its conjugate Cap_M^* [143], defined by $Cap_M^*(B) = 1 - Cap_M(X - B)$ is a belief function in sense of [144]. Beliefs about the

evolution process ($\omega(t)$) conform to a time-homogeneous Markov structure. In standard models, this would involve a stochastic kernel giving conditional probabilities. We assume that beliefs conditional on $\omega(t)$ are too vague to be represented by a probability measure and are represented instead by a family of probability measures whose lower envelope is Cap^*_M.

7.4.3 Stochastic Reachability Analysis

Let us consider $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$, as in the previous subsection, the realization of a stochastic hybrid system H. To address the reachability problem assume that we have a given set $E \in \mathcal{B}(X)$ and a horizon time $T > 0$. Let us define

$$\begin{aligned} Reach_T(E) &= \{\omega \in \Omega \mid \exists t \in [0,T] : x_t(\omega) \in E\} \\ Reach_\infty(E) &= \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in E\}. \end{aligned} \tag{7.23}$$

These two sets are the sets of trajectories of H, which reach the set E (the flow that enters E) in the interval of time $[0,T]$ or $[0,\infty)$. The reachability problem consists of determining the probability of such sets. That means we have to determine $\mathbb{P}(T_E < T)$ or $\mathbb{P}(T_E < \infty)$. In this way, the reachability problem is related with the computation of the capacities associated to the processes M_T and M, where M_T is the process M "killed" after the time T (see [62] for the details about the killed process).

On the other hand, we would like to characterise the sets

$$\begin{aligned} Reach^{init}_T(E) &= \{x \in X | \exists \omega \in \Omega, \exists t \in [0,T] : \ \phi(t,\omega,x) \in E\} \\ Reach^{init}_\infty(E) &= \{x \in X | \exists \omega \in \Omega, \exists t \in [0,\infty) : \ \phi(t,\omega,x) \in E\} \end{aligned}$$

where $\phi(t,\omega,x)$ is a trajectory of M starting with $x \in X$. These are thought of as sets of initial points, which give trajectories of M with nonempty intersection with E.

Lemma 49 *For any measurable set $E \in \mathcal{B}$ and for $T > 0$, we have*

$$Reach^{init}_T(E) = \{x \in X | \sup_{t \in [0,T]} P_t I_E(x) > 0\}.$$

Proposition 50 *If M has the càdlàg property and G is an open set of X then*

$$Reach_{\infty}^{init}(G) = \{x \in X | V_x(G) > 0\}.$$

Remark 20 *The measure V_x does not have enough ability for our purposes: a trajectory ω that reaches the set E is accounted for every 'visit' in E. This weakness is eliminated when considering the measure $\mathbb{P}(T_E < \infty)$.*

7.4.4 Stochastic Bisimulation

In this section we define a stochastic bisimulation concept which preserves the probabilities of reachable sets.

Let $(X^{(1)}, \mathcal{B}(X^{(1)}))$ and $(X^{(2)}, \mathcal{B}(X^{(2)}))$ be analytic spaces and let $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ be a relation such that $\Pi^1(\mathcal{R}) = X^{(1)}$ and $\Pi^2(\mathcal{R}) = X^{(2)}$. Let $\mathcal{R}^1$ and $\mathcal{R}^2$ be the equivalence relations induced on $X^{(1)}$ and $X^{(2)}$, respectively, by $\mathcal{R}$ (see formula (7.13)). Consider the quotient spaces $(X^{(i)}/_{\mathcal{R}^i}, \mathcal{B}^*(X^{(i)}))$, $i = 1, 2$.

We define a bijective mapping

$$\begin{aligned} \varphi &: X^{(1)}/_{\mathcal{R}^1} \to X^{(2)}/_{\mathcal{R}^2} \\ \varphi([x^1]) &= [x^2] \end{aligned}$$

provided that $(x^1, x^2) \in \mathcal{R}$ for some $x^1 \in [x^1]$ and some $x^2 \in [x^2]$.

We say that the relation $\mathcal{R}$ is *measurable* if for all $A^1 \in \mathcal{B}^*(X^{(1)})$ we have $\varphi(A^1) \in \mathcal{B}^*(X^{(2)})$ and vice versa, i.e. φ is a homeomorphism. This concept of measurable relation coincides with the weak measurable relation from Subsection 7.3.4. Then the real measurable functions defined on $X^{(1)}/_{\mathcal{R}^1}$ can be identified with those defined on $X^{(2)}/_{\mathcal{R}^2}$ through the homeomorphism φ. We can write

$$\mathcal{B}^{*b}(X^{(1)}) \overset{\varphi}{\cong} \mathcal{B}^{*b}(X^{(2)}).$$

Moreover, these functions can be thought as real functions defined on $X^{(1)}$ or $X^{(2)}$ measurable with respect to $\mathcal{B}^*(X^{(1)})$ or $\mathcal{B}^*(X^{(2)})$.

