Encyclopaedia of Proof Systems

http://ProofSystem.github.io/Encyclopedia/

Preface

1^{st} Edition

The **Encyclopedia of Proof Systems** aims at providing a reliable, technically accurate, historically informative, concise, uniform and convenient central repository of proof systems for various logics. The goal is to facilitate the exchange of information among logicians, in order to foster and accelerate the development of proof theory and automated deduction.

Preparatory work for the creation of the Encyclopedia, such as the implementation of the LaTeX template and the setup of the Github repository, started in October 2014, triggered by the call for workshop proposals for the 25th Conference on Automated Deduction (CADE). Christoph Benzmüller, CADE's conference chair, and Jasmin Blanchette, CADE's workshop co-chair, encouraged me to submit a workshop proposal and supported my alternative idea to organize instead a special poster session based on encyclopedia entries. I am thankful for their encouragement and support.

In December 2014, Björn Lellmann, Giselle Reis and Martin Riener kindly accepted my request to beta-test the template and the instructions I had created. They submitted the first few example entries to the encyclopedia and provided valuable feedback, for which I am grateful. Their comments were essential for improving the templates and instructions before the public announcement of the encyclopedia.

In July 2015, Julian Röder's assistance was essential for the successful organization of the poster session at CADE. Cezary Kaliszyk and Andrei Paskevitch kindly allowed me to organize a discussion session as part of the Proof Exchange for Theorem Proving (PxTP) workshop, where the participants provided useful feedback and many ideas for improvements. Discussions with Lev Beklemishev, Björn Lellmann, Tomer Libal, Roman Kuznets, Sergei Soloviev, Valeria de Paiva and Anna Zamansky also brainstormed many ideas for improving the organization and structure of the encyclopedia.

In the few months that preceded CADE, as many as 64 entries, spanning a wide range of deduction styles and logics, have been submitted by 34 contributors. Although large for a single event, these numbers are still small compared to the vast number of proof systems that have been invented and to the number of people who work on logical calculi nowadays. Therefore, this community-wide initiative is only at the beginning and the encyclopedia intends to remain open to submissions for a long time.

October 2016 *Bruno Woltzenlogel Paleo*

Preface

2^{nd} Edition

In December 2014, I had the honor to submit one of the first entries to the **Encyclopedia of Proof Systems** at the request of Bruno Woltzenlogel Paleo. Less than one year later, the Encyclopedia already counted 64 entries, which were presented at a poster session during CADE-25. After this successful event, Bruno has kindly invited me to co-organize with him a workshop during the Brasilia Spring on Automated Deduction, formed by the conferences TABLEAUX, ITP (Interactive Theorem Proving) and FroCoS (Frontiers of Combining Systems), in September 2017.

The EPS workshop comprised of presentations of new entries by the authors, an open discussion about the Encyclopedia (suggestion of improvements and long-term goals), and a hands-on session for active contributions. The workshop was accompanied by a poster session where the newest entries were displayed. We would like to thank Katalin Bimbó, Serenella Cerrito, Clare Dixon, Reiner Hähnle, Rolf Hennicker, Ullrich Hustadt, Björn Lellmann, João Marcos, Renate Schmidt, and Yoni Zohar for participating in the workshop and contributing to the discussions. There was a wide variety of interesting and accessible talks about proof systems in different areas, and many suggestions of new entries and features for the Encyclopedia. We would also like to thank Cláudia Nalon for all her support with the logistics of the workshop and for organizing a great conference.

In total, 29 new entries were submitted to the Encyclopedia of Proof Systems. Once again, a wide range of calculi is represented, such as resolution, sequent, axiomatic, display, and natural deduction. In addition to different logics (e.g., temporal, paraconsistent, hybrid, epistemic, etc.), there are calculi for different systems as well, such as unification and structured specifications. We are particularly happy to include in this new edition Hilbert's, Bernay's and Ackermann's calculi, thanks to Richard Zach. Many people had expressed that those historically important systems deserved an entry in the Encyclopedia. Now they finally have a place here.

This second edition of the Encyclopedia of Proof Systems book extends the first edition with the 34 new entries. Additionally, with the aim of encouraging practical applications of proof systems, a new meta-data tag for implementations or formalizations of an entry is now available.

New proof systems are proposed each day, so the Encyclopedia will always be open for new contributions. With almost 100 entries on the most diverse systems, this effort of knowledge organization can only succeed as a joint effort of the community. We are grateful for the support we have received so far and hope the Encyclopedia continues to grow in the years to come.

December 2017 *Giselle Reis*

Preface

3^{rd} Edition

The third edition of the Encyclopedia of Proof Systems has only 2 new entries compared to the second edition, but it has a key technical improvement in its LaTeX code.

Editing a large document in LaTeX composed of many smaller and independently authored chapters can become complicated, because the LaTeX system has limits on the number of packages and alphabets that can be used simultaneously within certain environments. In the case of the Encyclopedia of Proof Systems, soon after the second edition, we reached some limits that prevented us from adding more entries. Fortunately, we are pleased to announce that we have managed to resolve this issue. The Encyclopedia of Proof Systems is ready to grow again and is once again open to more submissions.

July 2022 *Bruno Woltzenlogel Paleo and Giselle Reis*

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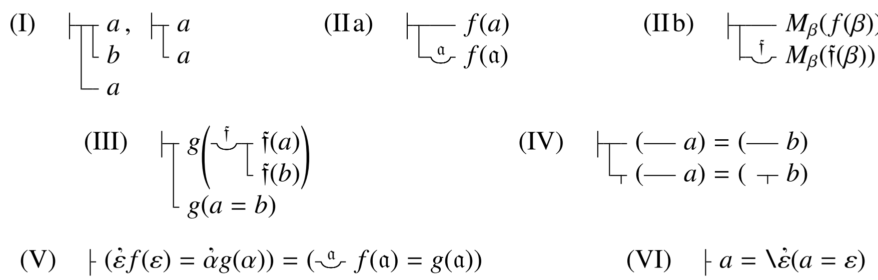
Part I
Proof Systems

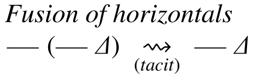
Frege's Concept-Script (*Grundgesetze der Arithmetik*) (1893)

Frege's six Basic Laws, as presented in his *Grundgesetze der Arithmetik* I (Jena, 1893):



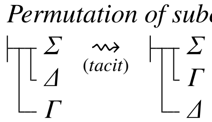
And his rules of inference:

1. *Fusion of horizontals*

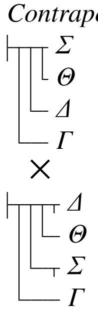


where the horizontal, —, is understood to include the horizontal-stroke portions of ⊤, ⌊, and ◡.

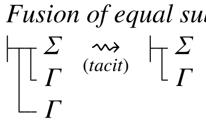
2. *Permutation of subcomponents*



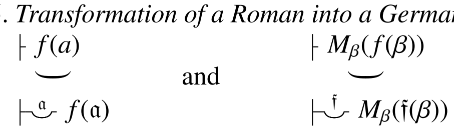
3. *Contraposition* [generalized]



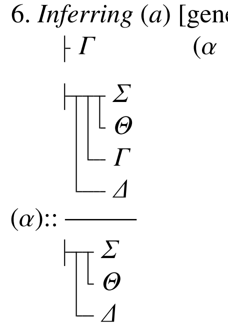
4. *Fusion of equal subcomponent*



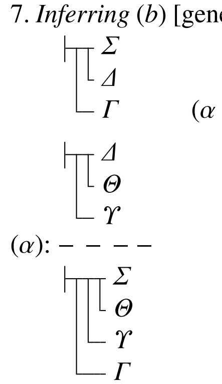
5. *Transformation of a Roman into a German letter*



6. *Inferring* (*a*) [generalized *modus ponens*]



7. *Inferring* (*b*) [generalized *hypothetical syllogism*]



Entry 2 by: Roy T. Cook and Philip A. Ebert and Marcus Rossberg

Frege's rules of inference, cont'd:

8. *Inferring* (c) [generalized *dilemma*]

```
├┬┬┬ Σ
 │││└ Υ
 ││└─ Δ
 │└── Γ          (α
├┬┬┬ Σ
 │││└ Ξ
 ││└┬ Δ
 │└── Θ
(α): -·-·-·-
├┬┬┬┬ Σ
 ││││└ Ξ
 │││└─ Θ
 ││└── Υ
 │└─── Γ
```

9. *Replacement of Roman letters*
Roman letters may uniformly be replaced by other Roman letters, constants, or complex expressions of the appropriate type. [Note that since there are no free variables (German letters or Greek vowels), no provision for illicit variable binding is required.]

10. *Replacement of German letters*
German letters (bound variables for quantification) may uniformly be replaced by other German letters of the appropriate type, provided the latter is free for the former.

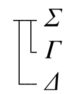
11. *Replacement of Greek vowels*
Greek vowels (bound variables for value-range notation) may uniformly be replaced by other Greek vowels, provided the latter is free for the former.