Definition 51 *Suppose we have the capacities $c_{X^{(1)}}$ and $c_{X^{(2)}}$ on analytic spaces $(X^{(1)}, \mathcal{B}(X^{(1)}))$*

and $(X^{(2)}, \mathcal{B}(X^{(2)}))$ *respectively. Suppose that we have a measurable relation* $\mathcal{R} \subset X^{(1)} \times X^{(2)}$. *The capacities* $c_{X^{(1)}}$ *and* $c_{X^{(2)}}$ *are called equivalent with respect to* $\mathcal{R}$ *if they define the same capacity on the quotient space of* $X^{(1)}$ *and* $X^{(2)}$, *i.e. if we have* $c_{X^{(1)}}(\pi^{-1}_{X^{(1)}}(A^1)) = c_{X^{(2)}}(\pi^{-1}_{X^{(2)}}[\varphi(A^1)])$ *for all* $A^1 \in \mathcal{B}(X^{(1)}/_{\mathcal{R}^1})$.

Suppose we have two Borel right Markov processes M^1 and M^2 with the state spaces $X^{(1)}$ and $X^{(2)}$.

Definition 52 *A measurable relation* $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ *is a bisimulation between* M^1 *and* M^2 *if their associated capacities* Cap_{M^1} *and* Cap_{M^2} *are equivalent with respect to* $\mathcal{R}$.

It is known that if two processes are symmetric and are defined on the same state space, the equality of their capacities implies that they are time changes of one another [78].

We can define now a *pseudometric with respect to a measurable relation* $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ between the processes M^1 and M^2 as follows:

$$d_{\mathcal{R}}(M^1, M^2) = \sup_{f \in \mathcal{B}^{*b}(X^{(1)})} |\int f dCap_{M^1} - \int f \circ \varphi dCap_{M^2}|$$

where $\mathcal{B}^{*b}(X^{(1)})$ is the set of bounded real $\mathcal{B}^*(X^{(1)})$-measurable functions on $X^{(1)}$.

Remark 21 *We can define a distance between two processes if and only if there exists a relation on the product of their state spaces* $X^{(1)} \times X^{(2)}$ *such that the two quotient spaces are homeomorphic. Or, equivalently, if there exists a third measurable space* $(Z, \mathcal{B}(Z))$ *and two surjective measurable mappings* $\phi_1 : X^{(1)} \to Z$ *and* $\phi_2 : X^{(2)} \to Z$ *then*

$$d(M^1, M^2) = \sup_{f \in \mathcal{B}^b(Z)} |\int f \circ \phi_1 dCap_{M^1} - \int f \circ \phi_2 dCap_{M^2}|$$

where $\mathcal{B}^b(Z)$ *is the set of bounded real* $\mathcal{B}^b(Z)$*-measurable functions on* Z.

Proposition 53 *A measurable relation* $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ *is a bisimulation between* M^1 *and* M^2 *if and only if*

$$d_{\mathcal{R}}(M^1, M^2) = 0$$

Suppose we have given two stochastic hybrid systems H and H' with the realizations M^1 and M^2 (with the state spaces $X^{(1)}$ and $X^{(2)}$).

Definition 54 *H and H' are bisimilar if there exists a measurable relation $\mathcal{R} \subset X^{(1)} \times X^{(2)}$ such that $\mathcal{R}$ is a bisimulation between M^1 and M^2.*

Proposition 55 *$\mathcal{R} \subset X^{(1)} \times X^{(2)}$ is a bisimulation relation between H and H' if and only if the probabilities of reachable events (7.23) associated to "saturated" (with respect to $\mathcal{R}$) Borel sets are equal, i.e.*

$$\mathbb{P}_{M^1}(T_E < \infty) = \mathbb{P}_{M^2}(T_{\varphi(E)} < \infty)$$

for all $E \in \mathcal{B}^*(X^{(1)})$.

The proof is a clear consequence of definition of a bisimulation relation between two Markov processes.

The above proposition shows that our definition of bisimulation between stochastic hybrid systems is natural since the probabilities of the reachable events are preserved. Then naturally, the reachability analysis of a stochastic hybrid system can be performed using much simpler stochastic hybrid systems bisimilar with the given one.

7.5 Some Remarks

In this chapter we have investigated some ways to define the concept of stochastic bisimulation for classes of Markov processes which can represent realizations of different models for stochastic hybrid systems. One way is based on the category theory: define an appropriate category of strong Markov processes defined on analytic spaces. The morphisms in this category are the zigzag morphims. A zigzag morphism between two Markov processes is a surjective (finely) continuous measurable functions between their state spaces which 'commutes' with the kernel operators of the processes considered. An important technical contribution is the proof that this stochastic bisimulation is indeed an equivalence relation. We also give a characterization of this bisimulation. Then we derive the properties characteristic to the bisimulation relation for GSHS.

The second approach to define stochastic bisimulation between Markov processes is based the concept of Choquet capacity. Each process can have associated, in a canonical way, a Choquet capacity, which for each measurable set of the state space is the probability to reach that set. Then a bisimulation relation between two processes is defined as a measurable relation that "preserves" the capacity. Further, we have employed this bisimulation to define bisimulation between stochastic hybrid systems whose realizations are Markov processes as above. This bisimulation is more robust since the computation of the capacities associated to our models is possible using Bayesian statistics algorithms.

From the verification and analysis of stochastic hybrid systems perspective, a concept of stochastic bisimulation can facilitate the way towards a model checking of stochastic hybrid systems.

The work presented in this chapter and the above discussion allow us to point out some possible research directions in the stochastic hybrid system framework:

- Use the stochastic bisimulation to get manageable sized system abstractions;
- Use the stochastic bisimulation to investigate the reachability problem;
- Make a comparative study of the different approaches on reachability analysis for stochastic hybrid systems:

1. the approach based on the hitting times and hitting probabilities for a target set [40];
2. the approach based on the so-called Dirichlet forms and excessive functions [41];
3. the approach based on Lyapunov function (for the switching diffusion processes) [164].

Chapter 8

Conclusions and Further Work

Many practical systems such as automobiles, chemical processes, and autonomous vehicles are best described by dynamics that comprise continuous state evolution within a mode of operation and discrete transitions from one mode to another, either controlled or autonomous. Such systems often interact with their environment in the presence of uncertainty and variability. Stochastic hybrid systems can model complex dynamics, uncertainty, and multiple modes of operations and they can support high-level control specifications that are required for design of autonomous or semi-autonomous applications. Stochastic hybrid system models have been successfully used to represent abrupt value changes in the parameters of economic systems, component/sensor failures in power, aircraft, and manufacturing systems, sudden system interconnection reorganization in power systems and large flexible structures, pilot commands in target tracking, and sudden environmental disturbances like clouds in isolation levels of a solar receiver. Meanwhile, this type of systems are used to model the random jump behaviour of integrated communication networks, which are used to transmit, on a single medium, different classes of traffic, such as voice, images and data. These models have been applied to describe the bahaviour of continuous systems driven by discrete event models.

Several modelling paradigms for stochastic hybrid systems have been already proposed in literature.

- A stochastic hybrid scheme that allows the continuous flows at each discrete location to be characterized by stochastic differential equations is described in [106]. An extension of

this model that satisfies the strong Markov property is presented in [41] and a method to study the reachability problem is proposed.

- A similar model based on Piecewise Deterministic Markov Processes is presented in [40] for studying a probabilistic reachability problem.

- Probabilistic hybrid automata are introduced in [101] for estimation and fault diagnosis.

- Communicating Piecewise Deterministic Markov Processes are proposed as compositional specifications for stochastic hybrid systems in [154] with emphasis on modelling concurrency.

- Applications of stochastic hybrid systems to air traffic management systems are presented in [137, 83].

- A stochastic hybrid system with application to communication networks is presented in [98, 100]. This SHS model has been developed from the necessity to provide accurate models for TCP (Transport Control Protocol) congestion control in communication networks. There exist various strategies to deal with congestion control of TCP, known as active queue management (AQM) policies (see [162] and the references therein). One commonly used AQM is the Random Early Detection (RED) [163]. The use of hybrid models to characterise the behaviour of congestion control was proposed first in [97], and after [75]. In [75], it is proposed a hybrid system model to capture the interaction between TCP and RED for a simple network connection experiencing congestion. In [100], it is constructed an SHS model (particular case of GSHS) for on-off TCP flows that considers both the congestion-avoidance and slow-starts modes and takes directly into account the distribution of the number of bytes transmitted.