Clarifications: Frege's $\neg$, $\zeta \rightarrow \xi$, $\forall\mathfrak{a}\, f\varphi(\mathfrak{a})$, $\forall\mathfrak{f}\, M_\beta(\mathfrak{f}(\beta))$, and $\zeta = \xi$ correspond, roughly, to contemporary negation, conditional, first-order universal quantification, second-order universal quantification (over unary functions), and identity. In the conditional, ζ is the *subcomponent* (the antecedent, in contemporary terms) and ξ is the *supercomponent* (the consequent). $\dot{\varepsilon}\varphi(\varepsilon)$ is a unary second-level function mapping functions to objects (the value-range operator), and $\backslash\xi$ is a unary function mapping objects to objects (the backslash operator, a kind of proto-definite description operator). It is important to note that negation is a *total* unary function mapping objects in the domain to truth values (which are included in the domain); hence, $\neg 2$ is a name of the True, thus $\vdash\neg 2$, as Frege notes in §6 of *Grundgesetze* I [2]. Likewise, $\zeta \rightarrow \xi$ names a binary function from objects to truth values, and $\forall\mathfrak{a}\, \varphi(\mathfrak{a})$ names a binary function from unary functions to truth values. Importantly, $\zeta = \xi$, a binary function from objects to truth values, does double duty: as the standard notion of identity, and as a biconditional, expressing that the two arguments name the same truth value. Interestingly, $\zeta = \xi$ does not name the same function as the conjunction of $\zeta \rightarrow \xi$ and $\xi \rightarrow \zeta$, although they agree when their inputs are truth values.

For easier legibility, some of the rules above are not given in the full generality in which Frege presents them. Frege notes that multiple embedded conditionals can be analyzed into supercomponent and subcomponent in multiple ways. Hence, we can analyze:

```
┬┬ Σ
│└ Γ
└─ Δ
```

as having Σ as supercomponent, and both Δ and Γ as subcomponents, or we can analyze it as having $\Gamma \rightarrow \Sigma$ as supercomponent, and Δ as subcomponent. Thus, *Permutation of Subcomponents* allows for the interchange of any two subcomponents on any (single) way of analyzing an expression into supercomponents and subcomponent(s) and, likewise, *Inferring (b)* (Generalized Hypothetical Syllogism) allows the replacement of any subcomponent Δ in one formula with all subcomponenents from a second formula whose supercomponent is Δ, on any way of analyzing those formulas into supercomponents and subcomponents. Similar comments apply to the other rules. This flexibility stems from the fact that, when read from a contemporary perspective, Frege's notation incorporates a systematic (and efficient!) ambiguity. We can understand:



as corresponding both to $(\Delta \rightarrow (\Gamma \rightarrow \Sigma))$ and as $((\Delta \wedge \Gamma) \rightarrow \Sigma)$ (note that these correspond to the two ways of dividing this formula into supercomponent and subconmponent(s).) Hence, a generalized version of exportation is built into the notation, and this justifies the flexibility of Frege's propositional rules of inference.

For more details on Frege's logic, see [4], especially the Translators' Introduction and the Appendix, "How to Read *Grundgesetze*", by Roy T. Cook.

History: The formal logic of *Grundgesetze* is an extension of the first formulation of what is essentially modern first- and higher-order predicate logic, which appeared in the earlier *Begriffsschrift* (1879) [1]. The system in *Begriffsschrift* is, setting aside the problematic treatment of substitution and of identity (see [3] for discussion), essentially modern second-order logic. *Grundgesetze* incorporates several innovations not found in the original system of *Begriffsschrift*, including a more sophisticated treatment of identity, and the value-range and backslash operators governed by Basic Laws V and VI respectively. As is well known, however, this expanded system falls prey to the Russell paradox.

Despite the inconsistency of the mature, *Grundgesetze* version of Frege's logic, the system in question represents a copernican revolution in the development of logic, resolving a number of issues that had been plaguing 19^{th} century work in logic, including:

- isolating the quantifier(s) as independent operators that applied to functions;
- unifying propositional logic and syllogistic (proto-quantificational) logic;
- analyzing logical operators as functions from arguments to truth values;
- formalizing propositions with multiple and embedded quantifiers;
- extending logical analysis to relations of arity $n > 1$, and to relations with arguments of multiple types.

These are made possible by Frege's innovation of analyzing sentences into function and argument, as opposed to the subject/predicate analysis as found in syllogistic. As a result of the resolution of these problems, the logic of *Grundgesetze* was the first formal system able to adequately formalize propositions of, and arguments in, contemporary mathematics.

[1] Gottlob Frege. *Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle a. d. Saale: Nebert, 1879.
[2] Gottlob Frege. *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet*. Jena: Pohle, 1893.
[3] George Boolos. "Reading the *Begriffsschrift*". In: *Mind* 94 (1985), pp. 331–344.
[4] Gottlob Frege. *Basic Laws Laws of Arithmetic.* Volume I & II. Trans. and ed. by Philip A. Ebert and Marcus Rossberg. Oxford: Oxford University Press, 2013.