- A particular class of stochastic hybrid systems with applications in the modelling of some chemical processes is presented in [99].

- A modelling framework and a simulation environment for concurrent stochastic hybrid systems is presented [20].

- For those hybrid systems, whose discrete state transitions depend on both deterministic and stochastic events, a suitable model called Discrete Hybrid Stochastic Automaton has been introduced in [18]. The Discrete Hybrid Stochastic Automaton model can be seen as a discrete time counterpart of the Piecewise Deterministic Markov process. The differences between these models are in the way randomness affects the continuous and the discrete dynamics in their interaction.

In this thesis we have developed a very general modelling framework for stochastic hybrid systems - General Stochastic Hybrid Systems. We have studied different mathematical characterizations of this general model (Markov property, strong Markov property, infinitesimal generator, cadlag property). These constitute the foundation of the stochastic reachability analysis for General Stochastic Hybrid Systems. As well, we have proposed Distributed Stochastic Hybrid Systems as compositional specifications for General Stochastic Hybrid Systems.

Another very important issue we have tackled in this thesis is the setting of the stochastic reachability problem as the main verification tool for stochastic hybrid systems. First achievement was to well-define the reachability problem in probabilistic framework. Then we have proposed some analytical solutions to estimate the reach set probabilities. Since the computation of these analytical solutions is hard because of the complicated mathematical formulas we were looking to find a way to simplify the stochastic hybrid system models preserving the reach set probabilities. In order to achieve this goal, we have introduced a concept of bisimulation for stochastic hybrid systems, which allows us to simplify the initial models and to deal with the reachability problem for much friendly probabilistic models. We believe that this stochastic bisimulation concept will constitute the basis for probabilistic model checking for stochastic hybrid systems.

In the following, we sketch some possible research directions which can be derived from the work presented in this thesis.

1. Study different ways to introduce abstracting equivalences for hybrid systems. We have seen that the categorical bisimulation concept for SHS, introduced in Chapter 7, is quite strong because it implies the equivalence of the stochastic processes which represent the semantics of the respective SHS. The bisimulation preserving the reach set probabilities,

developed in the second part of the same chapter, represents a significant step in the abstractization process for SHS. We need weaker versions of bisimulations which preserve only the properties we are interested in. One way to get this goal is to consider SHS bisimulations which preserve different statistical parameters of the respective SHS (first order moments, variance, expectation of some hitting times, etc). This direction can lead to very nice approximation results for SHS since the statistical analysis provides us a rich collection of tools which allow to estimate the behaviour of a stochastic process from its statistical parameters.

2. Investigate the decidable classes of SHS. It is clear that we can not obtain decidability results in the general framework of SHS. This goal is very demanding and may seem impossible to achieve without being restrictive in some way or another. Therefore, we propose an incremental approach. Start, for example, in a first instance, with PHA with general probability distributions and aim to generalize the results from [152]. After that, in order to deal with stochastic differential equations we have to use approximation results for diffusion processes. In this direction, the results developed in [116] can be really useful since they offer the possibility to approximate a particular class of SHS, with diffusions between jumps, by Markov Decision Processes. For the later ones, there exists a well-developed theory of abstractions (see, e.g., the survey [109]).

3. Set up the foundations of formal specification of Stochastic Hybrid Systems. One candidate to specifying formally SHS seems to be process algebra and its stochastic and hybrid extensions. This approach requires a bisimulation concept for SHS that we have introduced in this thesis and we have extended it to general Markov processes. However, from the control and software engineering perspectives a model theoretic approach like Z might be also suitable.

4. The above research directions should not be seen disjoint. The ultimate goal is to combine the results from computer science, mathematics, and control theory, to develop methodologies for verifying temporal properties of SHS. In our view, however, what is still missing is a unified framework (although such a framework may not necessarily be the only one that can be proposed) that can directly handle systems with hybrid dynamics, nonlinear-

ity, uncertainty, constraints, stochasticity, and so on. Moreover, many of the currently available techniques suffer from scalability issues: their computational cost grows exponentially with respect to the system size. We have to mention that the first proposed computational method that can provide a verifiable upper bound on the reach probability is that proposed in [139]. Needless to say, the area of verification for stochastic hybrid systems is still in its infancy, and we expect to see many more developments in the upcoming years.

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